THE ASYMPTOTIC BEHAVIOUR OF THE SOLUTION OF THE FILTRATION EQUATION

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ABSTRACT

It is proved that the self-similar solution of the nonlinear equation of filtration gives the asymptotic representation of the solution of the Cauchy problem for the same equation.

Consider the Cauchy problem for the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

when the initial data are of compact support. It is a known result that the solution of this problem behaves asymptotically as a fundamental solution of the same equation as $t \to \infty$. (This can be proved easily by means of the Poisson integral formula.) We propose to prove here the same result for the solution of the Cauchy problem for the equation of unsteady filtration

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u^{\lambda+1}}{\partial x^2} \qquad (\lambda > 0)^{\dagger} \tag{1}$$

Equation (1) is invariant under the following group of transformations:

$$u' = cu, \ x' = l^{-1}x, \ t' = l^{-2}c^{-\lambda}t. \tag{2}$$

This important observation enables us to find a self-similar solution $w_E(x, t)$ of (1) that satisfies the initial condition

$$w_E(x, 0) = E\delta(x)^{\dagger\dagger}$$

† The asymptotic behaviour of the solution of this problem was considered in the paper of Barenblatt and Zeldovich [4]. In this paper, several terms of the asymptotic representation of the solution for large time were stated without proof.

†† δ (x) denotes the Dirac measure.

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with an arbitrary constant E. Such a solution is of the form

$$w_E(x,t) = E^{2/(\lambda+2)}t^{-1/(\lambda+2)}\Phi(xE^{-\lambda(\lambda+2)}t^{-1/(\lambda+2)}) \tag{3}$$

where

$$\Phi(\xi) = \begin{cases} a(\lambda)(\xi_0^2 - \xi^2)^{1/\lambda} & \text{if } \xi \leqq \xi_0 \\ 0 & \text{if } \xi > \xi_0 \end{cases} \tag{4}$$

and $\xi_0 = \xi_0(\lambda)$, cf. [10].

The proof of the theorem given below is also based on the existence of a group of transformations of type (2).

We consider the Cauchy problem for Eq. (1) in the half-plane

$$S = \{(x,t) \colon x \in R^1, \ 0 \leqq t < \infty\}$$

with the initial condition

$$u\big|_{t=0} = u_0(x), \tag{5}$$

where $u_0(x)$ is a continuous non-negative function with compact support. Assume, for simplicity, that the function $[u_0(x)]^{\lambda+1}$ satisfies the Lipshitz condition.

DEFINITION. A function $u(x,t)$ defined and bounded in S is called a generalized solution of the Cauchy problem (1), (5) in S if it satisfies the following conditions:

(i) $u(x,t)$ is continuous and non-negative.

(ii) there exists a generalized bounded derivative $(\partial u^{\lambda+1})/(\partial x)$.

(iii) for any continuously differentiable function $f(x,t)$ with compact support

$$\int\int_S \left(u\frac{\partial f}{\partial t} - \frac{\partial u^{\lambda+1}}{\partial x}\frac{\partial f}{\partial x}\right)dxdt + \int_{-\infty}^{\infty} u_0(x)f(x,0)dx = 0. \tag{6}$$

The existence and uniqueness of the generalized solution of the problem (1), (5) was proved in the paper of Oleinik, Kalasnikov, and Czou, Yui-Lin [7]. See also Aronson [1, 2, 3].

As was shown in [7], the generalized solution $u(x,t)$ of the problem (1), (5) is a generalized solution of the Cauchy problem in any half-plane $(x \in R^1, \ 0 \leqq t_0 \leqq t < \infty)$ with the initial condition

$$u\big|_{t=t_0} = u(x, t_0).$$

It was also proved in [7] that $[u(x,t)]^{\lambda+1}$ is a Lipshitz function with respect to x.

It is easy to verify that the self-similar solution (3) is a generalized solution of a corresponding Cauchy problem in any half-plane $(x \in R^1, \ 0 < t_0 \leqq t < \infty)$.

The aim of this paper is to prove the following theorem.

