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CROSSED PRODUCTS OVER PRIME RINGS

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ABSTRACT

In this paper we obtain necessary and sufficient conditions for the crossed product $R * G$ to be prime or semiprime under the assumption that R is prime. The main techniques used are the Δ-methods which reduce these questions to the finite normal subgroups of G and a study of the X-inner automorphisms of R which enables us to handle these finite groups. In particular we show that $R * G$ is semiprime if R has characteristic 0. Furthermore, if R has characteristic $p > 0$, then $R * G$ is semiprime if and only if $R * P$ is semiprime for all elementary abelian p-subgroups P of $\Delta^+(G) \cap G_{\mathrm{inn}}$.

Let G be a multiplicative group and let R be a ring with 1. Then a crossed product $R * G$ of G over R is an associative ring determined by G, R and certain other parameters. To be more precise, for each $x \in G$ there exists an element $\bar{x} \in R * G$ and every element $\alpha \in R * G$ is uniquely writable as a finite sum

$$\alpha = \sum_{x \in G} r_x \bar{x}$$

with $r_x \in R$. The addition in $R * G$ is the obvious one and the multiplication is given by the formulas

$$\bar{x}\bar{y} = t(x, y)\overline{xy},$$

$$r\bar{x} = \bar{x}r^{\bar{x}}$$

for all $x, y \in G$ and $r \in R$. Here $t : G \times G \to U$ is a map from $G \times G$ to the group of units U of R and, for fixed $x \in G$, the map ${}^{\bar{x}} : r \to r^{\bar{x}}$ is an automorphism of R.

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It is a simple exercise to determine the relations on t and the automorphisms $^{\bar{x}}$ which make $R*G$ associative. Furthermore one knows that $R*G$ has an identity element namely $1 = t(1,1)^{-1}\bar{1}$, that each $\bar{x}$ is invertible and indeed that

$$\mathfrak{G} = \{u\bar{x} \mid u \in U, x \in G\}$$

is a multiplicative group of units in $R*G$. Thus the equation $r\bar{x} = \bar{x}r^{\bar{x}}$ is equivalent to $\bar{x}^{-1}r\bar{x} = r^{\bar{x}}$ and hence the automorphism $^{\bar{x}}$ is merely conjugation by a unit in $R*G$. In fact it is clear that $\mathfrak{G}$ acts on R by conjugation. In general, $R*G$ does not contain an isomorphic copy of G. However we do have $R \subseteq R*G$ by way of the embedding $r \to r1$ and then $U \triangleleft \mathfrak{G}$ with $\mathfrak{G}/U \simeq G$.

Certain special cases of this construction warrant additional mention. The simplest is the case in which $t(x,y) = 1$ for all x, y and $r^{\bar{x}} = r$ for all x, r. This is the ordinary group ring which we denote by $R[G]$. Again if we assume that $t(x,y) = 1$ for all x, y, but if we allow an action of G on R, then we obtain a skew group ring, usually denoted by RG. In both of these cases, since $\bar{x}\bar{y} = \overline{xy}$, $R*G$ contains a copy of G and, by setting $\bar{x} = x$, we identify G as a subgroup of the units of $R*G$. Finally, if we assume that $r^{\bar{x}} = r$ for all r, x, then we obtain a twisted group ring $R^t[G]$. Here it is clear that each $t(x,y)$ must belong to the center of R.

Recently there has been a growing ring theoretic interest in crossed products and in particular in skew group rings of finite groups. This is due mainly to their relationship to a possible Galois theory for rings. For example, if G is a finite group of automorphisms of R, then the skew group ring RG contains all the necessary ingredients of the theory, namely G, R and the fixed ring R^G. Thus there is now a body of results concerning these rings when G is finite. On the other hand, there is a technique which has proved fruitful in the study of ordinary group algebras which can frequently reduce problems from infinite groups to the finite case. It is the aim of this paper to show that these Δ-methods can also apply to yield theorems on crossed products at least when R is prime.

