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SADDLE POINTS AND INSTABILITY OF NONLINEAR HYPERBOLIC EQUATIONS†

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ABSTRACT

A number of authors have investigated conditions under which weak solutions of the initial-boundary value problem for the nonlinear wave equation will blow up in a finite time. For certain classes of nonlinearities sharp results are derived in this paper. Extensions to parabolic and to abstract operator equations are also given.

1. Introduction

Consider the nonlinear hyperbolic equation

$$u_{tt} = \Delta u + f(u)$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = v_0(x) \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \partial D,$$

with $t > 0$ and $x \in D$, where D is a smoothly bounded domain in $\mathbf{R}^n$. A number of authors (J. B. Keller [3], D. H. Sattinger [9], [10], K. Jörgens [2], M. Tsutsumi [13], [14], [15], R. T. Glassey [1], H. A. Levine [5], [6], [7], and others [4]) have investigated conditions on the initial data and nonlinearity f for which the solutions of (1.1) blow up in a finite time. In this paper we establish a number of sharp results in this direction. In addition, we derive these results for weak solutions of (1.1). We hope that our work will contribute to a better intuitive understanding of the phenomena of instability.

In order to describe the results it will be convenient first to consider the one-dimensional mechanical analogue of (1.1), namely

$$\ddot{x} = -x + f(x), \tag{1.2}$$

where x is a real number. Equation (1.2) describes a mechanical system with one degree of freedom, while (1.1) may be thought of as a system with an infinite number of degrees of freedom. Let

$$F(x) = \int_0^x f(s)\,ds.$$

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The potential energy for Equation (1.2) is

$$V(x) = \frac{x^2}{2} - F(x).$$

Suppose that V has the qualitative shape shown in Fig. 1: a local minimum at $x = 0$ and a local maximum at $x = x_1$.

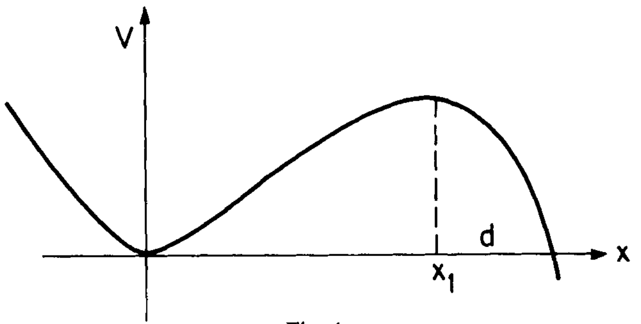

Fig. 1

The set

$$W = \{x : V(x) < d, x < x_1\}$$

describes a potential well—an interval containing the origin. The total energy of (1.2) (kinetic plus potential) is

$$E = \frac{\dot{x}^2}{2} + V(x).$$

E is of course conserved under the motion; if $E(0) < d$ and $x(0) \in W$, then, by the conservation of energy, $x(t)$ must lie in W for all $t > 0$. On the other hand, if $E(0) < d$ and $x > x_1$, then $x(t) > x_1$ for all t. The solution can never cross into W because, to do so, its total energy would have to be greater than d.

Now let us make the additional assumption that

$$xf(x) > c\,|x|^{p+1} \quad \text{whenever} \quad x > x_1$$

for some $c > 0$ and $p > 1$. This simply says that Equation (1.2) is strictly nonlinear for $x > x_1$. Then, under these assumptions, it is easily demonstrated that $x(t) \to +\infty$ in a finite time. This result is sharp in that it gives a precise description of the initial data for which (1.2) has global bounded solutions, and those for which the solution tends to infinity in a finite time.

In this paper we extend the above results to the infinite dimensional case (1.1). The potential energy associated with (1.1) is the functional

$$J(u) = \int_D \left\{ \frac{|\nabla u|^2}{2} - F(u) \right\} dx,$$

where

$$F(u) = \int_0^u f(s) ds.$$

Under certain assumptions on $f(u)$ (which apply, for example, in the cases $f(u) = u^p$, or $u|u|^{p-1}$, $p < (n+2)/(n-2)$) we prove that (1.1) has a local minimum at the origin, a potential well W, and a saddle point w. The potential well has a positive depth d; and, just as in the finite dimensional case, if $\mathscr{E}(0) < d$ and u_0 lies outside W, then the solution u to (1.1) tends to infinity in finite time (in the L_2 norm).

It has previously been shown [10] that, if u_0 lies inside W and $\mathscr{E}(0) < d$, then $u(t) \in W$ for all t and (1.1) has a global solution.

On the other hand, Levine and Tsutsumi have proved a number of interesting instability theorems[†]. The principal innovations in the present paper are the construction of the saddle points of J, and the use of related properties of J to obtain the sharp conditions for instability mentioned in the previous paragraphs.

In Section 2 we discuss the characterization of the potential well W by certain differential-integral inequalities. In addition, we prove the existence of saddle points of J (unstable critical points) by a direct method. The method may possibly be applicable to other problems in the calculus of variations, for example, the construction of unstable minimal surfaces or surfaces of constant mean curvature.

In the homogeneous case, $f(u) = u^p$ or $f(u) = u|u|^{p-1}$, there is a direct connection between the solution w to

$$\Delta w + f(w) = 0, \quad w = 0 \quad \text{on} \quad \partial D$$

(which is the Euler equation for the critical points of J) and an associated Sobolev inequality.

If for a smoothly bounded domain D in $\mathbf{R}^n$ we define the norms

[†] Generally speaking, the class of nonlinear functions $f(u)$ considered in this paper is more restrictive than that considered by Levine and Tsutsumi. However, in this more restrictive class the results of the present paper are considerably sharper than those of Levine and Tsutsumi.

(1.3) $$\| u \| = \left[\int_D |\nabla u|^2 dx \right]^{1/2}$$

and

(1.4) $$| u |_p = \left[\int_D | u |^p dx \right]^{1/p}$$

for $1 < p < \infty$, then the associated Sobolev inequality is given by

$$| u |_p \leqq \gamma_p \| u \| .$$

In fact, the function w is an extremal to the Sobolev problem

$$\gamma_p = \inf_u \frac{\| u \|}{| u |_p}$$

over functions u with finite Dirichlet norm which vanish on the boundary. The depth d of the potential well can be computed exactly in terms of the Sobolev constant γ_p, and one obtains

$$\gamma_p = \left[\frac{2p}{p-2} d \right]^{(p-2)/2p},$$

provided $p < 2n/(n-2)$ $(x \in \mathbf{R}^n)$. These connections are discussed in Section 3.

In Section 4 we prove the instability results. These are proved by obtaining appropriate second order differential inequalities on

$$M(t) = \int_D u^2 dx.$$

In Section 5 we discuss similar results for parabolic problems, and in Section 6 we discuss generalizations to abstract second order Cauchy problems of the type

$$Pu_{tt} + Qu = f(u),$$

where Q is a positive definite operator on a Hilbert space and f is a gradient operator.

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2. Potential wells in function space

In this section we first impose conditions on the nonlinear function $f(u)$ which will insure that all non-trivial critical points of J are unstable equilibria for (1.1)[†]. We characterize these critical points as extrema of a variational problem, proving first the uniqueness and constant sign of the extrema (assumed to exist) and then establishing existence of the extrema. The latter proof actually establishes the existence of a potential well W of depth $d > 0$.

We assume throughout that D is a smoothly bounded domain in $\mathbf{R}^n$. We denote by $\mathring{H}_1$ the closure of the class $C_0^\infty(D)$ under the norm $\| \ \|$, introduced in (1.4). The reader will recall the Sobolev embedding theorems which state that (a): for $p \leqq 2n/(n-2)$ there is a constant S_p such that $|u|_p \leqq S_p \|u\|$ for all $u \in \mathring{H}_1$; and (b): the injection from $\mathring{H}_1$ into L_p is compact for $p < 2n/(n-2)$. The latter statement means that every bounded set in $\mathring{H}_1$ contains a subsequence which converges in L_p.