THEOREM. *Let $u(x,t)$ be a generalized solution of the Cauchy problem* (1) (5) *and*

$$\int_{-\infty}^{\infty} u_0(x)\,dx = E_0. \tag{7}$$

Then

$$t^{1/(\lambda+2)}\left|u(x,t) - w_{E_0}(x,t)\right| \to 0 \quad as \quad t \to \infty$$

uniformly with respect to $x \in R^1$.

We start with the following lemma.

LEMMA 1. *Let $u(x,t)$ be a generalized solution of the Cauchy problem* (1), (5). *Then there exists a constant E_1 such that*

$$u(x,t) \leqq w_{E_1}(x,t+1) \ \ in \ \ S. \tag{8}$$

PROOF. We first prove (8) for $t = 0$, i.e. that

$$u_0(x) \leqq w_{E_1}(x,1) = E_1^{2/(\lambda+2)}\phi(xE_1^{-\lambda/(\lambda+2)}) \tag{9}$$

or sufficiently large E_1.

Let b be a constant such that $u_0(x) = 0$ for $|x| \geqq b$. It follows from (4) that $\phi(\xi)$ is positive for $\xi < \xi_0$ and that it increases monotonically when $|\xi|$ decreases. Therefore $w_E(x,1) > 0$ for $|x| < b$ provided that $bE^{-\lambda/(\lambda+2)} < \xi_0$ and by choosing E sufficiently large, the function $w_E(x,1)$ will become larger than $u_0(x)$. Thus (9) is proved.

Now, using the monotonic dependence of the generalized solution of the Cauchy problem on the initial data proved in [7], we obtain (8).

Set $u_k(x,t) = ku(kx, k^{\lambda+2}t)$, $(k > 0)$. Now $u_k(x,t)$ is a generalized solution of Eq. (1) for the initial data $u_k(x,0) = ku_0(kx)$.

Since $u_k(x,0) \to E_0\delta(x)$ as $k \to \infty$, it is reasonable to expect that $u_k(x,t) \to w_{E_0}(x,t)$ as $k \to \infty$. To show that this is indeed the case, we consider the functions $u_k(x,t)$ in the strip $(x \in R^1, 0 \leqq t \leqq T)$ where T is an arbitrary fixed constant.

From (8), we get $ku(kx, k^{\lambda+2}t) \leqq kw_{E_1}(kx, k^{\lambda+2}t + 1)$. Hence

$$u_k(x,t) \leqq w_{E_1}(x, t + k^{-(\lambda+2)}). \tag{10}$$

Let $\tau \in (0, T/2)$. It follows from (10) that there exist constants $C_1(\tau)$ and $C_2(\tau)$ independent of k such that

$$\max_{x \in R^1} u_k(x,\tau) \leqq C_1(\tau) \tag{11}$$

and

$$\int_{-\infty}^{\infty} [u_k(x,\tau)]^{\lambda+2} dx \leqq C_2(\tau). \tag{12}$$

Let $S_\tau = \{(x,t): x \in R^1, \tau \leqq t \leqq T\}$ We shall need the following lemma.

LEMMA 2. *For any* $\tau \in (0, T/2)$ *and any* $k > 0$, *there exists a sequence of smooth functions* $u_{\tau,k,n}(x,t)$ $(n = 1, 2, \cdots)$ *having the following properties*:

a) $u_{\tau,k,n}(x,t)$ *is defined in the region* $S_{\tau,n} = \{(x,t): |x| \leqq n, \tau \leqq t \leqq T\}$ *and is a classical solution of equation* (1) *in the same region*;

b) $u_{\tau,k,n}(\pm n, t) = C_1(\tau) + 1$ *for* $\tau \leqq t \leqq T$,

$$0 \leqq u_{\tau,k,n}(x,t) \leqq C_1(\tau) + 1, \tag{13}$$

$$\int_{-n}^{n} [u_{\tau,k,n}(x,\tau)]^{\lambda+2} dx \leqq C_3(\tau) \tag{14}$$

with a constant $C_3(\tau)$ *depending only on* τ;

c) *at every point* $(x,t) \in S_\tau$, $\lim_{n\to\infty} u_{\tau,k,n}(x,t) = u_k(x,t)$.

