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CONTACT METRIC MANIFOLDS SATISFYING A NULLITY CONDITION

BY

DAVID E. BLAIR

Department of Mathematics
Michigan State University
East Lansing, MI 48824, USA

AND

THEMIS KOUFOGIORGOS

Department of Mathematics
University of Ioannina
Ioannina 45110, Greece

AND

BASIL J. PAPANTONIOU

Department of Mathematics
University of Patras
Patras 26110, Greece

Dedicated to Professor Chorng-Shi Houh on his 65th birthday

ABSTRACT

This paper presents a study of contact metric manifolds for which the characteristic vector field of the contact structure satisfies a nullity type condition, condition (∗) below. There are a number of reasons for studying this condition and results concerning it given in the paper: There exist examples in all dimensions; the condition is invariant under D-homothetic deformations; in dimensions > 5 the condition determines the curvature completely; and in dimension 3 a complete classification is given, in particular these include the 3-dimensional unimodular Lie groups with a left invariant metric.

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1. Introduction

It is well known that there exist contact metric manifolds, $M^{2n+1}(\varphi,\xi,\eta,g)$, for which the curvature tensor R and the direction of the characteristic vector field ξ satisfy $R(X,Y)\xi = 0$, for any vector fields X, Y on M^{2n+1}. For example, the tangent sphere bundle of a flat Riemannian manifold admits such a structure [2]. Applying a D-homothetic deformation [11] to a contact metric manifold with $R(X,Y)\xi = 0$ we obtain a contact metric manifold satisfying

$$(*)\qquad R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

where κ, μ are constants and $2h$ is the Lie derivative of φ in the direction ξ. An essential characteristic of the class of contact metric structures defined by $(*)$ is that the form of $(*)$ is invariant under a D-homothetic deformation. The existence and the invariance of $(*)$ have been our motivation in studying this kind of manifold.

Section 2 is devoted to preliminaries on contact metric manifolds. In Section 3 we prove that for $\kappa < 1$, the curvature tensor is completely determined by the condition $(*)$. As a consequence, we draw the conclusion that these manifolds have constant scalar curvature. In Section 4 we study the three-dimensional case ($n = 1$) more extensively and we prove that these manifolds are either Sasakian or locally isometric to one of the following Lie groups: SU(2) (or SO(3)), SL$(2,R)$ (or O(1,2)), E(2), E(1,1) with a left invariant metric. We remark that the Heisenberg group carries a natural Sasakian structure.

Finally, in Section 5 we prove that the standard contact metric structure of the tangent sphere bundle T_1M satisfies the condition $(*)$ if and only if the base manifold is of constant sectional curvature.

2. Preliminaries on contact manifolds

A differentiable $(2n+1)$-dimensional manifold M^{2n+1} is called a **contact manifold** if it carries a global differential 1-form η such that $\eta\wedge(d\eta)^n \neq 0$ everywhere on M^{2n+1}. This form η is usually called the **contact form** of M^{2n+1}. It is well known that a contact manifold admits an **almost contact metric structure** (φ,ξ,η,g), i.e. a global vector field ξ, which will be called the **characteristic vector field**, a $(1,1)$ tensor field φ and a Riemannian metric g such that

$$(2.1)\qquad \varphi^2 = -\mathrm{Id} + \eta\otimes\xi, \quad \eta(\xi) = 1,$$

$$(2.2)\qquad g(\varphi X,\varphi Y) = g(X,Y) - \eta(X)\eta(Y),$$

for any vector fields X, Y on M^{2n+1}. Moreover, (φ, ξ, η, g) can be chosen such that $d\eta(X,Y) = g(X, \varphi Y)$ and we then call the structure a **contact metric structure** and the manifold M^{2n+1} carrying such a structure is said to be a **contact metric manifold**. As a consequence of the above relations we have

$$\eta(\xi) = 1, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0, \quad d\eta(\xi, X) = 0. \tag{2.3}$$

Denoting by L and R, Lie differentiation and the curvature tensor, respectively, we define the operators l and h by

$$lX = R(X,\xi)\xi, \quad hX = \frac{1}{2}(L_\xi\varphi)X. \tag{2.4}$$

The $(1,1)$ tensors h and l are self-adjoint and satisfy

$$h\xi = 0, \quad l\xi = 0, \quad \mathrm{Tr}h = \mathrm{Tr}h\varphi = 0, \quad h\varphi = -\varphi h. \tag{2.5}$$

Since h anti-commutes with φ, if X is an eigenvector of h corresponding to the eigenvalue λ, then φX is also an eigenvector of h corresponding to the eigenvalue $-\lambda$.

If ∇ is the Riemannian connection of g, then

$$\nabla_X\xi = -\varphi X - \varphi hX, \tag{2.6}$$

$$\varphi l\varphi - l = 2(h^2 + \varphi^2), \tag{2.7}$$

$$\nabla_\xi\varphi = 0, \tag{2.8}$$

$$\nabla_\xi h = \varphi - \varphi l - \varphi h^2, \tag{2.9}$$

$$g(R(\xi,X)Y,Z) = g((\nabla_X\varphi)Y,Z) + g((\nabla_Z\varphi h)Y - (\nabla_Y\varphi h)Z, X), \tag{2.10}$$

$$\begin{aligned} 2(\nabla_{hX}\varphi)Y = &- R(\xi,X)Y - \varphi R(\xi,X)\varphi Y + \varphi R(\xi,\varphi X)Y \\ &- R(\xi,\varphi X)\varphi Y + 2g(X+hX,Y)\xi - 2\eta(Y)(X+hX). \end{aligned} \tag{2.11}$$

Formulas (2.6)–(2.8) occur in [2], (2.9) in [4] and (2.10), (2.11) in [10].

A contact metric manifold, $M^{2n+1}(\varphi,\xi,\eta,g)$, for which ξ is a Killing vector field is called a K-**contact manifold**. It is well known that a contact metric manifold is K-contact if and only if $h = 0$. Moreover, on a K-contact manifold, $R(X,\xi)\xi = X - \eta(X)\xi$.

A contact structure on M^{2n+1} gives rise to an almost complex structure on the product $M^{2n+1} \times R$. If this structure is integrable, then the contact metric manifold is said to be **Sasakian**. Equivalently, a contact metric manifold is Sasakian if and only if

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y. \tag{2.12}$$

Moreover, on a Sasakian manifold

$$(\nabla_X\varphi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.13}$$

Note that a Sasakian manifold is K-contact, but the converse holds only if $\dim M^{2n+1} = 3$.

A contact metric manifold is said to be **η-Einstein** if

$$Q = a\mathrm{Id} + b\eta \otimes \xi \tag{2.14}$$

where Q is the Ricci operator and a, b are smooth functions on M^{2n+1}.

The Riemannian connection ∇ of the metric g is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) =& Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ & - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]). \end{aligned} \tag{2.15}$$

The sectional curvature $K(\xi, X)$ of a plane section spanned by ξ and a vector X orthogonal to ξ is called a **ξ-sectional curvature**, while the sectional curvature $K(X, \varphi X)$ is called a φ-sectional curvature. Finally, the (κ,μ)-nullity distribution of a contact metric manifold $M^{2n+1}(\varphi,\xi,\eta,g)$ for the pair $(\kappa,\mu) \in R^2$ is a distribution

$$\begin{aligned} N(\kappa,\mu) : p \to N_p(\kappa,\mu) = \{Z \in T_pM \mid R(X,Y)Z =& \kappa(g(Y,Z)X - g(X,Z)Y) \\ & + \mu(g(Y,Z)hX - g(X,Z)hY)\}. \end{aligned}$$

So, if the characteristic vector field ξ belongs to the (κ,μ)-nullity distribution, we have

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

For more details concerning contact manifolds and related topics we refer the reader to [2].