Consider the functional J on $\mathring{H}_1$ defined by

$$J(u) = \frac{1}{2}\|u\|^2 - \int_D F(u)\,dx.$$

This functional, which may be regarded as the potential energy functional for the infinite dimensional dynamical system (1.1), is well defined provided that $|F(u)| = O(|u|^p)$ as $|u| \to \infty$ for $p \leqq 2n/(n-2)$. The critical points of J are functions w in $\mathring{H}_1$ which satisfy the Euler equation

$$\Delta w + f(w) = 0; \qquad w = 0 \quad \text{on} \quad \partial D, \tag{2.1}$$

where $f = F'$. Since (2.1) is elliptic, the critical points of J are regular functions on D, provided f is regular.

Throughout this paper we make the following assumptions:

(i) $F(u) = \int_0^u f(s)\,ds$.

(ii) f is C^1 and $f(0) = f'(0) = 0$; $f \not\equiv 0$ in a neighborhood of the origin.

(iii) (a) f is monotone and is convex for $u > 0$, concave for $u < 0$; or (b) f is convex.

(iv) $(p+1)F(u) \leqq uf(u)$, and $|uf(u)| \leqq \gamma |F(u)|$, where $2 < (p+1) \leqq \gamma < 2n/(n-2)$.

[†] The fact that J has non-zero critical points has been established under various hypotheses by others (see, e.g., A. Ambrosetti and P. Rabinowitz, J. Functional Analysis **14** (1973) and papers cited therein). We include a simple proof in this section for completeness.

REMARK. In all the proofs that follow, the two cases (iii) (a) and (b) must be treated slighly differently.

LEMMA 2.1. *Under the conditions* (i)–(iv) *above we have*

$$|F(u)| = O(|u|^\gamma); \tag{2.2}$$

and in case (iii) (a)

$$u(uf' - f) \geqq 0, \tag{2.3}$$

with equality holding only for $u = 0$. *The inequality* (2.3) *also holds in case* (iii) (b) *for* $u \geqq 0$.

PROOF. The growth condition (2.2) is obtained by integrating the inequality $|uf(u)| \leqq \gamma |F(u)|$, using the fact that $F' = f$. The inequality (2.3) follows by noting that the quantity

$$f(u) - uf'(u)$$

is the y-intercept of the tangent line to the graph of f at the point $(u, f(u))$.

The second variation of J at a critical point w is the quadratic functional

$$\delta^2 J[v] = \frac{1}{2}\left\{\|v\|^2 - \int_D f'(w)v^2 dx\right\}.$$

A necessary condition that w be a local minimum of J is that $\delta^2 J$ be positive definite. This is the case at the origin ($w = 0$), since there $\delta^2 J[v] = \|v\|^2/2$. At a non-trivial critical point, however, $\delta^2 J$ is not positive definite. In fact, noting that w satisfies the boundary conditions and so is an admissible trial function, we compute

$$\begin{aligned}\delta^2 J[w] &= \frac{1}{2}\left\{\|w\|^2 - \int_D f'\ w)w^2 dx\right\} \\ &= -\frac{1}{2}\int_D w[\Delta w + f'(w)w]\,dx \\ &= -\frac{1}{2}\int_D w[wf'(w) - f(w)]\,dx.\end{aligned}$$

If f satisfies (iii) (a), then from (2.3) we see that $\delta^2 J[w] < 0$. If f is convex then, by the maximum principle, $w > 0$, since

$$\Delta w = -f(w) < 0;$$

therefore, again by (2.3), $\delta^2 J[w] < 0$. (Incidentally, note that if f is positive homogeneous of degree p then $wf'(w) = pf(w)$; and so

$$\delta^2 J[w] = -\frac{1}{2}(p-1)\int_D wf(w)\,dx = -\frac{1}{2}(p-1)\|w\|^2 < 0.$$

The reader will note that the proofs which follow are particularly simple in case f is homogeneous.)

Thus, under conditions (i)–(iv) on f, all non-trivial critical points are *a priori* unstable equilibria for the hyperbolic problem (1.1). The origin, however, is at least formally a stable equilibrium. We are going to show that under the growth condition (iv) $w = 0$ is a local minimum of J and that the depth of the potential well is positive.

Let u be an arbitrary element of $\mathring{H}_1$ and consider the real valued function of λ

$$j(\lambda) = J(\lambda u)$$

and

$$j'(\lambda) = \lambda\|u\|^2 - \int_D uf(\lambda u)\,dx$$

$$j''(\lambda) = \|u\|^2 - \int_D u^2 f'(\lambda u)\,dx.$$

Therefore $j(0) = j'(0) = 0$ and $j''(0) = \|u\|^2 > 0$. Thus for any $u \in \mathring{H}_1, j(\lambda)$ is a convex function of λ for small λ. Let us show that under our assumptions on f and F, $j(\lambda)$ has a unique positive critical point $\lambda^* = \lambda^*(u)$.

LEMMA 2.2. *If f satisfies* (i)–(iv) *then for any $u \in \mathring{H}_1$, $u \not\equiv 0$,*

(a) $\lim_{\lambda\to+\infty} j(\lambda) = -\infty$;

(b) *there is a unique* $\lambda^* = \lambda^*(u) > 0$ *such that* $j'(\lambda^*) = 0$;

(c) $j''(\lambda^*) < 0$,

In case (iii) (b) *we also assume* $u \geqq 0$.

PROOF. We first consider the case (iii) (a). From the inequality $(p+1)F(u) \leqq uf(u)$ (F and uf are nonnegative in this case) we obtain the growth condition

$$F(u) \geqq B\,|u|^{p+1} \quad \text{for} \quad |u| \geqq 1,$$

where $B = \min\{F(1), F(-1)\}$. Accordingly

$$j(\lambda) = \frac{\lambda^2\|u\|^2}{2} - \int_D F(\lambda u)\,dx$$

$$\leqq \lambda^2\frac{\|u\|^2}{2} - B\,|\lambda|^{p+1}\int_{D\cap\{u \geqq 1/\lambda\}} |u|^{p+1}\,dx.$$

Since $(p+1) > 2$, $j(\lambda) \to -\infty$ as $\lambda \to \infty$.

The case (iii) (b) may be handled in the same way as (iii) (a), since we assume in addition that $u > 0$.

To prove (b), suppose that there are two roots, say $\lambda_1 < \lambda_2$, of $j'(\lambda) = 0$. (The existence of λ^* is guaranteed by the facts that $j(\lambda) \to -\infty$ as $\lambda \to \infty$ and that $j(\lambda)$ is convex for small λ.) Then

$$\lambda_1 \| u \|^2 - \int_D uf(\lambda_1 u)\, dx = 0$$

and

$$\lambda_2 \| u \|^2 - \int_D uf(\lambda_2 u)\, dx = 0.$$

Eliminating $\| u \|^2$ from these two equations, we get

$$\int_D u \left[\frac{f(\lambda_2 u)}{\lambda_2} - \frac{f(\lambda_1 u)}{\lambda_1} \right] dx = 0.$$

Putting $w = \lambda_1 u$ and $\lambda = \lambda_2 / \lambda_1$, this condition can be rewritten as

$$\int_D w[f(\lambda w) - \lambda f(w)]\, dx = 0, \tag{2.4}$$

where $\lambda > 1$. If f satisfies (iii) (a), or if f satisfies (iii) (b) and $u > 0$, then it is easily seen that the integrand in (2.4) does not change sign and does not vanish identically. Therefore (2.4) is impossible and (b) is proved.

To prove (c) we note that

$$0 = j'(\lambda^*) = \lambda^* \| u \|^2 - \int_D uf(\lambda^* u)\, dx;$$

and so

$$\begin{aligned} j''(\lambda^*) &= \| u \|^2 - \int_D u^2 f'(\lambda^* u)\, dx \\ &= (\lambda^*)^{-2} \int_D \lambda^* u \,\{f(\lambda^* u) - (\lambda^* u) f'(\lambda^* u)\}\, dx \\ &< 0 \end{aligned}$$

by (2.3) in either of cases (iii) (a) or (b). This completes the proof of Lemma 2.2.