The proof of the lemma is similar to the proof of the theorem of the existence of the generalized solution in [7]. The plan of the proof follows.

Let $\phi_{\tau,k}(x) = [u_k(x,\tau)]^{\lambda+1}$. Then

$$|\phi_{\tau,k}(x+\Delta x) - \phi_{\tau,k}(x)| \leqq M(k,\tau)\Delta x \tag{15}$$

where $M(k,\tau)$ is a constant depending on k and τ.

From (11), (12) and (15), we conclude that there exists a sequence of infinitely differentiable functions $\phi_{\tau,k,n}(x)$ $(n = 1, 2, \cdots)$ with the following properties: $\phi_{\tau,k,n}(x) \to \phi_{\tau,k}(x)$ as $n \to \infty$, and the convergence is uniform in every bounded interval of x; $0 < \phi_{\tau,k,n+1}(x) \leqq \phi_{\tau,k,n}(x) \leqq [C_1(\tau)+1]^{\lambda+1}$, $\phi_{\tau,k,n}(x) = [C_1(\tau)+1]^{\lambda+1}$ for $x \geqq n-1$,

$$\left|\frac{d\phi_{\tau,k,n}}{dx}\right| \leqq M(k,\tau) + 1, \text{ and } \int_{-n}^{n} [\phi_{\tau,k,n}(x)]^{(\lambda+2)/(\lambda+1)} dx \leqq C_3(\tau)$$

with a constant $C_3(\tau)$ depending on $C_1(\tau)$ and $C_2(\tau)$.

Substituting $u = v^{1/(\lambda+1)}$ into (1), we get

$$\frac{\partial v}{\partial t} = (\lambda+1) v^{\lambda/(\lambda+1)} \frac{\partial^2 v}{\partial x^2}. \tag{16}$$

Consider the first boundary problem for Eq. (16) in the rectangle $S_{\tau,n}$

$$v(x,t)\big|_{t=\tau} = \phi_{\tau,k,n}(x), \quad v(\pm n, t) = [C_1(\tau)+1]^{\lambda+1}. \tag{17}$$

This problem has a solution $v_{\tau,k,n}(x,t)$ for every n and $0<\inf_x\phi_{\tau,k,n}(x)\leqq v_{\tau,k,n}(x,t)\leqq[C_1(\tau)+1]^{\lambda+1}$. Compare the functions $v_{\tau,k,n}(x,t)$ and $v_{\tau,k,n+1}(x,t)$ in the region $S_{\tau,n}$. Let $\Gamma_{\tau,n}$ be that part of the boundary of $S_{\tau,n}$ consisting of sides $t=\tau$, $x=n$, $x=-n$. Using (17), we find

$$v_{\tau,k,n}\Big|_{\Gamma_{\tau,n}}\geqq v_{\tau,k,n+1}\Big|_{\Gamma_{\tau,n}}.$$

It can be verified that the difference $(v_{\tau,k,n}-v_{\tau,k,n+1})$ is a solution of an equation for which the maximum principle holds. Hence $v_{\tau,k,n}(x,t)\geqq v_{\tau,k,n+1}(x,t)$ in $S_{\tau,n}$, and we conclude that the sequence $\{v_{\tau,k,n}(x,t)\}$ $(n=1,2,\cdots)$ is a monotonically decreasing sequence of positive functions. The same is true for the sequence $u_{\tau,k,n}=(v_{\tau,k,n})^{1/(\lambda+1)}$. We deduce that at every point $(x,t)\in S_\tau$,

$$\lim_{n\to\infty}u_{\tau,k,n}(x,t)=\tilde{u}_{\tau,k}(x,t),$$

$$\lim_{n\to\infty}v_{\tau,k,n}(x,t)=\tilde{v}_{\tau,k}(x,t)=[\tilde{u}_{\tau,k}(x,t)]^{\lambda+1}.$$