3. Contact manifolds satisfying $R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$

Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a contact metric manifold. By a **D_a-homothetic deformation** [11] we mean a change of structure tensors of the form

$$\bar\eta = a\eta, \quad \bar\xi = \frac{1}{a}\xi, \quad \bar\varphi = \varphi, \quad \bar g = ag + a(a-1)\eta\otimes\eta \tag{3.1}$$

where a is a positive constant. It is well known that $M^{2n+1}(\bar\varphi,\bar\eta,\bar\eta,\bar g)$ is also a contact metric manifold. By direct computations we can see that the curvature tensor and the tensor h transform in the following manner:

$$\bar h = \frac{1}{a}h$$

and

$$\begin{aligned} a\bar R(X,Y)\bar\xi =& R(X,Y)\xi \\ &- (a-1)[(\nabla_X\varphi)Y - (\nabla_Y\varphi)X + \eta(X)(Y+hY) - \eta(Y)(X+hX)] \\ &+ (a-1)^2[\eta(Y)X - \eta(X)Y]. \end{aligned} \tag{3.2}$$

On the other hand, the tangent sphere bundle of a flat Riemannian manifold admits a contact metric structure satisfying $R(X,Y)\xi = 0$ [2, p.137]. Moreover, it is also well known ([10] or [13]) that a contact metric manifold with $R(X,Y)\xi = 0$ satisfies

$$(\nabla_X\varphi)Y = g(X+hX,Y)\xi - \eta(Y)(X+hX). \tag{3.3}$$

Suppose now that $M^{2n+1}(\varphi,\xi,\eta,g)$ is a contact metric manifold with $R(X,Y)\xi = 0$. Using (3.1) and (3.3), we obtain from (3.2)

$$\bar R(X,Y)\bar\xi = \frac{a^2-1}{a^2}(\bar\eta(Y)X - \bar\eta(X)Y) + \frac{2(a-1)}{a}(\bar\eta(Y)\bar hX - \bar\eta(X)\bar hY).$$

This fact raises the question of the classification of contact metric manifolds satisfying this condition or, more generally, the condition

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY). \tag{3.4}$$

Moreover, it is easy to check that a D_a-homothetic deformation of a contact metric manifold satisfying (3.4) yields a new contact metric manifold with characteristic vector field belonging to the $(\bar\kappa,\bar\mu)$-nullity distribution, where

$$\bar\kappa = \frac{\kappa + a^2 - 1}{a^2} \quad \text{and} \quad \bar\mu = \frac{\mu + 2a - 2}{a}.$$

Thus the type of (3.4), i.e. the (κ,μ)-nullity condition for ξ, remains invariant under a D_a-homothetic deformation. This is one more reason to study contact metric manifolds satisfying (3.4).

We now state our main results. The following Theorem informs us that the curvature tensor of a contact metric manifold is completely determined by the condition (3.4).

THEOREM 1: *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a contact metric manifold with ξ belonging to the (κ,μ)-nullity distribution. Then $\kappa\leq 1$. If $\kappa=1$, then $h=0$ and M^{2n+1} is a Sasakian manifold. If $\kappa<1$, M^{2n+1} admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$ determined by the eigenspaces of h, where $\lambda=\sqrt{1-\kappa}$. Moreover,*

$$\begin{aligned}
R(X_\lambda,Y_\lambda)Z_{-\lambda} &= (\kappa-\mu)[g(\varphi Y_\lambda,Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda,Z_{-\lambda})\varphi Y_\lambda],\\
R(X_{-\lambda},Y_{-\lambda})Z_\lambda &= (\kappa-\mu)[g(\varphi Y_{-\lambda},Z_\lambda)\varphi X_{-\lambda} - g(\varphi X_{-\lambda},Z_\lambda)\varphi Y_{-\lambda}],\\
R(X_\lambda,Y_{-\lambda})Z_{-\lambda} &= \kappa g(\varphi X_\lambda,Z_{-\lambda})\varphi Y_{-\lambda} + \mu g(\varphi X_\lambda,Y_{-\lambda})\varphi Z_{-\lambda},\\
R(X_\lambda,Y_{-\lambda})Z_\lambda &= -\kappa g(\varphi Y_{-\lambda},Z_\lambda)\varphi X_\lambda - \mu g(\varphi Y_{-\lambda},X_\lambda)\varphi Z_\lambda,\\
R(X_\lambda,Y_\lambda)Z_\lambda &= [2(1+\lambda)-\mu][g(Y_\lambda,Z_\lambda)X_\lambda - g(X_\lambda,Z_\lambda)Y_\lambda],\\
R(X_{-\lambda},Y_{-\lambda})Z_{-\lambda} &= [2(1-\lambda)-\mu][g(Y_{-\lambda},Z_{-\lambda})X_{-\lambda} - g(X_{-\lambda},Z_{-\lambda})Y_{-\lambda}],
\end{aligned}$$

where X_λ, Y_λ, $Z_\lambda\in D(\lambda)$ and $X_{-\lambda}$, $Y_{-\lambda}$, $Z_{-\lambda}\in D(-\lambda)$.

A consequence of Theorem 1 is the following Theorem:

THEOREM 2: *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a contact metric manifold with ξ belonging to the (κ,μ)-nullity distribution. If $\kappa<1$, then for any X orthogonal to ξ*

(1) *the ξ-sectional curvature $K(X,\xi)$ is given by*

$$K(X,\xi)=\kappa+\mu g(hX,X)=\begin{cases}\kappa+\lambda\mu, & \text{if } X\in D(\lambda),\\ \kappa-\lambda\mu, & \text{if } X\in D(-\lambda),\end{cases}$$

(2) *the sectional curvature of a plane section (X,Y) normal to ξ is given by*

$$K(X,Y)=\begin{cases}\text{(i)}\ \ 2(1+\lambda)-\mu, & \text{for any } X,Y\in D(\lambda),\ n>1,\\ \text{(ii)}\ -(\kappa+\mu)(g(X,\varphi Y))^2, & \text{for any unit vectors}\\ & X\in D(\lambda),\ Y\in D(-\lambda),\\ \text{(iii)}\ 2(1-\lambda)-\mu, & \text{for any } X,Y\in D(-\lambda),\ n>1,\end{cases}$$

(3) *M has constant scalar curvature, given by $S=2n[2(n-1)+\kappa-n\mu]$.*

Especially for $n=1$ we have the following classification:

THEOREM 3: *Let $M^3(\varphi, \xi, \eta, g)$ be a complete contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution. Then M^3 is either:*

(i) *A Sasakian manifold ($\kappa = 1, h = 0$), or*

(ii) *Locally isometric to one of the following Lie groups with a left invariant metric:* SU(2) *(or* SO(3)*),* SL(2, R) *(or* O(1, 2)*),* E(2) *(the group of rigid motions of the Euclidean 2-space),* E(1, 1) *(the group of rigid motions of the Minkowski 2-space).*

Moreover, this structure can occur on SU(2) *or* SO(3) *when $1 - \lambda - \mu/2 > 0$ and $1 + \lambda - \mu/2 > 0$, on* SL(2, R) *or* O(1, 2) *when $1 - \lambda - \mu/2 < 0$ and $1 + \lambda - \mu/2 > 0$ or $1 - \lambda - \mu/2 < 0$ and $1 + \lambda - \mu/2 < 0$, on* E(2) *when $1 - \lambda - \mu/2 = 0$ and $\mu < 2$, including a flat structure when $\mu = 0$, and on* E(1, 1) *when $1 + \lambda - \mu/2 = 0$ and $\mu > 2$.*

The special case $\mu = 0$ of Theorems 1, 2 and 3 has been studied in [1], [6] and [7].

THEOREM 4: *The standard contact metric structure on the tangent sphere bundle T_1M satisfies the condition that ξ belongs to the (κ, μ)-nullity distribution if and only if the base manifold M is of constant sectional curvature.*

The proofs of these theorems depend largely on several lemmas and propositions, which we now prove.

LEMMA 3.1: *Let $M^{2n+1}, (\varphi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution. Then:*

(3.5) (i) $[l\varphi - \varphi l] = 2\mu h\varphi$,

(3.6) (ii) $h^2 = (\kappa - 1)\varphi^2$, $\kappa \leq 1$ *and* $\kappa = 1$ *iff* M^{2n+1} *is Sasakian,*

(3.7) (iii) $R(\xi, X)Y = \kappa(g(X, Y)\xi - \eta(Y)X) + \mu(g(hX, Y)\xi - \eta(Y)hX)$,

(3.8) (iv) $Q\xi = (2n\kappa)\xi$, Q *is the Ricci operator,*

(3.9) (v) $(\nabla_X\varphi)Y = g(X + hX, Y)\xi - \eta(Y)(X + hX)$,

(vi) $$(\nabla_X h)Y - (\nabla_Y h)X = (1 - \kappa)[2g(X, \varphi Y)\xi + \eta(X)\varphi Y - \eta(Y)\varphi X] + (1 - \mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX],$$

for any vector fields X, Y on M^{2n+1}.