Having proved that there exists a unique $\lambda^* = \lambda^*(u)$ such that

$$\frac{d}{d\lambda} J(\lambda u) \bigg|_{\lambda = \lambda^*} = 0,$$

we now define

$$H(u) = J(\lambda^* u).$$

We may think of $H(u)$ as the highest level attained when leaving the "potential well" along a ray in the direction u. We now define the depth d of the potential well by

$$d = \inf_{u \neq 0} H(u).$$

Equivalently, if we always normalize u so that $\lambda^* = 1$, that is, so that

$$K(u) \stackrel{\text{def}}{=} \| u \|^2 - \int_D uf(u)dx = 0,$$

then our variational problem may be written

$$d = \inf J(u) \tag{2.5}$$

subject to the constraints

$$\| u \| \neq 0, \ K(u) = 0. \tag{2.6}$$

Cleary every non-trivial critical point of J is an extremal of (2.5)–(2.6), since every such critical point of J satisfies the constraint (2.6). The existence of extremals of (2.5)–(2.6) will be shown below. We now prove:

THEOREM 2.3. *Let f satisfy conditions* (i)–(iv). *Then any extremal of* (2.5)–(2.6) *is a critical point of J. If an extremal exists in case* (iii) (b), *then it is unique and positive. In case* (iii) (a) *no extremal can change sign and there can exist at most one extremal W_1 under the additional constraint $u \geqq 0$ in* (2.6) *and at most one extremal W_2 under the additional constraint $u \leqq 0$. If an extremal exists and f is odd, then $W_1 = -W_2$.*

PROOF. Let us first show that if f is convex then any extremal of (2.5)–(2.6) is non-negative. Suppose u is an extremal, so that $d = J(u)$ and $K(u) = 0$. If u changes sign then $K(|u|) < K(u) = 0$. By Lemma 2.2 (b) there exists a unique $\bar{\lambda} \geqq 0$ such that $K(\bar{\lambda}|u|) = 0$. Since $K(|u|) < 0$, $\bar{\lambda}$ must be less than one, and since $K(\lambda u) > 0$ for $\lambda < 1$ and

$$\frac{d}{d\lambda} J(\lambda u) = K(\lambda u) > 0 \quad \text{for} \quad \lambda < 1,$$

we have

$$d = J(u) > J(\bar{\lambda}u) > J(\bar{\lambda}|u|).$$

Thus the function $\bar{\lambda}|u|$ satisfies the constraint $K(\bar{\lambda}|u|) = 0$ and $L(\bar{\lambda}|u|) < d$, in contradiction to the definition of d. Therefore u cannot change sign.

The above result can be rephrased in the following way. Let

$$d' = \inf J(u)$$

subject to

$$K(u) = 0 \quad \text{and} \quad u \geqq 0 .$$

Then $d \leqq d'$, since d is obtained by minimizing over a larger admissible class. On the other hand, by the argument of the preceding paragraph, one can see that $d' \leqq d$.

In fact, given a function u such that $K(u) = 0$, we choose $\bar{\lambda} < 1$ such that $K(\bar{\lambda} \mid u \mid) = 0$. Then, as above, $K(\bar{\lambda} \mid u \mid) = 0$ and $J(\bar{\lambda} \mid u \mid) < J(u)$. Therefore $d' = d$.

A similar argument works in case (iii) (a). If u is an extremal and u changes sign, then there is a $\bar{\lambda} < 1$ such that $K(\bar{\lambda} \mid u \mid) = 0$ and $J(\bar{\lambda} \mid u \mid) < J(u)$, contradicting the hypothesis that u was an extremal. To obtain the positive and negative extremals, we minimize over the classes of positive and negative functions respectively.

Let us now show that an extremal of (2.5)–(2.6) is an extremal of J.

The Euler equation associated with (2.5)–(2.6) is

$$(\Delta w + f(w)) + \Lambda(2\Delta w + wf'(w) + f(w)) = 0 ,$$

or

$$(1 + 2\Lambda)(\Delta w + f(w)) - \Lambda(f(w) - wf'(w)) = 0 , \tag{2.7}$$

where Λ is the Lagrange multiplier. Multiplying (2.7) by w and integrating, we obtain

$$\Lambda \int_D w \{f(w) - wf'(w)\} \, dx = 0 , \tag{2.8}$$

since

$$\| w \|^2 - \int_D wf(w) \, dx = 0$$

by (2.6).

If f satisfies (iii) (a), then the integral in (2.8) cannot vanish as a consequence of (2.3). The same argument applies when f is convex, since we have already seen in that case that $w \geq 0$. Thus, in either case, $\Lambda = 0$ and w satisfies the Euler equation $\Delta w + f(w) = 0$. If f is convex then $\Delta w \leqq 0$ and $w > 0$ by the strong maximum principle. If f is monotone and $w \geqq 0$, then $\Delta w \leqq 0$ and again $w > 0$ by the strong maximum principle. Similarly, if $w \leqq 0$ then w is in fact strictly negative.

To prove uniqueness in case (iii) (b) we use a result known as "Serrin's Sweeping Statement" (see [11], p. 40; [12], p. 15). A statement of Serrin's theorem is given in the appendix of the present paper. To apply the theorem let w be a solution of (2.1) and consider the family $\{\lambda w\}_{\lambda \geqq 1}$. From the convexity of f for $w > 0$ we obtain

$$\Delta(\lambda w) + f(\lambda w) \leqq \lambda(\Delta w + f(w)) = 0$$

for $\lambda \geqq 1$. Let u be any other solution of (2.1). If $u(x) > w(x)$ somewhere in D, choose λ so large that $u(x) \leqq \lambda w(x)$. Then, from the first part of Serrin's theorem, $u(x) < w(x)$, since $v = \lambda w = 0$ on ∂D. Thus $u(x) \leqq w(x)$ everywhere in D. Similarly, applying the second statement of the theorem to the family of lower solutions $\{\lambda w\}_{\lambda \leqq 1}$, we see that $u(x) \geqq w(x)$ everywhere in D as well, so that $u \equiv w$.

In case (iii) (a) we prove by similar arguments the uniqueness of the positive and negative extremals separately.

Corollary 2.4. *Let f satisfy* (i), (ii), *and the growth condition* $|f(u)| = O(|u|^p)$ *where* $p + 1 \leqq 2n/(n-2)$. *If w is an extremal of* (2.5)–(2.6) *for which*

$$\frac{d^2}{d\lambda^2} J(\lambda w)\bigg|_{\lambda=1} < 0, \tag{2.9}$$

then w is a critical point of J.

Proof. We have

$$\| w \|^2 - \int_D wf(w)\,dx = 0,$$

$$\| w \|^2 - \int_D w^2 f'(w)\,dx < 0.$$

Substracting, we get

$$\int_D w[f(w) - wf'(w)]\,dx < 0. \tag{2.10}$$

Referring to the proof of Theorem 2.3, we see that again we may conclude that $\Lambda = 0$, and so w is a critical point of J.

Corollary 2.4 applies in general — that is, f need not satisfy any of the special condition (iii). Below, in Theorem 2.6, we prove that $d > 0$ and demonstrate the existence of extremals of the variational problem (2.5)–(2.6). We first prove

Lemma 2.5. *Under assumptions* (i), *and* (ii), *and a growth condition on f,* ($|f(u)| = O(|u|^p)$ *where* $(p+1) \leqq 2n/(n-2)$), *the functionals J and K are continuous on* $\mathring{H}_1$.

PROOF. It suffices to show that the functionals

$$\int_D F(u)\,dx \quad \text{and} \quad \int_D uf(u)\,dx$$

are continuous on $\mathring{H}_1$. We shall demonstrate this fact for the first functional.

By the mean value theorem we have

$$F(u)-F(v)=\int_0^1 (u-v)f(u+\tau(v-u))\,d\tau\,;$$

hence

$$\left|\int_D \{F(u)-F(v)\}dx\right| \leqq \int_D \left\{\int_0^1 |u-v|\,|f(u+\tau(v-u))|\,d\tau\right\}dx$$

$$\leqq \int_0^1 \left\{\int_D |f(u+\tau(v-u))|^r dx\right\}^{1/r}\left\{\int_D |u-v|^s dx\right\}^{1/s} d\tau$$

$(r^{-1}+s^{-1}=1)$. Since $|f(u)|=O(|u|^p)$ where $(p+1)\leqq 2n/(n-2)$, we may take $r=1+p^{-1}$ and $s=p+1$. Then the above difference is dominated by

$$|u-v|_{p+1}\int_0^1\left\{\int_D |v+\tau(u-v)|^{p+1}dx\right\}^{p/(p+1)} d\tau$$

$$\leqq S_{p+1}\|u-v\|\int_0^1 (|v+\tau(u-v)|_{p+1})^p\,d\tau$$

$$\leqq C(p,|u|_{p+1},|v|_{p+1})\|u-v\|,$$

where C is a constant depending on p, $|u|_{p+1}$, and $|v|_{p+1}$. This inequality shows in fact that J is Lipschitz continuous on $\mathring{H}_1$.