Now it is necessary to show that $\tilde{u}_{\tau,k}(x,t)$ is a generalized solution of (1). For that purpose we must check that there exists a generalized derivative $\partial\tilde{v}_{\tau,k}/\partial x$. It is easy to see that the function $\partial v_{\tau,k,n}/\partial x$ is a solution of an equation for which the maximum principle holds. Hence, in $S_{\tau,n}$,

$$\left|\frac{\partial v_{\tau,k,n}}{\partial x}\right|\leqq\max_{\Gamma_{\tau,n}}\left|\frac{\partial v_{\tau,k,n}}{\partial x}\right|.$$

We have for $t=0$

$$\left|\frac{\partial v_{\tau,k,n}}{\partial x}\right|\leqq M(k,\tau)+1.$$

It follows from (17) that

$$\frac{\partial v_{\tau,k,n}}{\partial x}\Big|_{x=n}\geqq 0.$$

Next, consider the function $Z_{\tau,k,n}(x,t)=v_{\tau,k,n}(x,t)-(x-n+1)[C_1(\tau)+1]^{\lambda+1}$. This function satisfies the equation

$$\frac{\partial Z_{\tau,k,n}}{\partial t}=(\lambda+1)(v_{\tau,k,n})^{\lambda/(\lambda+1)}\frac{\partial^2 Z_{\tau,k,n}}{\partial x^2}.$$

Therefore, the minimum of $Z_{\tau,k,n}$ in the region $\{(x,t)\colon n-1\leqq x\leqq n,\ \tau\leqq t\leqq T\}$ can be only at $t=\tau$, $x=n-1$ or $x=n$.

We have $Z_{\tau,k,n}(x,\tau)\geqq 0$, $Z_{\tau,k,n}(n-1,t)>0$, and $Z_{\tau,k,n}(n,t)=0$. Thus

$$\left.\frac{\partial Z_{\tau,k,n}}{\partial x}\right|_{x=n} \leqq 0$$

and

$$(18) \qquad 0 \leqq \left.\frac{\partial v_{\tau,k,n}}{\partial x}\right|_{x=n} \leqq [C_1(\tau)+1]^{\lambda+1}.$$

The same reasoning applies to $x = -n$. Hence

$$\max_{S_{\tau,n}} \left|\frac{\partial v_{\tau,k,n}}{\partial x}\right| \leqq \max[M-1),(C_1+1)^{\lambda+1}].$$

Therefore the function $\tilde{v}_{\tau,k}(x,t)$ is a Lipshitz function with respect to x and there exists a generalized derivative $(\partial\tilde{v}_{\tau,k})/(\partial x)$. Using this, it is possible to prove that the function $u_{\tau,k}(x,t)$ is continuous and satisfies the corresponding integral identity. The equality $\tilde{u}_{\tau,k}(x,t) = u_k(x,t)$ in S_τ follows from the uniqueness theorem proved in [7].

Now we prove the following lemma.

LEMMA 3. *Let* $\tau \in (0, T/2)$ *and* $R_\tau = \{(x,t): |x| \leqq 1/\tau, 2\tau \leqq t \leqq T\}$. *There exist constants* $C_4(\tau)$ *and* $C_5(\tau)$ *independent of* k *such that for every* $k > 0$

$$(19) \qquad \iint_{R_\tau} \left(\frac{\partial u_k^{\lambda+1}}{\partial x}\right)^2 dxdt \leqq C_4(\tau),$$

$$(20) \qquad \iint_{R_\tau} \left(\frac{\partial u_k^{\lambda+1}}{\partial t}\right)^2 dxdt \leqq C_5(\tau).$$

PROOF. Let $u_{\tau,k,n}(x,t)$ and $v_{\tau,k,n}(x,t)$ be the functions defined in Lemma 2. We have

$$\iint_{S_{\tau,n}} \frac{\partial u_{\tau,k,n}}{\partial t} v_{\tau,k,n} dxdt = \iint_{S_{\tau,n}} \frac{\partial^2 v_{\tau,k,n}}{\partial x^2} v_{\tau,k,n} dxdt.$$