Proof: (i) By definition of the operator l and $h\xi = 0$ one easily proves that

$$lX = \kappa(X - \eta(X)\xi) + \mu hX$$

for any vector field X. Replacing X by φX and at the same time applying φ we get

$$(**) \qquad l\varphi X = \kappa\varphi X + \mu h\varphi X \quad \text{and} \quad \varphi lX = \kappa\varphi X + \mu\varphi hX.$$

Subtracting these and using $h\varphi = -\varphi h$, the required result is immediate.

(ii) Using (2.7), anti-commutativity of φh, the relation $(**)$, $h\xi = 0$ and the first of (2.1), we deduce that $h^2 = (\kappa - 1)\varphi^2$. Now since h is symmetric and $\varphi^2 = -\mathrm{Id} + \eta \otimes \xi$, $\kappa \leq 1$. Moreover, $\kappa = 1$ iff $h = 0$ and, by using (3.4), this is equivalent to (2.12). This completes the proof of (3.6).

(iii) This is an immediate consequence of (3.4) and $g(R(\xi, X)Y, Z) = g(R(Y, Z)\xi, X)$.

(iv) Let $\{e_i\}$, $i = 1, \dots, 2n+1$ be a local orthonormal basis of M^{2n+1}. Then the definition of the Ricci operator Q, (3.7), $\mathrm{Tr}h = 0$ and $h\xi = 0$ give $Q\xi = (2n\kappa)\xi$.

(v) Using (3.7), $\varphi\xi = 0$, $\eta \circ \varphi = 0$, (2.11) is reduced to

$$(\nabla_{hX}\varphi)Y = \kappa(\eta(Y)X - g(X, Y)\xi) - \eta(Y)(X + hX) + g(X + hX, Y)\xi.$$

Replacing now, in this equation, X by hX and using $\varphi^2 = -\mathrm{Id} + \eta \otimes \xi$, (2.8) and (3.6), we get

$$(\kappa - 1)[(\nabla_X\varphi)Y - g(X + hX, Y)\xi + \eta(Y)(X + hX)] = 0,$$

which is the required result for $\kappa < 1$. On the other hand, by (3.6), M^{2n+1} is Sasakian for $\kappa = 1$ and so (2.13) is valid. Hence (3.9) also has meaning for $\kappa = 1$.

(vi) Using (3.9) and the symmetry of h we get, for any vector fields X, Y, Z,

$$(\nabla_Z\varphi h)Y - (\nabla_Y\varphi h)Z = \varphi((\nabla_Z h)Y - (\nabla_Y h)Z)$$

and hence (2.10) is reduced to

$$R(Y, Z)\xi = \eta(Z)(Y + hY) - \eta(Y)(Z + hZ) + \varphi((\nabla_Z h)Y - (\nabla_Y h)Z).$$

Comparing this equation with (3.4), we have

$$(3.11) \qquad \begin{aligned} \varphi((\nabla_Z h)Y - (\nabla_Y h)Z) =& (\kappa - 1)(\eta(Z)Y - \eta(Y)Z) \\ &+ (\mu - 1)(\eta(Z)hY - \eta(Y)hZ). \end{aligned}$$

Using now (2.6) and the symmetry of h and $\nabla_X h$, by straightforward computation we get

$$g((\nabla_Z h)Y - (\nabla_Y h)Z, \xi) = 2(\kappa - 1)g(Y, \varphi Z). \tag{3.12}$$

Acting now by φ on (3.11) and using (3.12), we get the required result. ∎

The following Lemma generalizes Lemma 3.2 of [12], which is valid for the Sasakian case.

LEMMA 3.2: *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution. Then for any vector fields X, Y, Z*

$$\begin{aligned}
&R(X,Y)\varphi Z - \varphi R(X,Y)Z = \\
&\{(1-\kappa)[\eta(X)g(\varphi Y, Z) - \eta(Y)g(\varphi X, Z)] \\
&\ + (1-\mu)[\eta(X)g(\varphi hY, Z) - \eta(Y)g(\varphi hX, Z)]\}\xi \\
&- g(Y + hY, Z)(\varphi X + \varphi hX) + g(X + hX, Z)(\varphi Y + \varphi hY) \\
&- g(\varphi Y + \varphi hY, Z)(X + hX) + g(\varphi X + \varphi hX, Z)(Y + hY) \\
&- \eta(Z)\{(1-\kappa)[\eta(X)\varphi Y - \eta(Y)\varphi X] \\
&\qquad + (1-\mu)[\eta(X)\varphi hY - \eta(Y)\varphi hX]\}.
\end{aligned} \tag{3.13}$$

Proof: Let P be a fixed point of M^{2n+1} and X, Y, Z local vector fields such that $(\nabla X)_P = (\nabla Y)_P = (\nabla Z)_P = 0$. The Ricci identity for φ:

$$R(X,Y)\varphi Z - \varphi R(X,Y)Z = (\nabla_X \nabla_Y \varphi)Z - (\nabla_Y \nabla_X \varphi)Z - (\nabla_{[X,Y]}\varphi)Z,$$

at the point P, takes the form

$$R(X,Y)\varphi Z - \varphi R(X,Y)Z = \nabla_X(\nabla_Y \varphi)Z - \nabla_Y(\nabla_X \varphi)Z. \tag{3.14}$$

On the other hand, combining (3.9) and (2.6) we have, at P,

$$\begin{aligned}
\nabla_X(\nabla_Y \varphi)Z - \nabla_Y(\nabla_X \varphi)Z = &g((\nabla_X h)Y - (\nabla_Y h)X, Z)\xi \\
&- \eta(Z)((\nabla_X h)Y - (\nabla_Y h)X) \\
&- g(Y + hY, Z)(\varphi X + \varphi hX) \\
&+ g(\varphi X + \varphi hX, Z)(Y + hY) \\
&+ g(X + hX, Z)(\varphi Y + \varphi hY) \\
&- g(\varphi Y + \varphi hY, Z)(X + hX).
\end{aligned}$$

Now equation (3.13) is a straightforward combination of the last equation, (3.14) and (3.10).

LEMMA 3.3: *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a contact metric manifold with $\xi\in(\kappa,\mu)$-nullity distribution. Then, for any vector fields X, Y, Z, W, we have*

$$\begin{aligned} g(\varphi R(\varphi X,\varphi Y)Z,\varphi W) =& g(R(X,Y)Z,W) \\ &+\eta(Y)\{(1-\kappa)[\eta(Z)g(W,X)-\eta(W)g(Z,X)] \\ &\qquad+(1-\mu)[\eta(Z)g(hW,X)-\eta(W)g(hZ,X)]\} \\ &-\eta(X)\{(1-\kappa)[(\eta(Z)g(W,Y)-\eta(W)g(Z,Y)] \\ &\qquad+(1-\mu)[\eta(Z)g(hW,Y)-\eta(W)g(hZ,Y)]\} \\ &+g(X,\varphi Z+\varphi hZ)g(W+hW,\varphi Y) \\ &-g(X,\varphi W+\varphi hW)g(Z+hZ,\varphi Y) \\ &-g(X,W+hW)g(Y,Z+hZ) \\ &+g(X,Z+hZ)g(Y,W+hW). \end{aligned} \tag{3.15}$$

Proof: The proof of this lemma is a direct calculation using the relations (2.2), (3.13), (3.4), $\eta\circ\varphi=0$, $\varphi\xi=0$ and $h\xi=0$. ∎

LEMMA 3.4: *Let $M^{2n+1}(\varphi,\xi,\eta,g)$ be a contact metric manifold with $\xi\in(\kappa,\mu)$-nullity distribution. Then, for any vector fields X, Y, Z, we have*

$$\begin{aligned} \varphi R(\varphi X,\varphi Y)\varphi Z+R(X,Y)Z =& \eta(X)\{\kappa[g(Y,Z)\xi-\eta(Z)Y] \\ &\qquad+(2-\mu)]\eta(Z)hY-g(hZ,Y)\xi]\} \\ &-\eta(Y)\{\kappa[g(X,Z)\xi-\eta(Z)X] \\ &\qquad+(2-\mu)[\eta(Z)hX-g(hZ,X)\xi]\} \\ &+2\{g(Y,Z)hX+g(hZ,Y)X \\ &\qquad-g(Z,X)hY-g(hZ,X)Y\}. \end{aligned} \tag{3.16}$$