For the most part we will be concerned with the problem of determining when $R*G$ is prime or semiprime under the assumption that R is prime. In Section 1 we develop the necessary Δ-methods to reduce these questions to the finite normal subgroups of G. In Section 2 we study X-inner automorphisms of R, amplifying known results on crossed products of finite groups and further reducing these problems to certain twisted group algebras. Finally in Section 3 we consider these twisted group algebras in detail and obtain the main result on the semiprimeness of $R*G$. In the course of this work we also obtain some facts on annihilator ideals as well as a sharpening of the Δ-methods in the case of twisted group rings. We remark that the hypothesis that R is prime is used at crucial points throughout this paper. In fact very little is true without this assumption. In a later paper we will consider what can be salvaged in case R is just semiprime.

Finally we note that even if R does not have a 1, it is still possible to define and study skew group rings RG. In fact, with just a little additional care, one can show that the main results of this paper also hold in this extended context. Nevertheless, we will assume throughout that $1 \in R$.

§1. Δ-methods

We consider a crossed product $R * G$ and introduce some notation. First, in view of the fact that $1 = t(1,1)^{-1}\bar{1}$, there is really no loss of generality in assuming that $\bar{1} = 1$. We will therefore make this assumption throughout the remainder of this paper.

Now if $\alpha = \Sigma\, r_x\bar{x} \in R * G$, then the support of α is defined to be

$$\operatorname{Supp} \alpha = \{x \in G \mid r_x \neq 0\}.$$

Thus $\operatorname{Supp}\alpha$ is a finite subset of G. If D is any subset of G, then we let

$$R * D = \{\alpha \in R * G \mid \operatorname{Supp}\alpha \subseteq D\}.$$

It is clear that $R * D$ is both a right and left R-submodule of $R * G$. Furthermore if $D = H$ is a subgroup of G, then $R * H$ is also a crossed product of H over R with corresponding twisting $t : H \times H \to U$ and automorphisms inherited from $R * G$. Observe that if $H \lhd G$, then $\mathfrak{G}$ acts by conjugation as automorphisms on $R * H$. Hence we see easily that $R * G$ is a crossed product of G/H over the ring $R * H$.

Again if D is a subset of G, we define the projection map $\pi_D : R * G \to R * D$ by

$$\pi_D\left(\sum_{x \in G} r_x\bar{x}\right) = \sum_{x \in D} r_x\bar{x}.$$

Thus π_D truncates $\alpha = \Sigma_{x\in G}\, r_x\bar{x}$ to just the partial sum of its terms with $x \in D$ and we call $\pi_D(\alpha)$ the segment of α in D. Observe that π_D is both a right and left R-module homomorphism, but even if $D = H$ is a subgroup, π_D need not be a ring homomorphism. In the special case in which $D = \langle 1\rangle$, the map $\pi_{\langle 1\rangle}$ is usually called the trace map and is denoted by tr.

If G is an arbitrary group, we define two characteristic subsets as follows:

$$\Delta = \Delta(G) = \{x \in G \mid [G : \mathrm{C}_G(x)] < \infty\}$$

and

$$\Delta^+ = \Delta^+(G) = \{x \in G \mid [G : \mathrm{C}_G(x)] < \infty \text{ and } x \text{ has finite order}\}.$$

The next two lemmas, for the most part due to B. H. Neumann, contain all the group theoretic information we need. See [6, lemmas 4.1.3, 4.2.1, 4.1.6 and 4.1.8] for proofs.

LEMMA 1.1. *Let $H_1, H_2, \cdots, H_n$ be subgroups of G.*

(i) *If each H_i has finite index in G, then so does $H_1 \cap H_2 \cap \cdots \cap H_n$.*

(ii) *If there exists a finite collection of elements $x_{ij} \in G$ such that $G = \bigcup_{ij} H_i x_{ij}$, then for some i, $[G : H_i] < \infty$.*

LEMMA 1.2. *Let G be a group. Then*

(i) *Δ and Δ^+ are both characteristic subgroups of G.*

(ii) *Δ/Δ^+ is torsion free abelian.*

(iii) *Any finite subset of Δ^+ is contained in a finite normal subgroup H of G with $H \subseteq \Delta^+$.*

We now reserve the symbols θ and θ^+ for the projection maps $\theta : R * G \to R * \Delta$ and $\theta^+ : R * G \to R * \Delta^+$. Furthermore if D is a finite subset of G we define $D_\Delta = D \cap \Delta$ and

$$\mathcal{T}(D) = \{x \in \mathrm{C}_G(D_\Delta) \mid x^{-1}(D \setminus D_\Delta)xD \cap D_\Delta D = \varnothing\}.$$

In other words, the latter condition asserts that the equation $x^{-1}d_1xd_2 = d_3d_4$ has no solution with $d_1 \in D \setminus D_\Delta$, $d_3 \in D_\Delta$ and $d_2, d_4 \in D$. The following lemma, in all its different variants, is essentially the Δ-method.