Hence

$$(21) \quad \frac{1}{\lambda+2}\iint_{S_{\tau,n}} \frac{\partial}{\partial t}(u_{\tau,k,n})^{\lambda+2} dxdt = \int_\tau^T \left.\frac{\partial v_{\tau,k,n}}{\partial x} v_{\tau,k,n} dt\right|_{-n}^{n} - \iint_{S_{\tau,n}} \left(\frac{\partial v_{\tau,k,n}}{\partial x}\right)^2 dxdt.$$

From (13), (14), (18) and (21), we deduce the existence of a constant $C_4(\tau)$ such that

$$(22) \qquad \iint_{S_{\tau,n}} \left(\frac{\partial v_{\tau,k,n}}{\partial x}\right)^2 dxdt \leqq C_4(\tau).$$

It follows from (22) that, for every k and n, we can find a point $\tau^* \in [\tau, 2\tau]$ (possibly dependent on k and n) such that

(23) $$\int_{-n}^{n}\left(\frac{\partial v_{\tau,k,n}}{\partial x}\right)^2 dx\Big|_{t=\tau^*} \leqq \frac{C_4(\tau)}{\tau}.$$

Let $S_{\tau^*,n} = \{(x,t): |x| \leqq n, \tau^* \leqq t \leqq T\}$. We have

(24) $$\iint_{S_{\tau^*,n}} \frac{\partial u_{\tau,k,n}}{\partial t}\frac{\partial v_{\tau,k,n}}{\partial t} dxdt = \iint_{S_{\tau^*,n}} \frac{\partial^2 v_{\tau,k,n}}{\partial x^2}\frac{\partial v_{\tau,k,n}}{\partial t} dxdt.$$

Using (17), we can write

(25) $$\iint_{S_{\tau^*,n}} \frac{\partial^2 v_{\tau,k,n}}{\partial x^2}\frac{\partial v_{\tau,k,n}}{\partial t} dxdt = \frac{1}{2}\int_{-n}^{n}\left(\frac{\partial v_{\tau,k,n}}{\partial x}\right)^2 dx\Big|_{t=\tau^*} - \frac{1}{2}\int_{-n}^{n}\left(\frac{\partial v_{\tau,k,n}}{\partial x}\right)^2 \Big|_{t=T}.$$

From (13), (23), (24) and (25) we see that

(26) $$\iint_{S_{\tau^*,n}}\left(\frac{\partial v_{\tau,k,n}}{\partial t}\right)^2 dxdt \leqq (\lambda+1)[C_1(\tau)+1]^{\lambda}C_4(\tau)/2\tau = C_5(\tau).$$

Using Lemma 2 and (22) and (26), we get (19) and (20). Thus, Lemma 3 is proved.

Now applying the Sobolev imbedding theorems (cf. [9]), we conclude that $\{u_k^{\lambda+1}\}$ is a compact subset of $L_2(R_\tau)$. Consequently, the set $\{u_k\}$ also is a compact subset of $L_2(R_\tau)$.

Until now, we have treated τ as an arbitrary constant. Let $\tau \to 0$. By using the diagonal process, we extract the subsequence $\{u_{k_i}\}$ $(k_i \to \infty)$ that converges in L_2 in every bounded region inside the strip $(x \in R^1, 0 < t < T)$. The limit function $u^*(x,t)$ is determined in the whole strip.

We shall prove now that

(27) $$u^*(x,t) = w_{E_0}(x,t)$$

where E_0 is a constant defined by (7).

Set $Z_\alpha(x,t) = w_{E_0}(x, t+\alpha)$. Let B be a sufficiently large constant so that for $k > 1$, $\alpha < 1$,

(28) $$u_k(x,t) = Z_\alpha(x,t) = 0 \text{ if } |x| \geqq B, 0 \leqq t \leqq T.$$

(Such a constant exists because of (4) and (8)). Let D_B be the region $\{|x| \leqq B+1, 0 \leqq t \leqq T\}$ and let $F(x,t)$ be an arbitrary infinitely differentiable function equal to zero near the boundary of region D_B. To prove (27), it suffices to show that for any $\varepsilon > 0$, there exist values k_0 and a_0 such that