Proof: In (3.13) replace X, Y by φX, φY respectively and take the inner product with φW. Then, using $\varphi h+h\varphi=0$, $\varphi\xi=0$, $h\xi=0$, (2.1), (2.2) and

$\eta \circ \varphi = 0$, we have

$$\begin{aligned} g(R(\varphi X, \varphi Y)\varphi Z, \varphi W) = & g(\varphi R(\varphi X, \varphi Y)Z, \varphi W) \\ & - g(\varphi Y - \varphi hY, Z)g(-X + hX, \varphi W) \\ & + g(\varphi X - \varphi hX, Z)g(-Y + hY, \varphi W) \\ & - g(Z, -Y + \eta(Y)\xi + hY) \\ & \quad \times [g(X, W) - \eta(X)\eta(W) - g(hX, W)] \\ & + g(Z, -X + \eta(X)\xi + hX) \\ & \quad \times [g(Y, W) - \eta(Y)\eta(W) - g(hY, W)]. \end{aligned}$$

Substitute (3.15) in this equation for $g(\varphi R(\varphi X, \varphi Y)Z, \varphi W)$, and use the fact that φ is anti-symmetric, h is symmetric, $h\varphi + \varphi h = 0$ and that the resulting equation is valid for every W, to give (3.16) by straightforward calculation. This completes the proof of the Lemma. ∎

It is well known that on a Sasakian manifold the Ricci operator Q commutes with φ. In our situation we have the following proposition:

PROPOSITION 3.5: *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with*

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY)$$

for any vector fields X, Y. Then

$$Q\varphi - \varphi Q = 2[2(n-1) + \mu]h\varphi. \tag{3.17}$$

Proof: Let $\{e_i, \varphi e_i, \xi\}$, $i = 1, \ldots, n$ be a local orthonormal φ-basis (see [2], p.22). Setting $Y = Z = e_i$ in (3.16), adding with respect to i and using $\eta(e_i) = 0$, we have

$$\begin{aligned} & \sum_{i=1}^{n} [\varphi R(\varphi X, \varphi e_i)\varphi e_i + R(X, e_i)e_i] \\ & \qquad = \eta(X)[n\kappa - (2-\mu)\sum_{i=1}^{n} g(he_i, e_i)]\xi \\ & \qquad + 2\{nhX + \sum_{i=1}^{n} [g(he_i, e_i)X - h(g(X, e_i)e_i) - g(hX, e_i)e_i]\}. \end{aligned}$$

On the other hand, setting $Y = Z = \varphi e_i$ in (3.16), adding with respect to i and using $\eta \circ \varphi = 0$, (2.1) and (2.2), we get

$$\sum_{i=1}^{n}[\varphi R(\varphi X, e_i)e_i + R(X, \varphi e_i)\varphi e_i] =$$
$$\eta(X)[n\kappa - (2-\mu)\sum_{i=1}^{n} g(h\varphi e_i, \varphi e_i)]\xi$$
$$+ 2\{nhX + \sum_{i=1}^{n}[g(h\varphi e_i, \varphi e_i)X - h(g(X, \varphi e_i)\varphi e_i) - g(hX, \varphi e_i)\varphi e_i]\}.$$

Adding now the last two equations and using the definition for Q, $h\xi = 0$ and $\mathrm{Tr}h = 0$, we have

$$\varphi(Q\varphi X - R(\varphi X, \xi)\xi) + QX - R(X,\xi)\xi = 2n\kappa\eta(X)\xi + 4(n-1)hX.$$

Using now (3.4), $\eta \circ \varphi = 0$ and $h\xi = 0$, we get

$$\varphi Q\varphi X + QX = 2\kappa n\eta(X)\xi + 2[2(n-1) + \mu]hX.$$

Finally, acting by φ and using (2.1) and $Q\xi = (2n\kappa)\xi$ as well as $\varphi\xi = 0$ and $\varphi h + h\varphi = 0$, we obtain (3.17) and the proof is completed. ∎

LEMMA 3.6: *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with $\xi \in (\kappa, \mu)$-nullity distribution. If $\kappa < 1$, then M^{2n+1} admits three mutually orthogonal and integrable distributions $D(0)$, $D(\lambda)$ and $D(-\lambda)$, defined by the eigenspaces of h, where $\lambda = \sqrt{1-\kappa}$.*

Proof: The proof of this lemma is similar to that of Proposition 5.1 of Tanno's paper [13] and hence we omit it.

We now state and prove the following proposition:

PROPOSITION 3.7: *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with*

$$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY), \quad \kappa < 1$$

for any vector fields X, Y.

(i) *If $X, Y \in D(\lambda)$ (resp. $D(-\lambda)$), then $\nabla_X Y \in D(\lambda)$ (resp. $D(-\lambda)$).*
(ii) *If $X \in D(\lambda)$, $Y \in D(-\lambda)$, then $\nabla_X Y$ (resp. $\nabla_Y X$) has no component in $D(\lambda)$ (resp. $D(-\lambda)$).*

Proof: In (3.10), replace Y by φY and take the inner product with Z to get

$$g((\nabla_X h)\varphi Y - (\nabla_{\varphi Y} h)X, Z) = 0$$

or, equivalently,

$$g(\nabla_X h\varphi Y - h\nabla_X \varphi Y - \nabla_{\varphi Y} hX + h\nabla_{\varphi Y} X, Z) = 0 \tag{3.18}$$

for any X, Y, Z orthogonal to ξ.

(i) Let $X, Y, Z \in D(\lambda)$ (resp., $D(-\lambda)$). Then equation (3.18) is reduced to $g(\nabla_X Z, \varphi Y) = 0$, since $\lambda \neq 0$ and $g(\varphi Y, Z) = 0$ by Lemma 3.6. On the other hand, use $\nabla_X \xi = -\varphi X - \varphi hX$ and take the inner product with Z to get $g(\nabla_X Z, \xi) = 0$. Applying now Lemma 3.6 we conclude that $\nabla_X Z \in D(\lambda)$ (resp., $D(-\lambda)$) for any $X, Z \in D(\lambda)$ (resp., $D(-\lambda)$).

(ii) Let $X, Z \in D(\lambda)$ and $Y \in D(-\lambda)$. Then from (i), $g(\nabla_X Y, Z) = -g(Y, \nabla_X Z)$ $= 0$ giving the second statement. ∎

Remark 3.1: It is obvious from Proposition 3.7 that $R(X,Y)Z \in D(\lambda)$ (resp. $D(-\lambda)$) for $X, Y, Z \in D(\lambda)$ (resp. $D(-\lambda)$).

LEMMA 3.8: *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with $\xi \in (\kappa, \mu)$-nullity distribution. Then for any vector fields X, Y we have*

(3.19)

$$(\nabla_X h)Y = \{(1-\kappa)g(X, \varphi Y) + g(X, h\varphi Y)\}\xi + \eta(Y)[h(\varphi X + \varphi hX)] - \mu\eta(X)\varphi hY.$$

Proof: Let $\kappa < 1$. Suppose $X, Y \in D(\lambda)$ (resp. $D(-\lambda)$). Then from Proposition 3.7 we have $\nabla_X Y \in D(\lambda)$ (resp. $D(-\lambda)$) and one easily proves that

$$(\nabla_X h)Y = 0. \tag{3.20}$$

Suppose now that $X \in D(\lambda)$ and $Y \in D(-\lambda)$. Let $\{e_i, \varphi e_i, \xi\}$, $i = 1, \ldots, n$ be an orthonormal φ-basis with $e_i \in D(\lambda)$ and so $\varphi e_i \in D(-\lambda)$. Then using