THEOREM 2.6. *Let f satisfy* (i)–(iv). *Then $d>0$; and, further, if $2<\gamma<2n/(n-2)$, there exists an extremal of the variational problem* (2.5)–(2.6). *If f is convex, there is a unique positive extremal, while in case* (iii) (a) *there are two extremals — one positive and one negative.*

PROOF. To prove that $d>0$ we establish a lower bound for $J(u)$ when u satisfies (2.6). As we have already seen in the proof of Theorem 2.3, it is sufficient to minimize over the class of non-negative functions in case (iii) (b), and as in case (iii) (b) we may assume in addition to (2.6) that $u\geqq 0$. Thus

$$J(u)=\frac{1}{2}\|u\|^2-\int_D F(u)\,dx$$

$$\geqq \frac{1}{2}\|u\|^2-\frac{1}{p+1}\int_D uf(u)\,dx\,;$$

hence from the constraint $K(u)=0$,

$$J(u) \geqq \frac{p-1}{2(p+1)} \| u \|^2. \tag{2.11}$$

On the other hand, $uf(u) \leqq \gamma F(u) \leqq \text{const.}\ |u|^\gamma$, so by Sobolev's inequality

$$|u|_\gamma^2 \leqq S_\gamma^2 \|u\|^2 = S_\gamma^2 \int_D uf(u)\,dx \leqq AS_\gamma^2 |u|_\gamma^\gamma$$

for some constant A. It follows that

$$|u|_\gamma^{\gamma-2} \geqq \frac{1}{AS_\gamma^2},$$

and hence that

$$\|u\| \geqq \frac{1}{S\gamma} |u|_\gamma \geqq \left(\frac{1}{AS_\gamma^\gamma}\right)^{1/(\gamma-2)}. \tag{2.12}$$

Therefore

$$d \geqq \frac{1}{2}\left(\frac{p-1}{p+1}\right)\left(\frac{1}{AS_\gamma^\gamma}\right)^{2/(\gamma-2)}.$$

We now prove the existence of extremals to (2.5)–(2.6). Let $\{u_n\}$ be a minimizing sequence. Thus

$$u_n \in \mathring{H}_1, \ \|u_n\| \neq 0, \ K(u_n) = 0,$$

and

$$\lim_n J(u_n) = d.$$

We first note that, from (2.11), $\{u_n\}$ is a bounded sequence in $\mathring{H}_1$. Therefore we may select a subsequence which converges strongly in L_γ for $\gamma < 2n/(n-2)$. Without loss of generality, we may assume that it is the original sequence $\{u_n\}$ which converges, and that this sequence also converges a.e. Denoting the limit function by w we have

$$|u_n - w|_\gamma \to 0,$$

$$\|w\| \geqq \varliminf_{n\to\infty} \|u_n\| = \lim_{n\to\infty} \|u_n\|.^\dagger$$

From (2.12) we see that

$$|w|_\gamma = \lim_{n\to\infty} |u_n|_\gamma \geqq S_\gamma \left(\frac{1}{AS_\gamma^\gamma}\right)^{1/(\gamma-2)};$$

hence $|w|_\gamma \neq 0$.

† Since $\int_D F(u_n)dx \to \int_D F(w)$ and $J(u_n) \to d$, we have $\lim_n \|u_n\|^2/2 = d + \int_D F(w)dx$, and the limit of $\|u_n\|$ exists.

If $\| w \| = \lim_{n\to\infty} \| u_n \|$, then $K(w) = 0$ and $J(w) = d$, and we are done. If $\| w \| < \lim_{n\to\infty} \| u_n \|$, then $K(w) < 0$ and $J(w) < d$. In that case, there exists by Lemma 2.2 a $\bar{\lambda} < 1$ such that $K(\bar{\lambda} w) = 0$, since for sufficiently small λ, $K(\lambda u) > 0$ for any $u \in \mathring{H}_1$, $u \neq 0$. For this choice of λ we have

$$
\begin{aligned}
J(\bar{\lambda} w) &= \frac{\bar{\lambda}^2}{2} \| w \|^2 - \int_D F(\bar{\lambda} w)\, dx \\
&\leqq \frac{\bar{\lambda}^2}{2} \lim_{n\to\infty} \| u_n \|^2 - \int_D F(\bar{\lambda} w)\, dx \\
&= d + \int_D \left\{ F(w) - F(\bar{\lambda} w) - \frac{1}{2}(1 - \bar{\lambda}^2)\, wf(w) \right\} dx .
\end{aligned}
$$

Let

$$
I(\lambda) = \int_D \left\{ F(w) - F(\lambda w) - \frac{1}{2}(1 - \lambda^2)\, wf(w) \right\} dx . \tag{2.13}
$$

We distinguish the cases (iii) (a) and (iii) (b). As already remarked, w may be assumed to be non-negative in case (iii) (b). Thus, in either case,

$$
I(0) = \int_D \left\{ F(w) - \frac{1}{2} wf(w) \right\} dx < 0
$$

from (iv) and

$$
I(1) = 0 .
$$

Moreover,

$$
I'(\lambda) = \int_D w[\lambda f(w) - f(\lambda w)]\, dx ,
$$

which is positive for $0 < \lambda < 1$ in either case. Therefore $I(\lambda) < 0$ for $0 < \lambda < 1$ and

$$
J(\bar{\lambda} w) < d \quad \text{while} \quad K(\bar{\lambda} w) = 0 ,
$$

in contradiction to the definition of d.

In case (iii) (a), of course, we may consider two variational problems — one over the positive and one over the negative functions, thereby obtaining a positive and negative extremal.

The interior of the potential well W is characterized by the integral inequalities

(i) $$J(u) < d$$

(ii) $$\frac{d}{d\lambda} J(\lambda u) \geqq 0 \quad \text{for} \quad 0 < \lambda \leqq 1.$$

It was shown by Sattinger [10] that, if the total energy of the initial data is less than d and u_0 lies in W, then (1.1) has global solutions in time. In this paper we exhibit an exterior region $\mathscr{E}$ in $\mathring{H}_1$ such that, if the initial position lies in $\mathscr{E}$ and the total energy is less than d, then the solution (1.1) goes to infinity in a finite time. Specifically, in case f is convex, $\mathscr{E}$ is characterized by

(2.14)

(i) $$J(u) < d$$

(ii) $$K(u) < 0.$$

In case (iii) (a) there are two regions $\mathscr{E}_1$ and $\mathscr{E}_2$ characterized by the inequalities in (2.14) with d replaced, respectively, by d_1 and d_2, where

$$d_1 = \inf J(u),$$

subject to

$$\| u \| \neq 0, \; K(u) = 0, \text{ and } u \geqq 0,$$

etc. Note that in all cases the regions $\mathscr{E}$ are unbounded.

The following lemma will be needed in Section 4.

LEMMA 2.7. *Let f satisfy conditions* (i)–(iv) *(with $(p+1) < 2n/(n-2)$), and suppose $\{u_n\}$ is a sequence in $\mathring{H}_1$ such that $K(u_n) \leqq 0$ and $K(u_n) \to 0$. Then, if $\| u_n \| \neq 0$ for all n,*

$$\varliminf_{n \to \infty} J(u_n) \geqq d.$$

PROOF. As in Theorem 2.6, we choose a strongly convergent subsequence in L_{p+1}, and let w denote the limit. Then $\| w \| \neq 0$ and $K(w) \leqq 0$. If $\lim_{n\to\infty} \| u_n \| = \| w \|$, then $K(w) = 0$ and $\lim_{n\to\infty} J(u_n) = J(w) \geqq d$. On the other hand, if $K(w) < 0$ we choose $\bar{\lambda}$ so that $K(\bar{\lambda} w) = 0$. Then, as in the proof of Theorem 2.6, we can show that

$$J(\bar{\lambda} w) < \varliminf_{n \to \infty} J(u_n) + I(\bar{\lambda}),$$

where $I(\lambda)$ is given by (2.13). In case (iii) (a), $I(\bar{\lambda}) < 0$; so if $\varliminf_{n\to\infty} J(u_n) < d$, then $J(\bar{\lambda} w) < d$ while $K(\lambda w) = 0$, contradicting the variational definition of d.