(29) $$\left|\iint_{D_B}(u_k - Z_\alpha)F\,dxdt\right| < \varepsilon \text{ for } k \geqq k_0,\ \alpha \leqq \alpha_0.$$

Let $f(x,t)$ be a continuous function with compact support which has continuous derivatives $\partial f/\partial t$, $\partial f/\partial x$ and $(\partial^2 f)/(\partial x^2)$. It follows from (6) that for every k and α,

$$\iint_S (u_k - Z_\alpha)\left[\frac{\partial f}{\partial t} + C_{k,\alpha}(x,t)\frac{\partial^2 f}{\partial x^2}\right] dxdt = \int_{-\infty}^{\infty} [Z_\alpha(x,0) - u_k(x,0)] f(x,0)\, dx \tag{30}$$

where

$$C_{k,\alpha}(x,t) = (\lambda + 1)\int_0^1 [\theta u_k(x,t) + (1-\theta) Z_\alpha(x,t)]^\lambda d\theta.$$

Let $C_{k,\alpha,p}(x,t)$ $(p = 1,2,\cdots)$ be a sequence of infinitely differentiable functions with the following properties: $C_{k,\alpha,p}(x,t) \geqq C_{k,\alpha}(x,t)$, $C_{k,\alpha,p}(x,t) > 0$ in D_B, and $C_{k,\alpha,p}(x,t) \to C_{k,\alpha}(x,t)$ uniformly in D_B.

From (30), we have

$$\begin{aligned} &\iint_S (u_k - Z_\alpha)\left(\frac{\partial f}{\partial t} + C_{k,\alpha,p}\frac{\partial^2 f}{\partial x^2}\right) dxdt \\ &= \iint_S (u_k - Z_\alpha)(C_{k,\alpha,p} - C_{k,\alpha})\frac{\partial^2 f}{\partial x^2}\, dxdt + \int_{-\infty}^{\infty} [Z_\alpha(x,0) - u_k(x,0)] f(x,0)\, dx. \end{aligned} \tag{31}$$

The inequality (29) will be proved if we find a function $f(x,t)$ for which (31) holds, and such that the absolute value of every integral in the right hand side of (31) is less than $\varepsilon/2$, and

$$\frac{\partial f}{\partial t} + C_{k,\alpha,p}\,\frac{\partial^2 f}{\partial x^2} = F(x,t). \tag{32}$$

To find such a function, consider a first boundary problem in D_B for Eq. (32)

$$f(x,T) = 0,\ f(B+1,t) = f(-B-1,t) = 0. \tag{33}$$

The problem (32), (33) has a solution $f_{k,\alpha,p}(x,t)$ for every k, α and p (cf. [8]). Set $f_{k,\alpha,p}(x,t) = 0$ for $|x| \leqq B+1$, $t \geqq T$. Applying the maximum principle to Eq.(32) we get

$$|f_{k,\alpha,p}(x,t)| \leqq M_1 \text{ in } D_B$$

where the constant M_1 depends only on $F(x,t)$. Next, multiplying the Eq. (32) by $(\partial^2 f_{k,\alpha,p})/(\partial x^2)$ and integrating on the region D_B, we get estimates

$$\iint_{D_B} C_{k,\alpha,p}\left(\frac{\partial^2 f_{k,\alpha,p}}{\partial x^2}\right)^2 dxdt \leqq M_2, \tag{34}$$

$$\frac{1}{2}\int_{-B}^{B}\left(\frac{df_{k,\alpha,p}(x,0)}{dx}\right)^2 dx \leqq M_2 \tag{35}$$

where M_2 also depends only on $F(x,t)$.