Proposition 3.7, $h\xi = 0$, $\varphi\xi = 0$, (2.1) and (2.6), we calculate

$$\begin{aligned} h\nabla_X Y &= h\left\{\sum_{i=1}^{n} g(\nabla_X Y, \varphi e_i)\varphi e_i + g(\nabla_X Y, \xi)\xi\right\} \\ &= \sum_{i=1}^{n} g(\nabla_X Y, \varphi e_i) h\varphi e_i \\ &= \lambda\varphi \sum_{i=1}^{n} g(\varphi\nabla_X Y, e_i) e_i \\ &= \lambda\varphi^2 \nabla_X Y \\ &= \lambda(-\nabla_X Y + g(\nabla_X Y, \xi)\xi) \\ &= \lambda(-\nabla_X Y - g(Y, \nabla_X \xi)\xi) \\ &= \lambda(-\nabla_X Y + g(Y, \varphi X + \varphi h X)\xi) \\ &= \nabla_X hY - \lambda(\lambda+1) g(X, \varphi Y)\xi, \end{aligned}$$

and so

$$(\nabla_X h)Y = \lambda(\lambda+1) g(X, \varphi Y)\xi. \tag{3.21}$$

Similarly we find

$$(\nabla_Y h)X = \lambda(\lambda-1) g(Y, \varphi X)\xi. \tag{3.22}$$

Suppose now that X, Y are arbitrary vector fields and write

$$X = X_\lambda + X_{-\lambda} + \eta(X)\xi$$

and

$$Y = Y_\lambda + Y_{-\lambda} + \eta(Y)\xi,$$

where X_λ (resp. $X_{-\lambda}$) is the component of X in $D(\lambda)$ (resp. $D(-\lambda)$). Then using (3.20), (3.21), (3.22) and $\nabla_\xi h = \mu h\varphi$, which follows from (3.10), we get by a direct computation

$$\begin{aligned} (\nabla_X h)Y =& \lambda^2[g(X_\lambda, \varphi Y_{-\lambda}) + g(X_{-\lambda}, \varphi Y_\lambda)]\xi + \lambda[g(X_\lambda, \varphi Y_{-\lambda}) - g(X_{-\lambda}, \varphi Y_\lambda)]\xi \\ &+ \eta(Y)(h(\varphi X + \varphi h X)) - \mu\eta(X)\varphi h Y. \end{aligned}$$

On the other hand, we easily find that

$$g(hX, \varphi Y) = \lambda[g(X_\lambda, \varphi Y_{-\lambda}) - g(X_{-\lambda}, \varphi Y_\lambda)]$$

and

$$g(hX, h\varphi Y) = \lambda^2[g(X_\lambda, \varphi Y_{-\lambda}) + g(X_{-\lambda}, \varphi Y_\lambda)].$$

These relations together with the previous one give the required equation (3.19). Note that for $\kappa = 1$ (and so $h = 0$), (3.19) is valid identically and the proof is completed. ∎

LEMMA 3.9: *Let $M^{2n+1}(\varphi, \xi, \eta, g)$ be a contact metric manifold with ξ belonging to the (κ, μ)-nullity distribution. Then for any vector fields X, Y, Z we have*

$$\begin{aligned} R(X,Y)hZ - hR(X,Y)Z =&\{\kappa[\eta(X)g(hY,Z) - \eta(Y)g(hX,Z)] \\ &+ \mu(\kappa-1)[\eta(Y)g(X,Z) - \eta(X)g(Y,Z)]\}\xi \\ &+ \kappa\{g(Y,\varphi Z)\varphi hX - g(X,\varphi Z)\varphi hY \\ &\quad + g(Z,\varphi hY)\varphi X - g(Z,\varphi hX)\varphi Y \\ &\quad + \eta(Z)[\eta(X)hY - \eta(Y)hX]\} \\ &- \mu\{\eta(Y)[(1-\kappa)\eta(Z)X + \mu\eta(X)hZ] \\ &\quad - \eta(X)[(1-\kappa)\eta(Z)Y + \mu\eta(Y)hZ] \\ &\quad + 2g(X,\varphi Y)\varphi hZ\}. \end{aligned} \tag{3.23}$$

Proof: The Ricci identity for h is

$$R(X,Y)hZ - hR(X,Y)Z = (\nabla_X\nabla_Y h)Z - (\nabla_Y\nabla_X h)Z - (\nabla_{[X,Y]}h)Z. \tag{3.24}$$

Using Lemma 3.8, the relations (3.6), $h\varphi + \varphi h = 0$ and the fact that $\nabla_X\varphi$ is antisymmetric, we get by direct calculation

$$\begin{aligned} (\nabla_X\nabla_Y h)Z =&\{(1-\kappa)g(\nabla_X Y, \varphi Z) \\ &- (1-\kappa)g((\nabla_X\varphi)Y, Z) \\ &+ g(\nabla_X Y, h\varphi Z) + g((\nabla_X h\varphi)Y, Z)\}\xi \\ &+ \{(1-\kappa)g(Y,\varphi Z) + g(Y, h\varphi Z)\}\nabla_X\xi \\ &+ g(Z, \nabla_X\xi)[h\varphi Y + (\kappa-1)\varphi Y] \\ &+ \eta(Z)\{(\nabla_X h\varphi)Y + h\varphi\nabla_X Y + (\kappa-1)[(\nabla_X\varphi)Y + \varphi\nabla_X Y]\} \\ &- \mu\{[\eta(\nabla_X Y) + g(Y, \nabla_X\xi)]\varphi hZ - \eta(Y)(\nabla_X\varphi h)Z\}. \end{aligned}$$

So, using also (3.19), (2.6) and (3.9), equation (3.24) yields

$$\begin{aligned} R(X,Y)hZ - hR(X,Y)Z \\ &= \{(\kappa-1)g((\nabla_X\varphi)Y - (\nabla_Y\varphi)X, Z) + g((\nabla_X h\varphi)Y - (\nabla_Y h\varphi)X, Z)\}\xi \\ &+ \{(1-\kappa)g(Y,\varphi Z) + g(Y, h\varphi Z)\}\nabla_X\xi \\ &- \{(1-\kappa)g(X,\varphi Z) + g(X, h\varphi Z)\}\nabla_Y\xi \\ &+ g(Z,\nabla_X\xi)[h\varphi Y + (\kappa-1)\varphi Y] \\ &- g(Z,\nabla_Y\xi)[h\varphi X + (\kappa-1)\varphi X] \\ &+ \eta(Z)\{\nabla_X h\varphi)Y - (\nabla_Y h\varphi)X + (\kappa-1)[(\nabla_X\varphi)Y - (\nabla_Y\varphi)X]\} \\ &- \mu\{\eta(Y)(\nabla_X\varphi h)Z - \eta(X)(\nabla_Y\varphi h)Z + 2g(X,\varphi Y)\varphi hZ\}. \end{aligned} \tag{3.25}$$

Using now (3.9), $h\xi = 0$ and Lemma 3.8, we get

$$\begin{aligned} (\nabla_X\varphi h)Y =& \{g(X,hY) + (\kappa-1)g(X, -Y + \eta(Y)\xi)\}\xi \\ &+ \eta(Y)\{hX + (\kappa-1)(-X + \eta(X)\xi)\} + \mu\eta(X)hY. \end{aligned}$$

Therefore, equation (3.25), by using (3.9) again, is reduced to (3.23) and the proof is completed. ∎

Proof of Theorem 1: The first part of the Theorem follows from (3.6) and Lemma 3.6. Let $\{e_i, \varphi e_i, \xi\}$, $i = 1, \ldots, n$ be an orthonormal basis of $T_P M$ at any point $P \in M$ with $e_i \in D(\lambda)$. Then we have

$$\begin{aligned} R(X_\lambda, Y_\lambda)Z_{-\lambda} =& \sum_{i=1}^{n}\{g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, e_i)e_i + g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, \varphi e_i)\varphi e_i\} \\ &+ g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, \xi)\xi. \end{aligned} \tag{3.26}$$

But since $\xi \in (\kappa,\mu)$-nullity distribution, using (3.4) we easily have

$$g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, \xi) = -g(R(X_\lambda, Y_\lambda)\xi, Z_{-\lambda}) = 0.$$

By Proposition 3.7 and Remark 3.1 we get

$$g(R(X_\lambda, Y_\lambda)Z_{-\lambda}, e_i) = -g(R(X_\lambda, Y_\lambda)e_i, Z_{-\lambda}) = 0.$$