LEMMA 1.3. *Let D be a finite subset of G with $H = \mathrm{C}_G(D_\Delta)$ and $T = \mathcal{T}(D)$. Then $[G : H] < \infty$ and for all $h_1, h_2, \cdots, h_m \in H$ we have $\bigcap_1^m Th_i \neq \varnothing$. Furthermore suppose $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_n \in R * D \subseteq R * G$ satisfy the identity*

$$x_1\bar{x}\beta_1 + \alpha_2\bar{x}\beta_2 + \cdots + \alpha_n\bar{x}\beta_n = 0$$

for all $x \in G$. Then we have

$$\theta(\alpha_1)^{\bar{y}}\beta_1 + \theta(\alpha_2)^{\bar{y}}\beta_2 + \cdots + \theta(\alpha_n)^{\bar{y}}\beta_n = 0$$

and

$$\theta(\alpha_1)^{\bar{y}}\theta(\beta_1) + \theta(\alpha_2)^{\bar{y}}\theta(\beta_2) + \cdots + \theta(\alpha_n)^{\bar{y}}\theta(\beta_n) = 0$$

for all $y \in T$.

PROOF. If $d \in D_\Delta$, then $[G : \mathbf{C}_G(d)] < \infty$. Hence since D_Δ is finite, Lemma 1.1(i) yields $[G : H] < \infty$. Now let

$$W = H \setminus T = \{x \in H \mid x^{-1}(D \setminus D_\Delta)xD \cap D_\Delta D \neq \emptyset\}.$$

Then for each $x \in W$ there exist $d_1 \in D \setminus D_\Delta$, $d_3 \in D_\Delta$ and $d_2, d_4 \in D$ with $x^{-1}d_1xd_2 = d_3d_4$. Now for fixed d_1, d_2, d_3, d_4, the set of solutions x of this equation is clearly either empty or a coset of $\mathbf{C}_H(d_1)$. Hence it follows that

$$W = \bigcup_{\substack{d \in D \setminus D_\Delta \\ j}} \mathbf{C}_H(d)w_{dj}$$

is a finite union of appropriate right cosets of these centralizers. But $[G : H] < \infty$ and $[G : \mathbf{C}_G(d)] = \infty$ since $d \not\in \Delta$ so $[H : \mathbf{C}_H(d)] = \infty$. Therefore Lemma 1.1(ii) implies that $W \neq H$. In fact for all $h_1, h_2, \cdots, h_m \in H$,

$$Wh_1 \cup Wh_2 \cup \cdots \cup Wh_m \neq H.$$

Thus since $H \setminus Wh_i = Th_i$, we conclude that $\bigcap_i Th_i \neq \emptyset$.

Now suppose $\alpha_1, \alpha_2, \cdots, \alpha_n, \beta_1, \beta_2, \cdots, \beta_n \in R * D \subseteq R * G$ satisfy the identity

$$\alpha_1\bar{x}\beta_1 + \alpha_2\bar{x}\beta_2 + \cdots + \alpha_n\bar{x}\beta_n = 0.$$

Write $\alpha_i = \alpha_i' + \alpha_i''$ with $\alpha_i' = \theta(\alpha_i)$. Then for all $x \in G$ we have

$$\sum_i \bar{x}^{-1}\alpha_i'\bar{x}\beta_i = -\sum_i \bar{x}^{-1}\alpha_i''\bar{x}\beta_i.$$