In case (iiib) we must proceed more carefully, since we cannot be certain that $w \geqq 0$. Suppose that f is convex and that $\{u_n\}$ is such that $K(u_n) \to 0$ while $K(u_n) \leqq 0$. Then

$$J(u_n) \geqq J(|u_n|)$$

and

$$0 \geqq K(u_n) \geqq K(|u_n|).$$

Let $u_n \to w$ and $|u_n| \to |w|$. Since $K(|w|) < 0$, we may choose $\bar{\lambda} < 1$ such that $K(\bar{\lambda}|w|) = 0$. Then

$$\begin{aligned}
J(\bar{\lambda}|w|) &= \frac{\bar{\lambda}^2}{2}\|w\|^2 - \int_D F(\bar{\lambda}|w|)\,dx \\
&\leqq \frac{\bar{\lambda}^2}{2}\varliminf_{n\to\infty}\|u_n\|^2 - \int_D F(\bar{\lambda}|w|)\,dx \\
&= \varliminf_{n\to\infty}\frac{\|u_n\|^2}{2} + \frac{1-\bar{\lambda}^2}{2}\varliminf_{n\to\infty}\|u_n\|^2 - \int_D F(\bar{\lambda}|w|)\,dx \\
&= \varliminf_{n\to\infty}\left\{J(u_n) + \frac{1-\bar{\lambda}^2}{2}\|u_n\|^2 + \int_D F(u_n)\,dx\right\} - \int_D F(\bar{\lambda}|w|)\,dx \\
&\leqq \varliminf_{n\to\infty}\left\{J(u_n) + \frac{1-\bar{\lambda}^2}{2}\int_D u_n f(u_n)\,dx + \int_D F(u_n)\,dx\right\} \\
&\qquad - \int_D F(\bar{\lambda}|w|)\,dx \\
&\leqq \varliminf_{n\to\infty} J(u_n) + \int_D \left\{\frac{1-\bar{\lambda}^2}{2} wf(w) + F(w) - F(\bar{\lambda}|w|)\right\} dx \\
&\leqq \varliminf_{n\to\infty} J(u_n) + \int_D \left\{\frac{1-\bar{\lambda}^2}{2}|w| f(|w|) + F(|w|) - F(\bar{\lambda}|w|)\right\} dx \\
&\leqq \varliminf_{n\to\infty} J(u_n).
\end{aligned}$$

Now if $\varliminf_{n\to\infty} J(u_n) < d$, then we have a function $\bar{\lambda}|w|$ such that $K(\bar{\lambda}|w|) = 0$ and $J(\bar{\lambda}|w|) < d$, which is impossible.

3. Sobolev constants

Although the following remarks are not essential to the development of our subject, we thought it worthwhile to clarify the relationship of the Sobolev constants to the preceding considerations about potential wells. Consider the nonlinear equation

$$\Delta w + |w|^{p-1} w = 0$$

(3.1)

$$w = 0 \quad \text{on} \quad \partial D.$$

If w is a solution of (3.1), then for all $\lambda > 0$, the family $(\lambda, \lambda^{1/(1-p)} w) = (\lambda, v(\lambda))$ is a one-parameter family of solutions of the equation

$$\Delta v + \lambda |v|^{p-1} v = 0$$

(3.2)

$$v = 0 \quad \text{on} \quad \partial D.$$

This is an immediate consequence of the homogeneity of the function $|u|^{p-1} u$. Now define

$$S_{p+1} = \inf \frac{\|u\|}{|u|_{p+1}}$$

over the class $u \in \mathring{H}_1$. The Euler equation for this homogeneous variational problem is

$$\Delta u + \lambda |u|^{p-1} u = 0, \tag{3.3}$$

where λ is a Lagrange multiplier.

Since the ratio $\|u\|/|u|_{p+1}$ is homogeneous, we can replace the extremal u by any scalar multiple of u and therefore by w itself, so that

$$S_{p+1} = \frac{\|w\|}{|w|_{p+1}}.$$

On the other hand,

$$\|w\|^2 = (|w|_{p+1})^{p+1}$$

from (3.1) and

$$d = \frac{1}{2}\|w\|^2 - \frac{1}{p+1}(|w|_{p+1})^{p+1}$$

$$= (|w|_{p+1})^{p+1}\left(\frac{p-1}{2(p+1)}\right).$$

Therefore

$$S_{p+1} = \frac{\|w\|}{|w|_{p+1}} = |w|_{p+1}^{(p-1)/2} = \left[\frac{2(p+1)}{p-1} d\right]^{(p-1)/2(p+1)}.$$

4. Finite blow-up time

In this section we prove two main results. First, we prove that if u is a weak solution of (1.1) such that $u_0 \in \mathscr{E}$ and $E(0) < d$, then u will blow up in a finite time. We then show that if the solution starts inside the potential well and $E(0) < d$, the kinetic energy $\int_D u_t^2\,dx$ will remain bounded for all time. Furthermore, u will remain bounded in $\mathring{H}_1$ for all t.

Under similar assumptions on $f(u)$, Sattinger [10] has already established global existence (see also Tsutsumi [13]) and boundedness (in certain norms) for solutions starting inside the potential well. We repeat some of the results here for completeness.

Since it is well known that classical solutions of nonlinear hyperbolic equations may not exist for all time no matter how smooth the data, coefficients, and geometry, we shall introduce a class of weak solutions. This is motivated at least in part by the fact that in many physical problems whose solutions are characterized by solutions of initial or initial-boundary value problems for nonlinear hyperbolic systems, the existence of a unique weak solution can sometimes be established in cases where classical solutions do not exist.

For simplicity we define the weak solution of (1.1) over the interval $[0, T)$, but it is to be understood throughout that T is either infinity or the limit of the existence interval.

We say that u is a weak solution of (1.1) on $[0, T)$ if it satisfies the following conditions:

(1) $u(t)$ is a weakly continuous mapping from $[0, T)$ to $\mathring{H}_1$; thus $\| u(t) \|$ and $| u(t) |_2$ are uniformly bounded on compact subsets of $[0, T)$.

(2) There is a weakly continuous mapping from $[0, T)$ to $L_2(D)$, denoted by u_t, such that

$$(4.1) \qquad (u, \varphi) \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} (u_t, \varphi)\, ds$$

for any t_1, t_2, $0 \leqq t_1 < t_2 < T$ and any $\varphi \in L_2(D)$. Here $(,)$ denotes the inner product on $L_2(D)$.

(3) For any $\varphi : [0, T) \to \mathring{H}_1$ with the same properties as u above,

$$(4.2) \qquad (u_t, \varphi) \Big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \{(u_t, \varphi_t) - ((\varphi, u)) + (\varphi, f(u))\}\, ds\,,$$

where $((,))$ denotes the inner product on $\mathring{H}_1$.

(4) The energy $E(t)$, defined by

$$E(t) = \frac{1}{2}(| u_t |_2^2 + \| u \|^2) - \int_D F(u)\, dx\,,$$

satisfies

$$E(t_2) \leqq E(t_1) \tag{4.3}$$

for any $t_1 < t_2 < T$.

From (4.1) it follows that the mapping $u(t)$ is weakly absolutely continuous (take φ to be a constant vector in $L_2(D)$). Moreover, since $|u_t|$ is bounded, $u(t)$ is weakly Lipschitz continuous in t, that is, $(u(t), \varphi)$ is Lipschitz continuous for any φ in L_2.