On the other hand, if $X \in D(\lambda)$ and $Y, Z \in D(-\lambda)$, then applying (3.23) we get

$$hR(X,Y)Z + \lambda R(X,Y)Z = -2\lambda\{\kappa g(X,\varphi Z)\varphi Y + \mu g(X,\varphi Y)\varphi Z\}$$

and, taking the inner product with $W \in D(\lambda)$, we have

$$g(R(X,Y)Z,W) = -\kappa g(X,\varphi Z)g(\varphi Y,W) - \mu g(X,\varphi Y)g(\varphi Z,W) \tag{3.27}$$

for any X, $W \in D(\lambda)$ and Y, $Z \in D(-\lambda)$. Using (3.27) and the first Bianchi identity we calculate

$$\begin{aligned}
&\sum_{i=1}^{n} g(R(X_\lambda,Y_\lambda)Z_{-\lambda},\varphi e_i)\varphi e_i \\
&= -\sum_{i=1}^{n} g(R(Y_\lambda,Z_{-\lambda})X_\lambda,\varphi e_i)\varphi e_i - \sum_{i=1}^{n} g(R(Z_{-\lambda},X_\lambda)Y_\lambda,\varphi e_i)\varphi e_i \\
&= \sum_{i=1}^{n}\{-\kappa g(Y_\lambda,\varphi^2 e_i)g(\varphi Z_{-\lambda},X_\lambda)\varphi e_i - \mu g(Y_\lambda,\varphi Z_{-\lambda})g(\varphi^2 e_i,X_\lambda)\varphi e_i\} \\
&\quad - \sum_{i=1}^{n}\{-\kappa g(Z_{-\lambda},\varphi Y_\lambda)g(\varphi X_\lambda,\varphi e_i)\varphi e_i - \mu g(Z_{-\lambda},\varphi X_\lambda)g(\varphi Y_\lambda,\varphi e_i)\varphi e_i\} \\
&= \kappa g(\varphi Z_{-\lambda},X_\lambda)\varphi\sum_{i=1}^{n} g(Y_\lambda,e_i)e_i + \mu g(Y_\lambda,\varphi Z_{-\lambda})\varphi\sum_{i=1}^{n} g(X_\lambda,e_i)e_i \\
&\quad + \kappa g(Z_{-\lambda},\varphi Y_\lambda)\varphi\sum_{i=1}^{n} g(X_\lambda,e_i)e_i + \mu g(Z_{-\lambda},\varphi X_\lambda)\varphi\sum_{i=1}^{n} g(Y_\lambda,e_i)e_i \\
&= \kappa\{g(\varphi Y_\lambda,Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda,Z_{-\lambda})\varphi Y_\lambda\} \\
&\quad + \mu\{g(\varphi X_\lambda,Z_{-\lambda})\varphi Y_\lambda - g(\varphi Y_\lambda,Z_{-\lambda})\varphi X_\lambda\} \\
&= (\kappa-\mu)\{g(\varphi Y_\lambda,Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda,Z_{-\lambda})\varphi Y_\lambda\}.
\end{aligned}$$

Therefore, (3.26) gives

$$R(X_\lambda,Y_\lambda)Z_{-\lambda} = (\kappa-\mu)[g(\varphi Y_\lambda,Z_{-\lambda})\varphi X_\lambda - g(\varphi X_\lambda,Z_{-\lambda})\varphi Y_\lambda].$$

The proof of the remaining cases are similar and will be omitted. ∎

Proof of Theorem 2:

(1) If we set $Y = \xi$ in relation (3.4), we get $R(X,\xi)\xi = \kappa X + \mu hX$ for X orthogonal to ξ from which, taking the inner product with X, we have $K(X,\xi) = \kappa + \mu g(hX,X)$, which is the required result. The special cases are obvious.

(2) This follows immediately from Theorem 1.

(3) Let $\{e_i, \varphi e_i, \xi\}$, $i = 1, \ldots, n$, be an orthonormal φ-basis with $e_i \in D(\lambda)$. Then from (1) or (3.8), $g(Q\xi, \xi) = 2n\kappa$, and from (2)

$$g(Qe_i, e_i) = (\kappa + \lambda\mu) + (n-1)(2(1+\lambda) - \mu) - (\kappa + \mu),$$
$$g(Q\varphi e_i, \varphi e_i) = (\kappa - \lambda\mu) + (n-1)(2(1-\lambda) - \mu) - (\kappa + \mu).$$

Therefore,

$$S = \mathrm{Tr}Q = \sum_{i=1}^{n}\{g(Qe_i, e_i) + g(Q\varphi e_i, \varphi e_i\} + g(Q\xi, \xi) = 2n(2(n-1) + \kappa - n\mu)$$

and the proof is completed. ∎

Remark 3.2: Using Theorem 1 one can easily prove that: In any contact metric manifold $M^{2n+1}(\varphi, \xi, \eta, g)$ with ξ belonging to the (κ, μ)-nullity distribution, the Ricci operator Q is given by

$$QX = [2(n-1) - n\mu]X + [2(n-1) + \mu]hX + [2(1-n) + n(2\kappa + \mu)]\eta(X)\xi$$

for any vector field X on M^{2n+1}. Especially for $\mu = 2(1-n)$, Q is of the form (2.14) and so M^{2n+1} is η-Einstein. ∎

4. Classification of the three-dimensional case

Let $M^3(\varphi, \xi, \eta, g)$ be a three-dimensional contact metric manifold with characteristic vector field ξ satisfying

(4.1) $$R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)hX - \eta(X)hY).$$

As we proved in Lemma 3.1, $\kappa \leq 1$. Suppose that X is a unit eigenvector of h, say $hX = \lambda X$, X orthogonal to ξ, where $\lambda = \sqrt{1-\kappa}$.

LEMMA 4.1: *For $\kappa < 1$, we have*

(i) $\nabla_X X = \nabla_{\varphi X}\varphi X = 0$,
(ii) $\nabla_X \varphi X = (\lambda + 1)\xi$,
(iii) $\nabla_{\varphi X} X = (\lambda - 1)\xi$,
(iv) $[X, \varphi X] = 2\xi$,
(v) $\nabla_X \xi = -(1+\lambda)\varphi X$,
(vi) $\nabla_\xi X = -\frac{1}{2}\mu\varphi X$,
(vii) $[\xi, X] = \left(1 + \lambda - \frac{1}{2}\mu\right)\varphi X$,
(viii) $[\varphi X, \xi] = \left(1 - \lambda - \frac{1}{2}\mu\right)X$.

Proof: Since X is a unit eigenvector of h belonging to $D(\lambda)$ and φX is a unit eigenvector of h belonging to $D(-\lambda)$, the relations in (i) are immediate consequences of Proposition 3.7(i) and the fact that $\dim D(\lambda) = \dim D(-\lambda) = 1$.

(ii) Because φX is unit we have $\nabla_X \varphi X$ orthogonal to φX. Moreover, since $\varphi X \in D(-\lambda)$, by Proposition 3.7(ii) we conclude that $\nabla_X \varphi X$ is parallel to ξ. But, using (2.6), $\varphi\xi = 0$ and (2.2), we have

$$g(\nabla_X \varphi X, \xi) = -g(\varphi X, \nabla_X \xi) = g(\varphi X, \varphi X + \varphi h X) = g(X, X + hX) = (\lambda + 1).$$

Therefore $\nabla_X \varphi X = (\lambda + 1)\xi$.

(iii) The proof is similar to that of (ii).

(iv) This is an immediate consequence of (ii) and (iii).

(v) This follows from (2.6).

(vi) By direct computation, using (i)–(iv) we have

$$R(X, \varphi X)X = \kappa \varphi X - 2\nabla_\xi X. \tag{4.2}$$

On the other hand, on any three-dimensional Riemannian manifold

$$\begin{aligned} R(X,Y)Z = & g(Y,Z)QX - g(X,Z)QY + g(QY,Z)X \\ & - g(QX,Z)Y - \frac{S}{2}(g(Y,Z)X - g(X,Z)Y) \end{aligned} \tag{4.3}$$

for any vector fields X, Y, Z. Moreover, using Remark 3.2 (for $n = 1$), we have

$$QX = \mu(\lambda - 1)X \tag{4.4}$$

and, using (4.4) and Proposition 3.5 (for $n = 1$), equation (4.3) gives

$$R(X, \varphi X)X = (\kappa + \mu)\varphi X. \tag{4.5}$$

Comparing (4.2) and (4.5) we get $\nabla_\xi X = -(\mu/2)\varphi X$.

(vii) This follows from (v) and (vi).

(viii) Using (2.6), (2.8) and (vi) above, we easily get (viii), completing the proof. ∎

Finally, to prove Theorem 3 we need the following result from Lie group theory (see e.g. [14, p.10]).

PROPOSITION 4.2: *Let M be an n-dimensional connected and simply connected manifold and let $X_1, \dots, X_n$ be complete vector fields which are linearly independent at each point of M and satisfy*

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k,$$

where the c_{ij}^k's are constant. Then, for each point $P \in M$, the manifold M has a unique Lie group structure such that P is the identity and the vector fields X_i are left invariant.