Observe that every group element in the support of the right hand term is of the form $x^{-1}d_1xd_2$ with $d_1 \in D \setminus D_\Delta$, $d_2 \in D$. Furthermore, if $x \in H = \mathbf{C}_G(D_\Delta)$, then every element in the support of the left hand term is of the form $x^{-1}d_3xd_4 = d_3d_4$ with $d_3 \in D_\Delta$, $d_4 \in D$. Thus if $x \in T$, then by definition of T we conclude that the right and left hand terms have disjoint supports and hence both must be zero. Since $\bar{x}^{-1}\alpha_i'\bar{x} = \theta(\alpha_i)^{\bar{x}}$ this yields $\Sigma_i\, \theta(\alpha_i)^{\bar{x}}\beta_i = 0$ for $x \in T$. Finally if we apply θ to this formula, we conclude immediately that $\Sigma_i\, \theta(\alpha_i)^{\bar{x}}\theta(\beta_i) = 0$ and the result follows.

For the most part we will only use the fact that $T \neq \emptyset$ in the above. For example, in the ordinary group ring situation we have

LEMMA 1.4. *Let $R[G]$ be an ordinary group ring. Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n$, $\beta_1, \beta_2, \cdots, \beta_n \in R[G]$ satisfy the identity*

$$\alpha_1x\beta_1 + \alpha_2x\beta_2 + \cdots + \alpha_nx\beta_n = 0$$

for all $x \in G$. Then we have

$$\theta(\alpha_1)\beta_1 + \theta(\alpha_2)\beta_2 + \cdots + \theta(\alpha_n)\beta_n = 0$$

and

$$\theta(\alpha_1)\theta(\beta_1) + \theta(\alpha_2)\theta(\beta_2) + \cdots + \theta(\alpha_n)\theta(\beta_n) = 0.$$

PROOF. Let D be the union of the supports of all α_i, β_i and apply the preceding lemma. Observe that $T \neq \emptyset$ and if $y \in T$ then y centralizes the supports of all $\theta(\alpha_i)$. Thus since y acts trivially on R, we see that $\theta(\alpha_i)^y = \theta(\alpha_i)$ and the result follows.

There are of course numerous applications of this fact in the study of ordinary group algebras. Many of these can be found in [6] and in particular in Chapter 4 of that book. Our concern here however is with the more complicated case of crossed products.

The whole difficulty in applying Δ-methods to crossed products is the presence of the conjugating elements $\bar{y}$ in the formulas of Lemma 1.3. Admittedly there are many such y which work, but the range of possible choices unfortunately varies with the supporting set D. The goal then is to eliminate these group elements y. The next lemma is given in a form which is useful even if R is not prime.

LEMMA 1.5. *Let A and B be ideals of $R * G$ with $AB = 0$. If $\theta(A)B \neq 0$ then there exists an element $\alpha \in A$ with*

(i) $\operatorname{tr} \alpha = a \neq 0$,

(ii) $\theta(\alpha)B \neq 0$,

(iii) *for all $\beta \in B$, there exists $y \in \mathbf{C}_G(\operatorname{Supp} \theta(\alpha))$ with*

$$Ra^{\bar{y}}R \cdot \theta(\alpha)\beta = 0.$$

PROOF. Since $\theta(A)B \neq 0$, we can choose $\theta(\alpha) \in \theta(A)$ of minimal support size with $\theta(\alpha)B \neq 0$. If $z \in \operatorname{Supp} \theta(\alpha)$, then $\theta(\bar{z}^{-1}\alpha)B = \bar{z}^{-1}\theta(\alpha)B \neq 0$ and $1 \in \operatorname{Supp} \theta(\bar{z}^{-1}\alpha)$. Thus we may clearly assume that $1 \in \operatorname{Supp} \theta(\alpha)$. Write $\alpha = \Sigma\, a_x \bar{x} \in A$ with $a_1 = \operatorname{tr} \alpha = a \neq 0$. Then of course α satisfies (i) and (ii) above.