Now because of (28), we can substitute in (31) the function $f = f_{k,\alpha,p}$. We obtain

$$\int\int_{D_B} (u_k - Z_\alpha)F(x,t)\,dxdt \tag{36}$$

$$= \int\int_{D_B} (u_k - Z_\alpha)(C_{k,\alpha,p} - C_{k\,\alpha}) \frac{\partial^2 f_{k,\alpha,p}}{\partial x^2} dxdt + \int_{-B}^{B} [Z_\alpha(x,0) - u_k(x,0)] f_{k,\alpha,p}(x,0)\,dx.$$

Next we prove that there are k_0 and α_0 such that

$$\left| \int_{-B}^{B} [Z_\alpha(x,0) - u_k(x,0)] f_{k,\alpha,p}(x,0) dx \right| \leqq \frac{\varepsilon}{2} \tag{37}$$

for $k \geqq k_0$, $\alpha \leqq \alpha_0$ and arbitrary p.

We have

$$\int_{-B}^{B} [Z_\alpha(x,0) - u_k(x,0)] f_{k,\alpha,p}(x,0)\,dx$$

$$= \int_{-B}^{B} [Z_\alpha(x,0) - u_k(x,0)]\,[f_{k,\alpha,p}(x,0) - f_{k,\alpha,p}(0,0)]\,dx$$

$$+ \int_{-B}^{B} [Z_\alpha(x,0) - u_k(x,0)] f_{k,\alpha,p}(0,0)\,dx$$

$$= \int_{-B}^{B} [Z_\alpha(x,0) - u_k(x,0)]\,[f_{k,\alpha,p}(x,0) - f_{k,\alpha,p}(0,0)]\,dx$$

because

$$\int_{-B}^{B} Z_\alpha(x,0)\,dx = \int_{-B}^{B} u_k(x,0)\,dx = E_0.$$

Let $\gamma \leqq (1/2M_2)(\varepsilon/4E_0)^2$. Then for $|x| \leqq \gamma$,

$$\left| f_{k,\alpha,p}(x,0) - f_{k,\alpha,p}(0,0) \right| = \left| \int_0^x \frac{df_{k,\alpha,p}(x,0)}{dx} dx \right| \leqq (2M_2)^{\frac{1}{2}} \gamma^{\frac{1}{2}} \leqq \varepsilon/4E_0. \tag{38}$$

Now let k_0 be sufficiently large and α_0 sufficiently small constants so that for $|x| \geqq \gamma$, $k \geqq k_0$ and $\alpha \leqq \alpha_0$, $u_k(x,0) = Z_\alpha(x,0) = 0$. Then using (38), we obtain

$$\left| \int_{-B}^{B} [Z_\alpha(x,0) - u_k(x,0)]\,[f_{k,\alpha,p}(x,0) - f_{k,\alpha,p}(0,0)]\,dx \right.$$

$$= \left| \int_{-\gamma}^{\gamma} [Z_\alpha(x,0) - u_k(x,0)]\,[f_{k,\alpha,p}(x,0) - f_{k,\alpha,p}(0,0)]\,dx \right.$$

$$\leqq \varepsilon/4E_0 \left[\int_{-\gamma}^{\gamma} Z_\alpha(x,0)dx + \int_{-\gamma}^{\gamma} u_k(x,0)\,dx \right] = \varepsilon/4E_0\, 2E_0 = \varepsilon/2$$

for $k \geqq k_0$, $\alpha \leqq \alpha_0$ and arbitrary p. This leads to (37).

Next we fix some values of $k \geqq k_0$ and $\alpha \leqq \alpha_0$. Using (34), we have

$$\left| \iint_{D_B} (u_k - Z_\alpha)(C_{k,\alpha,p} - C_{k,\alpha}) \frac{\partial^2 f_{k,\alpha,p}}{\partial x^2}\, dxdt \right|$$

$$\leqq \max \left| u_k - Z_\alpha \right| \left\{ \iint_{D_B} \frac{(C_{k,\alpha,p} - C_{k,\alpha})^2}{C_{k,\alpha,p}} dxdt \right\}^{\frac{1}{2}}$$

$$\times \left\{ \iint_{D_B} C_{k,\alpha,p} \left(\frac{\partial^2 f_{k\,\alpha\,p}}{\partial x^2} \right)^2 dxdt \right\}^{\frac{1}{2}}$$

$$\leqq \max \left| u_k - Z_\alpha \right| M_2^{\frac{1}{2}} [2(B+1)T]^{\frac{1}{2}} \max \left| C_{k,\alpha,p} - C_{k,\alpha} \right| \leqq \varepsilon/2$$

if p is large enough. Hence we get (29) and also (27). It follows from (27) that for every sequence where $k \to \infty$

$$u_k(x,t) \to w_{E_0}(x,t) \tag{39}$$

in L_2 in every bounded region inside the strip $(x \in R^1, 0 < t < T)$.