Proof of Theorem 3: We distinguish the cases $\kappa = 1$ and $\kappa < 1$. When $\kappa = 1$, then by using Lemma 3.1 we conclude that M^3 is a Sasakian manifold. Suppose now $\kappa < 1$. Let X be a unit eigenvector of h orthogonal to ξ with corresponding eigenvalue $\lambda = \sqrt{1-\kappa} > 0$. Then, as is proved in Lemma 4.1, there exist three mutually orthonormal vector fields ξ, X, φX such that

$$(4.6) \quad [X, \varphi X] = 2\xi, \quad [\varphi X, \xi] = \left(1 - \lambda - \frac{\mu}{2}\right) X, \quad [\xi, X] = \left(1 + \lambda - \frac{\mu}{2}\right) \varphi X,$$

where $(\lambda, \mu) \in R^2$. Let $\xi = e_1$, $X = e_2$ and $\varphi X = e_3$. It is known that ξ is defined globally on M^3. Going to the universal covering space $\bar{M}^3$ if necessary, we have global vector fields, which we also denote by e_1, e_2 and e_3, satisfying the conditions of Proposition 4.2 above. Hence $\bar{M}^3$ has a unique Lie group structure. So, relations (4.6) may be written as

$$(4.7) \quad [e_2, e_3] = 2e_1, \quad [e_3, e_1] = \left(1 - \lambda - \frac{\mu}{2}\right) e_2, \quad [e_1, e_2] = \left(1 + \lambda - \frac{\mu}{2}\right) e_3.$$

On the other hand, in [9, p. 307] J. Milnor gave a complete classification of three-dimensional manifolds admitting the Lie algebra structure

$$[e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad [e_1, e_2] = c_3 e_3.$$

Comparing this and (4.7) we have

$$(4.8) \qquad c_1 = 2, \quad c_2 = 1 - \lambda - \frac{\mu}{2}, \quad c_3 = 1 + \lambda - \frac{\mu}{2}.$$

So, the signs of c_2 and c_3 vary. Since $c_1 = 2 > 0$, the possible combinations of the signs of c_1, c_2 and c_3, the associated solution sets and the corresponding Lie groups are indicated in Table 1, where:

$D_I = \{(\lambda, \mu) \in R^2 \mid c_2 > 0, \ c_3 > 0\}$. The special case $\mu = 0$, $0 < \lambda < 1$ has been studied in [7].

$D_{II} = \{(\lambda, \mu) \in R^2 \mid c_2 < 0, \ c_3 > 0\}$. The special case $\mu = 0$, $\lambda > 1$ has been studied in [7].

$D_{III} = \{(\lambda, \mu) \in R^2 \mid c_2 = 0, \ \mu < 2\}$. The special case $\mu = 0$, $\kappa = 0$, M^3 is flat [3].

$D_{IV} = \{(\lambda, \mu) \in R^2 \mid c_3 = 0, \ \mu > 2\}.$
$D_V = \{(\lambda, \mu) \in R^2 \mid c_2 < 0, \ c_3 < 0\}.$

Table 1

c_1	c_2	c_3	Associated region	Associated Lie group
+	+	+	D_I	SU(2) or SO(3)
+	+	−	$\emptyset$	−
+	−	+	D_{II}	SL(2, R) or O(1, 2)
+	−	−	D_V	SL(2, R) or O(1, 2)
+	+	0	$\emptyset$	−
+	0	+	D_{III}	·E(2)
+	−	0	D_{IV}	E(1, 1)
+	0	−	$\emptyset$	−

Conversely, we will exhibit the contact metric structure on the above Lie groups such that (4.1) is satisfied. The method which we will use is that of D. Blair and H. Chen [7] and, for the sake of completeness, we will repeat some necessary relations from [7]. We consider the general Lie algebra structure on these manifolds:

(4.9) $$[e_2, e_3] = c_1 e_1, \quad [e_3, e_1] = c_2 e_2, \quad [e_1, e_2] = c_3 e_3.$$

Let $\{w_i\}$ be the dual 1-forms to the vector fields $\{e_i\}$. Using (4.9) we get

$$dw_1(e_2, e_3) = -dw_1(e_3, e_2) = \frac{c_1}{2} \neq 0 \quad \text{and} \quad dw_1(e_i, e_j) = 0$$

for $(i, j) \neq (2, 3), (3, 2)$. It is easy to check that w_1 is a contact form and e_1 is the characteristic vector field. Defining a Riemannian metric g by $g(e_i, e_j) = \delta_{ij}$, then, because we must have $dw_1(e_i, e_j) = g(e_i, \varphi e_j)$, φ has the same matrix as dw_1 with respect to the basis e_i. Moreover, for g to be an associated metric, we must have $\varphi^2 = -\mathrm{Id} + w_1 \otimes e_1$. So for $(\varphi, e_1 w_1, g)$ to be a contact metric structure we must have $c_1 = 2$. The unique Riemannian connection ∇ corresponding to g is given by (2.15). So we easily get, using $c_1 = 2$ and (4.9),

$$\nabla_{e_1} e_1 = 0, \quad \nabla_{e_2} e_2 = 0, \quad \nabla_{e_3} e_3 = 0,$$
$$\nabla_{e_1} e_2 = \frac{1}{2}(c_2 + c_3 - 2) e_3, \quad \nabla_{e_2} e_1 = \frac{1}{2}(c_2 - c_3 - 2) e_3,$$
$$\nabla_{e_1} e_3 = -\frac{1}{2}(c_2 + c_3 - 2) e_2, \quad \nabla_{e_3} e_1 = \frac{1}{2}(2 + c_2 - c_3) e_2.$$

But we also know that

$$\nabla_{e_2} e_1 = -\varphi e_2 - \varphi h e_2.$$

Comparing now those two relations for $\nabla_{e_2} e_1$ and using $\varphi e_1 = 0$, $\varphi e_3 = -e_2$ we conclude that

$$he_2 = \frac{c_3 - c_2}{2} e_2 \quad \text{and hence} \quad he_3 = -\frac{c_3 - c_2}{2} e_3.$$

Thus $\{e_i\}$ are eigenvectors of h with corresponding eigenvalues $\{0, \lambda, -\lambda\}$ where $\lambda = (c_3 - c_2)/2$. Moreover, by direct calculation we have

$$R(e_2, e_1)e_1 = \left\{1 - \frac{(c_3 - c_2)^2}{4}\right\} e_2 + (2 - c_2 - c_3)he_2,$$
$$R(e_3, e_1)e_1 = \left\{1 - \frac{(c_3 - c_2)^2}{4}\right\} e_3 + (2 - c_2 - c_3)he_3,$$

and

$$R(e_2, e_3)e_1 = 0.$$

Putting

$$\kappa = 1 - \frac{(c_3 - c_2)^2}{4} \leq 1 \quad \text{and} \quad \mu = 2 - c_2 - c_3$$

we conclude, from these relations, that e_1 belongs to the (κ, μ)-nullity distribution, for any c_2, c_3. If we choose $c_2 = c_3$ then we have the Sasakian case ($\kappa = 1$, $h = 0$), while for $c_2 \neq c_3$ we have the desired structure ($\kappa < 1$, $\mu \in R$), and the proof is completed. Note that for the special Sasakian case $c_1 = 2$, $c_2 = c_3 = 0$, the group is the Heisenberg group [9, 14 ch. 7]. ∎

5. The tangent sphere bundle

The natural contact metric structure on the tangent sphere bundle $\pi: T_1M \to M$ of a manifold M is described in Chapter VII of [2] and in [5]. In particular, the characteristic vector field ξ is horizontal and, as a hypersurface of the tangent bundle TM, the Weingarten map annihilates horizontal vectors. Thus on T_1M, $R(X, Y)\xi$ can be computed by the formulas for the curvature of TM which were computed by Kowalski [8] and which we now describe.