Fix $\beta \in B$. If $y \in \mathbf{C}_G(\operatorname{Supp} \theta(\alpha))$, then $\alpha^{\bar{y}} = \Sigma\, a_x^{\bar{y}} \bar{x}^{\bar{y}} \in A$ and $\operatorname{Supp} \theta(\alpha^{\bar{y}}) = \operatorname{Supp} \theta(\alpha)$. Since $\bar{1} = 1$, this now implies that for all $r \in R$

$$\gamma = a^{\bar{y}} r \alpha - \alpha^{\bar{y}} r a \in A$$

with $|\mathrm{Supp}\,\theta(\gamma)| < |\mathrm{Supp}\,\theta(\alpha)|$. Hence, by the minimality of $|\mathrm{Supp}\,\theta(\alpha)|$ we have $\theta(\gamma)B = 0$. In particular, $\theta(\gamma)\beta = 0$ yields the identity

$$a^{\bar{y}} r\theta(\alpha)\beta = \theta(\alpha)^{\bar{y}} ra\beta$$

for all $r \in R$.

Set $D = \mathrm{Supp}\,\alpha \cup \mathrm{Supp}\,\beta$. Then $\alpha, s\beta \in R * D$ for all $s \in R$ and then for all $g \in G$ we have

$$x\bar{g}(s\beta) \in AB = 0.$$

Hence, by Lemma 1.3,

$$\theta(\alpha)^{\bar{y}} s\beta = 0$$

for all $y \in T = \mathcal{T}(D)$. Note that $T \neq \varnothing$ and that $T \subseteq \mathbf{C}_G(D_\Delta) \subseteq \mathbf{C}_G(\mathrm{Supp}\,\theta(\alpha))$. Thus setting $s = ra = r\,\mathrm{tr}\,\alpha$ in the above and applying our preceding identity we have

$$0 = \theta(\alpha)^{\bar{y}} ra\beta = a^{\bar{y}} r\theta(\alpha)\beta.$$

Since this holds for all $r \in R$, we obtain finally

$$Ra^{\bar{y}}R \cdot \theta(\alpha)\beta = 0$$

and (iii) is proved.

We remark that if $H \triangleleft G$ and if I is an ideal of $R * G$ then

$$I \subseteq \pi_H(I)(R * G) = (R * G)\pi_H(I)$$

(see [6, lemma 1.1.5]). Hence if $I \neq 0$, then $\pi_H(l) \neq 0$ and in particular $\theta(I) \neq 0$.

LEMMA 1.6. *Let R be a prime ring.*

(i) *If A and B are ideals of $R * G$ with $AB = 0$, then $\theta(A)B = 0$.*

(ii) *If $A_1, A_2, \cdots, A_n$ are ideals of $R * G$ with $A_1A_2\cdots A_n = 0$, then*

$$\theta(A_1)\theta(A_2)\cdots\theta(A_n) = 0.$$

PROOF. Suppose $AB = 0$. If $\theta(A)B \neq 0$ then Lemma 1.5 applies. But observe that if $Ra^{\bar{y}}R \cdot \theta(\alpha)\beta = 0$, then the nonzero ideal $Ra^{\bar{y}}R$ of R annihilates all coefficients in $\theta(\alpha)\beta$ and hence since R is prime we have $\theta(\alpha)\beta = 0$. Thus $\theta(\alpha)B = 0$. Since this contradicts the definition of α, we conclude that $\theta(A)B = 0$ and (i) follows.

For (ii) we show by induction on n that if $A_1A_2\cdots A_nB = 0$, then $\theta(A_1)\theta(A_2)\cdots\theta(A_n)B = 0$. The case $n = 1$ is given above and for $n > 1$ we conclude by induction from $A_1A_2\cdots A_{n-1}(A_nB) = 0$ that

$$\theta(A_1)\cdots\theta(A_{n-1})A_nB=0.$$

Hence if C is the ideal

$$C=(R*G)\theta(A_1)\cdots\theta(A_{n-1})A_n$$

then $CB=0$. We now conclude from (i) that $\theta(C)B=0$ and since clearly $\theta(C)\supseteq\theta(A_1)\cdots\theta(A_{n-1})\theta(A_n)$, the induction step is proved. Finally (ii) follows immediately by taking $B=R*G$.

As will be apparent soon, the above lemma yields the necessary reduction from $R*G$ to $R*\Delta$. The next two results on crossed products of ordered groups facilitate the further reduction to $R*\Delta^+$. The second of these, due to J. Roseblade, is a generalization of [6, theorem 4.3.16] and [8, theorem 5.5]. We thank Dr. Roseblade for allowing us to include this lemma here.