Putting $\varphi = u$ in (4.2) we get

$$(u_t, u)\big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \{|u_t|^2 - \|u\|^2 + (u, f(u)\} \, ds. \tag{4.4}$$

LEMMA 4.1. *Let*

$$M(t) = \int_D u^2(x, t)\, dx,$$

where u is a weak solution of (1.1). *Then $\ddot{M}$ exists a.e. in* $[0, T)$, *and $\dot{M}(t)$ is Lipschitz continuous there.*

PROOF. Let $Q(t, s) = (u(t), u(s))$. Since $u(t)$ is weakly absolutely continuous, and u_t is weakly continuous,

$$\dot{M}(t) = \left(\frac{\partial}{\partial t} Q(t, s) + \frac{\partial}{\partial s} Q(t, s)\right)\bigg|_{s=t} = 2(u_t, u),$$

and so from (4.4) we have

$$\dot{M}(t_2) - \dot{M}(t_1) = 2\int_{t_1}^{t_2} \{|u_t|^2 - \|u\|^2 + (u, f(u))\} \, ds,$$

for all $0 \leqq t_1 < t_2 < T$. Since each term in the integrand is bounded on compact subsets of $[0, T)$, we see that $\dot{M}(t)$ is Lipschitz continuous on such sets. Therefore, $\ddot{M}$ exists a.e. in $[0, T)$, and

$$\ddot{M}(t) = 2\{|u_t|_2^2 - \|u\|^2 + (u, f(u))\} \quad \text{a.e.} \tag{4.5}$$

We establish now the following lemma:

LEMMA 4.2. *Let $\mathscr{E}$ be a region of $\mathring{H}_1$ satisfying* (2.14). *Then $\mathscr{E}$ is invariant under the flow of* (1.1), *provided*

$$E(0) \equiv \frac{1}{2}[\|u_0\|^2 + |v_0|_2^2] - \int_D F(u_0)\, dx < J(w) = d.$$

(This condition provides that the total energy of the initial data is less than the potential of the saddle point on the boundary of $\mathscr{E}$.)

PROOF. Let $u(t)$ denote the weak solution of (1.1) with initial data $u(0) = u_0, u_t(0) = v_0$ (properly interpreted). By the energy inequality we must have $J(u(t)) < E(0)$. To check that $u(t)$ remains in $\mathscr{E}$, we proceed as follows: If u leaves $\mathscr{E}$ at time $t = t_0$ then we must have

$$K(u(t_0)) = 0 .$$

In fact, let $t_n \to t_0^-$. Then $K(u(t_n)) \leqq 0$. By the lower semi-continuity of the norm $\| \ \|$,

$$K(u(t_0)) \leqq \varliminf_{n\to\infty} K(u(t_n)) \leqq 0 .$$

If $K(u(t_0)) < 0$, then $u(t_0) \in \mathscr{E}$. On the other hand, if $K(u(t_0)) = 0$, then, by the variational definition of d, we must have $J(u(t_0)) \geqq d$. This, however, is impossible, since it violates the energy inequality.

We can now prove one of the main instability theorems.

THEOREM 4.3. *Let f satisfy the conditions* (i)–(iv), *and let W denote the corresponding potential well associated with the potential energy of* (1.1). *If $u_0 \in \mathscr{E}$ and*

$$E(0) < d ,$$

then $|u|_2 \to \infty$ at a finite time.

PROOF. Let

$$M(t) = |u|_2^2 .$$

Then

$$\dot M(t) = 2(u_t, u) .$$

By (4.5) we have

$$\ddot M = 2|u_t|_2^2 + 2\int_D uf(u)\,dx - 2\|u\|^2 . \tag{4.6}$$

Since u lies in $\mathscr{E}$ for all t in the existence interval, than

$$\int_D uf(u)\,dx - \|u\|^2 \geqq 0 , \tag{4.7}$$

and so $\ddot M \geqq 0$. Moreover, by assumption (iv)

$$\ddot M \geqq 2|u_t|_2^2 + 2(p+1)\int_D F(u)\,dx - 2\|u\|^2 .$$

From the energy inequality (4.3) it follows that

$$\int_D F(u)\,dx \geqq \frac{1}{2}[\| u \|^2 + | u_t |_2^2] - E(0).$$

Thus,

$$\ddot{M} \geqq 2 | u_t |_2^2 - 2\| u \|^2 + 2(p+1)\left\{ \frac{1}{2}[| u_t |_2^2 + \| u \|^2] - E(0) \right\},$$

i.e.

$$\ddot{M} \geqq (p+3) | u_t |_2^2 + (p-1)\| u \|^2 - 2(p+1)E(0). \tag{4.8}$$

From the variational inequality

$$\| u \|^2 \geqq \lambda_1 | u |_2^2 = \lambda_1 M, \tag{4.9}$$

where λ_1 is the principal eingenvalue of the Laplacian, we get

$$\ddot{M} \geqq (p+3) | u_t |_2^2 + \lambda_1(p-1)M - 2(p+1)E(0).$$

Since M is a convex function of t, it follows that if these exists a time t_1 such that $\dot{M}(t_1) > 0$, then $M(t)$ is increasing for all $t > t_1$ (within the interval of existence). In that case, the quantity

$$\lambda_1(p-1)M - 2(p+1)E(0)$$

will eventually became positive, and will remain positive thereafter. Thus for large enough t we would have

$$\ddot{M} \geqq (p+3) | u_t |_2^2,$$

and

$$M\ddot{M} - \frac{p+3}{4}\dot{M}^2 \geqq (p+3)\left[[M | u_t |_2^2 - \left(\int_D uu_t dx \right)^2 \right] \geqq 0.$$

Since

$$(M^{-\alpha})^{\cdot\cdot} = -\frac{\alpha}{M^{\alpha+2}}(M\ddot{M} - (\alpha+1)\dot{M}^2),$$

we see that for $\alpha = (p-1)/4$ we have $(M^{-\alpha})^{\cdot\cdot} \leqq 0$. Therefore $M^{-\alpha}$ is concave for sufficiently large t, and there exists a finite time T for which $M^{-\alpha} \to 0$. In other words,

$$\lim_{t \to T^-} M(t) = \infty.$$

These arguments are justified by the fact that $\dot{M}$ is absolutely continuous, as we can see from assumptions (1)–(3).

The proof will be complete once we have shown that $\dot{M} > 0$ for some t. Suppose $\dot{M} \leqq 0$ for all t. Then since $M > 0$ and M is convex, M must tend to a finite, positive limit as $t \to \infty$. (M cannot tend to zero because then for large t, u would lie inside the potential well, which, as we have seen, cannot happen.) Therefore, there is a sequence $\{t_n\}$ such that, as $t_n \to \infty$, $M \to A > 0$, $\dot{M} \to 0$ and $\ddot{M} \to 0$. From (4.6) and (4.7) we see that

$$\lim_{t_n \to \infty} | u_{t_n} |_2^2 = 0 .$$

But from the energy inequality we have

$$\frac{1}{2}\left[\| u \|^2 + | u_t |_2^2 - 2 \int_D F(u)\, dx \right] \leqq E(0) .$$

Thus, as $t_n \to \infty$,

$$\left[\frac{1}{2} \| u \|^2 - \int_D F(u)\, dx \right] \to B \leqq E(0) ,$$

where B is a constant. On the other hand, from (4.6) we may conclude that, as $t_n \to \infty$,

$$\int_D uf(u)\, dx - \| u \|^2 \to 0 .$$

We now apply Lemma 2.7 to the family $\{u(t);\ t \geqq t_*\}$; we see that

$$\liminf_{t_n \to \infty} J(u(t_n)) \geqq d > E(0);$$

but this contradicts the energy assumption. It follows therefore that $M(t) \to \infty$ in a finite time, and the proof of Theorem 4.3 is complete.

Suppose now that the solution starts inside the potential well, i.e. $K(u_0) > 0$; suppose further that $E(0) < d$. By arguments similar to those used in the proof of Lemma 4.2, we can show that, provided additional assumptions are made on f which will insure that $K(t)$ is continuous (see, e.g., Lemma 2.5), then W is invariant under the flow of (1.1), i.e. $K(u) > 0$ for all t. However, by assumption (iv),

$$J(u) \geqq \frac{1}{2} \| u \|^2 - \int_D \frac{1}{p+1} uf(u)\, dx = \frac{1}{p+1} K(u) + \frac{p-1}{2(p+1)} \| u \|^2 \geqq 0 .$$

Thus, since we know [10] that the solution exists for all time, it follows that at any time t

$$E(t)=\frac{1}{2}\int_D u_t^2\,dx+J(u)\leqq E(0),$$

which in turn implies that

$$\frac{1}{2}\int_D u_t^2\,dx\leqq E(0),\quad \text{for all } t.$$

Also, since

$$E(0)\geqq E(t)\geqq\frac{1}{2}\int_D u_t^2\,dx+\frac{1}{p+1}K(u)+\frac{(p-1)}{2(p+1)}\|u\|^2,$$

we observe in fact that

$$\|u\|^2\leqq\hat{M},\quad \text{for all } t,$$

and since $K(u)$ and $J(u)$ are both non-negative, there exist constants M_i such that

$$\int_D uf(u)\,dx\leqq M_1,\ \int_D F(u)\,dx\leqq M_2,\quad \text{for all } t.$$

It follows then by (3.6) that

$$|u|_2^2\leqq\frac{1}{\lambda_1}\|u\|^2\leqq\hat{M}/\lambda_1,$$

and hence that u is bounded in L_2 for all t. We have thus established the boundedness of u_t in L_2 and the boundedness of u in $\mathring{H}_1$ for all t.