Let G, D and $\boldsymbol{R}$ denote the Riemannian metric, the Levi-Civita connection and the curvature tensor on the base manifold M, and $\bar{\pi}: TM \to M$ the projection map. D induces a horizontal subbundle in TM and the connection map $K : TTM \to TM$ is given by

$$KX^H = 0, \quad K(X_t^V) = X_{\bar{\pi}(t)},$$

where $t \in TM$ and X^H and X^V denote the horizontal and vertical lifts of vector fields on M. $\bar{g}(X,Y) = G(\bar{\pi}_* X, \bar{\pi}_* Y) + G(KX, KY)$ is then a Riemannian metric on TM and its curvature $\bar{R}$ is given by

$$\begin{aligned}
&\bar{R}(X^V,Y^V)Z^V = 0,\\
&(\bar{R}(X^V,Y^V)Z^H)_t = \left(\boldsymbol{R}(X,Y)Z + \frac{1}{4}\boldsymbol{R}(t,X)\boldsymbol{R}(t,Y)Z - \frac{1}{4}\boldsymbol{R}(t,Y)\boldsymbol{R}(t,X)Z\right)_t^H,\\
&(\bar{R}(X^H,Y^V)Z^V)_t = -\left(\frac{1}{2}\boldsymbol{R}(Y,Z)X + \frac{1}{4}\boldsymbol{R}(t,Y)\boldsymbol{R}(t,Z)X\right)_t^H,\\
&(\bar{R}(X^H,Y^V)Z^H)_t = \left(\frac{1}{2}\boldsymbol{R}(X,Z)Y + \frac{1}{4}\boldsymbol{R}(\boldsymbol{R}(t,Y)Z,X)t\right)_t^V + \frac{1}{2}((D_X\boldsymbol{R})(t,Y)Z)_t^H,\\
&(\bar{R}(X^H,Y^H)Z^V)_t = \left(\boldsymbol{R}(X,Y)Z + \frac{1}{4}\boldsymbol{R}(\boldsymbol{R}(t,Z)Y,X)t - \frac{1}{4}\boldsymbol{R}(\boldsymbol{R}(t,Z)X,Y)t\right)_t^V\\
&\qquad + \frac{1}{2}((D_X\boldsymbol{R})(t,Z)Y - (D_Y\boldsymbol{R})(t,Z)X)_t^H,\\
&(\bar{R}(X^H,Y^H)Z^H)_t = \frac{1}{2}((D_Z\boldsymbol{R})(X,Y)t)_t^V + \left(\boldsymbol{R}(X,Y)Z + \frac{1}{4}\boldsymbol{R}(t,\boldsymbol{R}(Z,Y)t)X\right.\\
&\qquad \left. + \frac{1}{4}\boldsymbol{R}(t,\boldsymbol{R}(X,Z)t)Y + \frac{1}{2}\boldsymbol{R}(t,\boldsymbol{R}(X,Y)t)Z\right)_t^H.
\end{aligned}$$

With respect to local coordinates $\{x^i\}$ on M and fibre coordinates $\{v^i\}$, the characteristic vector field is given by

$$\xi = 2v^i\left(\frac{\partial}{\partial x^i}\right)^H.$$

On T_1M for a vertical vector U and a horizontal vector X orthogonal to ξ, hU and hX are given by

$$hU_t = U_t - (\boldsymbol{R}(KU,t)t)^V \quad \text{and} \quad hX_t = -X_t + (\boldsymbol{R}(\pi_* X,t)t)^H \tag{5.1}$$

(cf. eq. (4.1) of [5]).

Proof of Theorem 4: First suppose that the base manifold is a Riemannian manifold of constant curvature c. Then from Kowalski's formulas it is easy to see that $R(X,Y)\xi = 0$ for X, Y orthogonal to ξ; for a vertical vector U, that $R(U,\xi)\xi = c^2U$ and, for a horizontal vector X orthogonal to ξ, that $R(X,\xi)\xi = (4c-3c^2)X$. Moreover, from equations (5.1), $hU = (1-c)U$ and $hX = (c-1)X$. Thus the curvature tensor on T_1M satisfies

$$R(X,Y)\xi = c(2-c)(\eta(Y)X - \eta(X)Y) - 2c(\eta(Y)hX - \eta(X)hY)$$

for all X, Y on T_1M.

Conversely, if the contact metric structure on T_1M satisfies the condition that ξ belongs to the (κ,μ)-nullity distribution, then

$$R(X,\xi)\xi = \kappa X + \mu hX \tag{5.2}$$

for any X orthogonal to ξ. Now, for a unit vector t on M define a symmetric operator $L_t : [t]^\perp \to [t]^\perp$ by $L_tX = \boldsymbol{R}(X,t)t$. Using (5.1) in (5.2) we see that

$$R(U,\xi)\xi = (\kappa+\mu)U - \mu(L_tKU)^V$$

and, in particular, that $R(U,\xi)\xi$ is vertical. On the other hand, computing $R(U,\xi)\xi$ by the Kowalski curvature formulas on TM we see that

$$R(U,\xi)\xi = -(\boldsymbol{R}(\boldsymbol{R}(t,KU)t,t)t)^V = (L_t^2KU)^V.$$

Thus the operator L_t satisfies the equation

$$L_t^2 + \mu L_t - (\kappa+\mu)I = 0.$$

Similarly, for a horizontal X orthogonal to ξ,

$$R(X,\xi)\xi = (\kappa-\mu)X + \mu(L_t\pi_*X)^H$$

and, from the Kowalski formulas,

$$R(X,\xi)\xi = (4L_t\pi_*X - 3L_t^2\pi_*X)^H,$$

giving

$$3L_t^2 + (\mu-4)L_t + (\kappa-\mu)I = 0.$$

Thus the eigenvalues a of L_t satisfy the two quadratic equations

$$a^2 + \mu a - (\kappa+\mu) = 0, \quad a^2 + \frac{\mu-4}{3}a + \frac{\kappa-\mu}{3} = 0.$$

If L_t had two eigenvalues, these quadratics imply that $\mu = -2$ and $\kappa = 1$, which implies that $h = 0$, i.e. the structure is K-contact. Moreover, $a = 1$ is now the only root and hence M is of constant curvature $+1$. As a side remark we recall a result of Tashiro [2, p. 136], that the contact metric structure on T_1M is K-contact if and only if the base manifold is of constant curvature $+1$. On the other hand, if L_t has only one eigenvalue, then M has constant curvature immediately.

References

[1] C. Baikoussis, D. E. Blair and T. Koufogiorgos, *A decomposition of the curvature tensor of a contact manifold satisfying* $R(X,Y)\xi = \kappa(\eta(Y)X - \eta(X)Y)$, Mathematics Technical Report, University of Ioannina, No 204, June 1992.

[2] D. E. Blair, *Contact manifolds in Riemannian geometry,* Lecture Notes in Mathematics **509**, Springer-Verlag, Berlin, 1976.

[3] D. E. Blair, *Two remarks on contact metric structures,* The Tôhoku Mathematical Journal **29** (1977), 319–324.

[4] D. E. Blair and J. N. Patnaik, *Contact manifolds with characteristic vector field annihilated by the curvature,* Bulletin of the Institute of Mathematics. Academia Sinica **9** (1981), 533–545.

[5] D. E. Blair, *When is the tangent sphere bundle locally symmetric?,* in *Geometry and Topology*, World Scientific, Singapore, 1989, pp. 15–30.

[6] D. E. Blair, T. Koufogiorgos and R. Sharma, *A classification of 3-dimensional contact metric manifolds with* $Q\varphi = \varphi Q$, Kodai Mathematical Journal **13** (1990), 391–401.

[7] D. E. Blair and H. Chen, *A classification of 3-dimensional contact metric manifolds with* $Q\varphi = \varphi Q$, *II,* Bulletin of the Institute of Mathematics. Academia Sinica **20** (1992), 379–383.

[8] O. Kowalski, *Curvature of the induced Riemannian metric on the tangent bundle,* Journal für die Reine und Angewandte Mathematik **250** (1971), 124–129.

[9] J. Milnor, *Curvature of left invariant metrics on Lie groups,* Advances in Mathematics **21** (1976), 293–329.

[10] Z. Olszak, *On contact metric manifolds,* The Tôhoku Mathematical Journal **31** (1979), 247–253.

[11] S. Tanno, *The topology of contact Riemannian manifolds,* Illinois Journal of Mathematics **12** (1968), 700–717.

[12] S. Tanno, *Isometric immersions of Sasakian manifolds in spheres,* Kodai Mathematical Seminar Reports **21** (1969), 448–458.

[13] S. Tanno, *Ricci curvatures of contact Riemannian manifolds,* The Tôhoku Mathematical Journal **40** (1988), 441–448.

[14] F. Trikerri and L. Vanhecke, *Homogeneous structure on Riemannian manifolds,* London Mathematical Society Lecture Note Series, 83, Cambridge Univ. Press, London, 1983.