The assertion of the theorem follows from the lemma:

Lemma 4: *Let $k \to \infty$. Then $u_k(x,1) \to w_{E_0}(x,1)$ uniformly with respect to $x \in R^1$.*

Proof. Let A be a constant so that $u_k(x,1) = w_{E_0}(x,1) = 0$ for $|x| \geqq A$, $k \geqq 1$. To prove this lemma, it is sufficient to prove that if $k \to \infty$,

$$u_k(x,1) \to w_{E_0}(x,1) \tag{40}$$

uniformly with respect to $x \in [-A, A]$. Suppose that $T = 1$. Then from (24) and (25),

$$\int_{-n}^{n} \left(\frac{\partial v_{\tau,k,n}}{\partial x} \right)^2 dx \Big|_{t=1} \leqq \frac{C_4(\tau)}{\tau}. \tag{41}$$

From (41), we conclude that

$$\int_{-A}^{A} \left(\frac{\partial u_k^{\lambda+1}}{\partial x} \right)^2 dx \Big|_{t=1} \leqq C_4(\tau)/\tau.$$

Hence, there is a subsequence $\{u_{k_i}^{\lambda+1}(x,1)\}$ $(k_i \to \infty)$ which converges uniformly with respect to $x \in [-A, A]$ (cf. [4]). The corresponding subsequence $u_{k_i}(x,1)$ also converges uniformly to the function $\tilde{u}(x,1)$. Now consider the integral identity for $u_{k_i}(x,t)$ in the half-plane $(x \in R^1, 1 \leqq t \leqq \infty)$. We get

$$(42) \qquad \int_1^\infty \int_{-\infty}^\infty \left(u_{k_i} \frac{\partial f}{\partial t} + u_{k_i}^{\lambda+1} \frac{\partial^2 f}{\partial x} \right) dxdt + \int_{-\infty}^\infty u_{k_i}(x,\, 1) f(x,1)\, dx = 0$$

for any infinitely differentiable function $f(x, t)$ with compact support. Using (39), we can pass to the limit in (42) and obtain

$$\int_1^\infty \int_{-\infty}^\infty \left[w_{E_0} \frac{\partial f}{\partial t} + w_{E_0}^{\lambda+1} \frac{\partial^2 f}{\partial x^2} \right] dxdt + \int_{-\infty}^\infty \tilde{u}(x,1) f(x,1)\, dx = 0.$$

Since $w_{E_0}(x,t)$ is a generalized solution of the Cauchy problem in the half-plane $(x \in R^1,\ 1 \leqq t < \infty)$, we conclude that

$$\int_{-\infty}^\infty w_{E_0}(x,1) f(x,1)\, dx = \int_{-\infty}^\infty \tilde{u}(x,1) f(x,1)\, dx.$$

Hence $\tilde{u}(x,1) = w_{E_0}(x,1)$ and (40) holds for every sequence of k. Thus Lemma 4 is proved.

From Lemma 4, we have

$$\left| ku(kx, k^{\lambda+2}) - w_{E_0}(x,1) \right| = \left| ku(kx, k^{\lambda+2}) - kw_{E_0}(kx, k^{\lambda+2}) \right| \to 0.$$

Setting $k = t^{1/(\lambda+2)}$, we obtain

$$t^{1/(\lambda+2)} \left| u(xt^{1/(\lambda+2)}, t) - w_{E_0}(xt^{1/(\lambda+2)}, t) \right| \to 0 \quad \text{if} \quad t \to \infty.$$

As this convergence is uniform with respect to $x \in R^1$, the assertion of the theorem follows.

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