If $E(0)>d$, a solution starting in W may or may not leave the potential well.

Other sufficient criteria for finite time blow up have been given by various authors ([1]–[7], [9], [10], [13]).

As a specific example consider the case $n\leqq 3$, $f(u)=u^3$. In this case

$$F(u)=\frac{u^4}{4},$$

so that $p=3$, $\gamma=4$ in (iv). We now have

$$\ddot{M}=2|u_t|_2^2+2|u^2|_2^2-2\|u\|^2.$$

From the arguments at the end of Section 3 it follows that

$$\|u\|^4\geqq 4d\,|u^2|_2^2,$$

which may be rewritten as

$$\| u \|^4 \geqq 4d \{ -4E(0) - 2 | u_t |_2^2 + 2 \| u \|^2 \},$$

or

$$[\| u \|^2 - 4d]^2 \geqq 16d \{d - E(0)\} + 8d \, | u_t |_2^2 .$$

This inequality clearly shows that if

$$E(0) < d$$

$$\| u_0 \|^2 > 4d,$$

then throughout the existence interval

$$\| u \|^2 > 4d.$$

The two inequalities

$$E(t) < E(0) < d$$

$$\| u \|^2 > 4d$$

then imply that

$$\| u \|^2 + 2 | u_t |_2^2 - | u^2 |_2^2 < 0,$$

which shows that

$$K(u) = \| u \|^2 - | u^2 |_2^2 < 0.$$

The concavity arguments then clearly imply blow up in a finite time. This example has been discussed by Tsutsumi [13].

5. Parabolic problems

In this section we consider weak solutions of

$$\begin{aligned} u_t &= \Delta u + f(u) \\ u(x,0) &= u_0(x) \qquad (5.1) \\ u &= 0 \text{ on } \partial D, \end{aligned}$$

where D is as before. We make the same assumptions on f as in the previous sections. We say that u is a weak solution of (5.1) on $[0, T)$ if it satisfies the conditions (1) and (2) used in defining a weak solution of (1.1), as well as the conditions:

(3′) For any $\varphi : [0, T) \to \mathring{H}_1$ with the properties of u given by (1) and (2),

$$(\varphi, u_t) + ((\varphi, u)) - (\varphi, f(u)) = 0. \qquad (5.2)$$

(4′) $J(u)$ satisfies the inequality

$$\int_0^t |u_t|^2 ds + J(u) \leqq J(u_0). \tag{5.3}$$

If we now define

$$M_1(t) = \int_0^t |u|_2^2 ds, \tag{5.4}$$

where u is a weak solution of (5.1), then the analogue of Lemma 4.1 follows directly, i.e.

LEMMA 5.1. *For $M_1(t)$ defined by* (5.4), *it follows that $\ddot{M}_1$ exists a.e. in* $[0,t)$ *and $\dot{M}_1(t)$ is Lipschitz continuous there.*

The proof is the same as the proof of Lemma 4.1.

The following analogue of Lemma 4.2 is likewise easily established:

LEMMA 5.2. *Let $\mathscr{E}$ be a region in $\mathring{H}_1$ satisfying* (2.14). *Then $\mathscr{E}$ is invariant under the flow of* (5.1), *provided*

$$J(u_0) < J(w) = d. \tag{5.5}$$

We now establish the main theorem of this section.

THEOREM 5.3. *If u is a weak solution of* (5.1), *$u_0 \in \mathscr{E}$ and $J(u_0) < d$, then $|u|_2 \to \infty$ in a finite time.*

To prove the theorem we assume the contrary and show that this leads to a contradiction.

From (5.4) it follows that

$$\dot{M}_1(t) = |u|_2^2 = |u_0|_2^2 + 2\int_0^t (u, u_t)\, ds.$$

Using (5.2) we rewrite this expression as

$$\dot{M}_1(t) = |u_0|_2^2 - 2\int_0^t \{\|u\|^2 - (u, f(u))\}\, ds,$$

and compute

$$\ddot{M}_1(t) = 2\{(u, f(u)) - \|u\|^2\} = -2K(u).$$

(The steps may be justified as in the previous section.) By Lemma 5.2, since $u_0 \in \mathscr{E}$, $K(u)$ must remain negative for all t in the existence interval. Thus

$\dot{M}_1(t) > 0$ and $\ddot{M}_1(t) \geqq 0$ for all $t \leqq T$. It is clear from the proof of Lemma 5.2 that $K(u) < 0$ for all finite time (in the interval of existence). To prove that $K(u_n) \equiv K(u(\cdot, t_n))$ cannot tend to zero through a sequence of t_n's tending to infinity, we invoke Lemma 2.7 which would then imply

$$\varliminf_{n\to\infty} J(u_n) \geqq d > J(u_0).$$

However, this contradicts (5.3), and thus we conclude that $K(u) < 0$ for all t in the interval of existence. From this it follows that as t increases $\dot{M}(t)$ will eventually become larger then any prescribed constant.

Now by assumption (iv)

$$\ddot{M}_1(t) \geqq 2(p+1) \int_D F(u)\,dx - 2\| u \|^2,$$

or, using the second of inequalities (5.3),

$$\begin{aligned}\ddot{M}_1(t) &\geqq 2(p+1) \int_0^t | u_\eta |_2^2\, d\eta + (p-1) \| u \|^2 - 2(p+1) J(u_0) \\ &\geqq 2(p+1) \int_0^t | u_\eta |_2^2\, d\eta + (p-1)\lambda_1 \dot{M}_1(t) - 2(p+1) J(u_0).\end{aligned}$$

We now form

$$\begin{aligned}&M_1\ddot{M}_1 - \frac{(p+1)}{2}\dot{M}_1^2 \\ &\quad = 2(p+1)\left\{\int_0^t | u |_2^2\, d\eta \int_0^t | u_\eta |_2^2\, d\eta - \left(\int_0^t \int_D u u_\eta\, dx d\eta\right)^2\right\} \\ &\qquad + (p-1)\lambda_1 M_1 \dot{M}_1 - (p+1) | u_0 |_2^2 \dot{M}_1 \\ &\qquad - 2(p+1) J(u_0) M_1 + \frac{p+1}{2} | u_0 |_2^4.\end{aligned}$$

The first term on the right is non-negative by Schwartz's inequality and the second term will eventually dominate the remainder. Thus for sufficiently large t, i.e. $t > \hat{t}$, the right hand side will be positive. This leads as before to

$$M_1^{-(p-1)/2}(t) \leqq M_1^{-(p-1)/2}(\hat{t}) \left\{1 - \left(\frac{p-1}{2}\right) \frac{\dot{M}_1(\hat{t})}{M_1(\hat{t})}(t - \hat{t})\right\},$$

which establishes the blow up in finite time.

6. Abstract problems

It is possible to generalize the results of the previous section and put them into an abstract setting. In this section we indicate how this is done.

Let D be a dense linear subspace of a Hilbert space H; denote by $(\,,)$ the scalar product on H and by $\|\ \|$ the corresponding norm. For simplicity we shall deal only with real Hilbert spaces. The extension to complex spaces is obvious. Let P and Q be linear operators mapping D into H. We assume that P and Q are positive definite, symmetric, and not necessarily bounded. The subspace D is also to be a Hilbert space and the injection $D \to H$ is assumed to be continuous.

We are interested in abstract equations of the form

$$Pu_{tt} = -Qu + f(u(t)) \quad \text{in} \quad (0,T) \tag{6.1}$$

$$u(0) = u_0, \quad u_t(0) = v_0,$$

where $f: D \to H$ is a gradient operator, i.e. f is the Fréchet derivative in the D norm of a scalar valued function $G: D \to R$, which is generally referred to as the potential of f. We assume throughout that P and Q do not depend on the parameter t.

Our conditions (i)–(iv) are now replaced by:

(i') $G(x) = \int_0^1 (f(\rho x), x)\, d\rho$, for all $x \in D$;

(ii') $f(x)$ has a strongly continuous symmetric Fréchet derivative f_x, for all $x \in D$, and $f(0) = 0$, $f_x(0) = 0$;

(iii') $(f_x \cdot x - f(x), x) \geqq 0$, for all $x \in D$;

(iv') $(x, f(x)) \geqq (p+1)\, G(x), p > 1$,

and $(x, f(x))$ is completely continuous with respect to (x, Qx). Furthermore, for some $\alpha > 1$ and for all $x, y \in D$,

$$(f(x), y)^2 \leqq k^2 (x, Qx)^\alpha (y, Qy), \quad k = \text{constant}. \tag{6.2}$$

Again it is possible to put the problem (6.1) into a weak setting; i.e., we say that u is a weak solution of (6.1) on $[0,T)$ if the following are satisfied:

(1) $u(t)$ is a weakly continuous mapping from $[0,T)$ to D.

(2) There is a weakly continuous mapping from $[0,T)$ to D, denoted by u_t, such that

$$(u, P\varphi)\,\big|_{t_1}^{t_2} = \int_{t_1}^{t_2} (u_t, P\varphi)\, ds$$

for any t_1, t_2 such that $0 \leqq t_1 \leqq t_2 < T$ and any $\varphi \in D$.

(3) (u, Pu), (u, Qu) and $\| u \|$ are uniformly bounded on compact subsets of $[0, T)$.

(4) For every $\varphi : [0, T) \to D$ with the same properties as u above,

$$(Pu_t, \varphi) \big|_{t_1}^{t_2} = \int_{t_1}^{t_2} \{(Pu_t, \varphi_t) - (\varphi, Qu) + (\varphi, f(u))\} \, dt.$$

(5) The energy $E(t)$ defined by

$$E(t) \equiv \frac{1}{2}\{(u_t, Pu_t) + (u, Qu)\} - G(u)$$

satisfies

$$E(t_2) \leqq E(t_1), 0 \leqq t_1 < t_2 < T.$$

With the conditions (i')–(iv') it follows easily as before that any non-trivial critical points are *a priori* unstable equilibria for (6.1).

In the proof of the analogue of Lemma 2.2 we use the fact that under the new conditions (i'), (ii') and (iv') we now have

$$G(\lambda u) \geqq \lambda^{p+1}(G(u), \quad \lambda \geqq 1$$

which is sufficient to guarantee that $j(\lambda) \to -\infty$ as $\lambda \to \infty$. The proof of the (b) and (c) parts of Lemma (2.2) follow as before, with integrals replaced by appropriate scalar products and derivatives replaced by Fréchet derivatives. In a similar manner the analogue of Theorem 2.3 is proved.

Lemma 2.5 does not follow directly in the abstract case. We now have

$$G(u) - G(v) = \int_0^1 (f(v + \lambda[u - v]), u - v) \, d\lambda,$$

or

$$|G(u) - G(v)| \leqq \int_0^1 |f(v + \lambda[u - v]), u - v)| \, d\lambda.$$

Making use of assumption (6.2), we obtain

$$|G(u) - G(v)| \leqq \int_0^1 \langle\langle v + \lambda\,[u - v]\rangle\rangle^\alpha d\lambda \;\; \langle\langle u - v\rangle\rangle$$

$$\leqq C(\langle\langle v\rangle\rangle^\alpha, \langle\langle u\rangle\rangle^\alpha) \;\; \langle\langle u - v\rangle\rangle,$$

where we have used the symbol

$$\langle\langle u - v\rangle\rangle = (u, Qu)^{1/2}.$$

It follows then that $J(u)$ is continuous in the Hilbert space formed by completing D in the norm $\langle\langle \;\; \rangle\rangle$. This statement clearly applies also to $K(u)$.

To prove that $d > 0$ we observe, as in the proof of Theorem 2.6, that

$$\begin{aligned} J(u) &= \frac{1}{2}(u, Qu) - G(u) \\ &\geqq \frac{1}{2}(u, Qu) - \frac{1}{p+1}(u, f(u)) \\ &= \frac{p-1}{2(p+1)}(u, Qu). \end{aligned}$$

But $(u, f(u))^2 \leqq k^2(u, Qu)^{\alpha+1} = k^2(u, f(u))^{\alpha+1}$
or

$$(u, f(u)) \geqq k^{-2(\alpha-1)}.$$

Thus

$$\begin{aligned} d &\geqq \frac{p-1}{2(p+1)}(u, Qu) \\ &\geqq \frac{(p-1)k^{-2(\alpha-1)}}{2(p+1)}. \end{aligned}$$

Because of the complete continuity assumption (iv), the proof of the existence of an extremal follows along the lines of the proof of Theorem 2.6, where $|u|_{p+1}$ is to be replaced by $(u, f(u))^{1/(\alpha+1)}$ and $\| u \|$ by $(u, Qu)^{1/2}$. The proof that in the abstract case $\mathscr{E}$ is invariant under the flow is an obvious extension of the proof of Lemma 4.2, and the analogue of Lemma 2.7 is established along the lines of the abstract version of Theorem 2.6.

The proof of blow up in finite time for $u_0 \in \mathscr{E}$ and $E(0) < d$ is a straightforward extension of the proof of Theorem 4.3. We require, however, the additional asssumption

$$(u, Qu) \geqq \mu(Pu, u).$$

We then set

$$M = (Pu, u),$$

from which are obtained

$$\dot{M} = 2(u, Pu_t)$$

and

$$\ddot{M} = 2[(u_t, Pu_t) - (u, Qu) + (u, f(u))].$$

Since u lies in $\mathscr{E}$ for all t, it follows that

$$(u, f(u)) - (u, Qu) \geqq 0,$$

which implies $\ddot{M} \geqq 0$. As before we are led to

$$\ddot{M} \geqq (p+3)(Pu_t, u_t) + \mu(p-1)M - 2(p+1)E(0),$$

and to

$$M\ddot{M} - \frac{p+3}{4}\dot{M}^2 \geqq M[\mu(p-1)M - 2(p+1)E(0)].$$

The implications of blow up in finite time follow as before.

Concluding remarks

It would clearly also be possible to put the parabolic problem in an abstract setting. It would likewise be possible with the appropriate assumptions on $f(u)$ and $g(u)$ to treat instead of (1.1) the more general problem

$$u_{tt} = \Delta u + f(u)$$

$$u(x,0) = u_0(x),\ u_t(x,0) = v_0(x)$$

$$\frac{\partial u}{\partial \nu} = g(u) \quad \text{on} \quad \partial D,$$

with $t > 0$, and $x \in D$. Here $\partial/\partial\nu$ denotes the normal derivative of u on ∂D. Such a problem could likewise be placed in a more general abstract setting. The case $f \equiv 0$ and $g(u) \not\equiv 0$ might present some difficulties, since any solution w of the corresponding equilibrium equations would have to satisfy $\oint_{\partial D} g(w)\,dS = 0$. In particular, if $g(w) = w^{2N+1}$ for some positive integer N, then w would have to change sign on ∂D.†

Appendix

Serrin's Sweeping Statements

THEOREM. *Suppose $v(x,\lambda) = v_\lambda$ is an increasing family of upper solutions of* (2.1) *on $a \leqq \lambda \leqq b$; that is,*

$$\Delta v(x,\lambda) + f(v(x,\lambda)) \leqq 0 \quad in \quad D.$$

If u is any solution of (2.1) *such that $u(x) \leqq v_b$ and $u \leqq v_a$ on ∂D, then either $u \equiv v_a$ or $u < v_a$ in D. Similarly, if v_λ is an increasing family of lower solutions*

† The authors wish to express their appreciation to the referees for their constructive comments and suggested improvements of the original manuscript.

$$\Delta v_\lambda + f(v_\lambda) \geqq 0 \quad in \quad D,$$

and u is a solution of (2.1) *such that* $u \geqq v_a$ *and* $u \geqq v_b$ *on* ∂D, *then either* $u \equiv v_b$ *or* $u > v_b$ *in* D.

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