Let $A\neq 0$ be an ideal of $R*G$. If k is the minimum support size of a nonzero element of A, we define $\tilde{A}$ to be the additive subset of A spanned by all $\alpha\in A$ with $|\operatorname{Supp}\alpha|=k$. It is clear that $\tilde{A}\neq 0$ and that $\tilde{A}$ is both a right and left R-submodule of A. Furthermore, since $|\operatorname{Supp}\alpha|=|\operatorname{Supp}\alpha\bar{x}|$ for all $x\in G$, we see that $\tilde{A}$ is an ideal of $R*G$ and that $\operatorname{tr}(\tilde{A})$ is a nonzero ideal of R.

LEMMA 1.7. *Let G be an ordered group and let A and B be nonzero ideals of $R*G$ with $AB=0$. Then* $\operatorname{tr}(\tilde{A})\cdot\operatorname{tr}(\tilde{B})=0$.

PROOF. Let $\alpha\in A$, $\beta\in B$ be elements of minimal support sizes n and m respectively. Since G is ordered, we can write

$$\alpha=a_1\bar{x}_1+a_2\bar{x}_2+\cdots+a_n\bar{x}_n,$$

$$\beta=b_1\bar{y}_1+b_2\bar{y}_2+\cdots+b_m\bar{y}_m$$

with $x_1<x_2<\cdots<x_n$, $y_1<y_2<\cdots<y_m$ and $a_i,b_j\in R$. Since $AB=0$ we have $\alpha\beta=0$ and from the group ordering it is trivial to see that $(a_1\bar{x}_1)(b_1\bar{y}_1)=0$.

Fix subscripts i, j. Now $(a_1\bar{x}_1)\beta\in B$ and by the above we have $|\operatorname{Supp}(a_1\bar{x}_1)\beta|<m$. Thus the minimality of m yields $(a_1\bar{x}_1)\beta=0$ and hence $(a_1\bar{x}_1)(b_j\bar{y}_j)=0$. Similarly $\alpha(b_j\bar{y}_j)\in A$ and $|\operatorname{Supp}\alpha(b_j\bar{y}_j)|<n$. Thus $\alpha(b_j\bar{y}_j)=0$ and $(a_i\bar{x}_i)(b_j\bar{y}_j)=0$ for all i,j. This clearly implies that $(\operatorname{tr}\alpha)(\operatorname{tr}\beta)=0$. Since $\tilde{A}$ is spanned additively by all such α and $\tilde{B}$ by all such β, we conclude that $\operatorname{tr}(\tilde{A})\cdot\operatorname{tr}(\tilde{B})=0$.

LEMMA 1.8. *Let G be an ordered group with $R*G$ semiprime. If A and B are ideals of $R*G$ with $AB=0$, then* $\operatorname{tr}(A)\cdot\operatorname{tr}(B)=0$.

PROOF. Let $P=\{x\in G\mid x\geqq 1\}$ so that P is a normal multiplicatively closed subset of G. We first study $A^+ = A \cap (R * P)$ and $B^+ = B \cap (R * P)$. It is clear that these are right and left R-submodules of $R * G$ which are $\mathfrak{G}$-invariant. Hence the same is true of $A_1 = \operatorname{tr} A^+$ and $B_1 = \operatorname{tr} B^+$. Since $AB = 0$ we have $A^+B^+ = 0$ and hence, by the ordering, we conclude immediately that $A_1B_1 = 0$.

Suppose I is a $\mathfrak{G}$-invariant ideal of R with $I^2 = 0$. Then it is easy to see that $I(R * G) = (R * G)I$ is a two sided ideal of $R * G$ with

$$[I(R * G)]^2 = (R * G)I^2(R * G) = 0.$$

Hence the semiprime assumption yields $I(R * G) = 0$ and $I = 0$. Two applications of this are as follows. First, let $C = \{r \in R \mid A_1^2 r = 0\}$. Then since A_1 is a $\mathfrak{G}$-invariant ideal of R, so is C, and since

$$(A_1C)^2 = A_1(CA_1)C \subseteq A_1^2C = 0$$

the above yields $A_1C = 0$. Similarly, if $D = \{r \in R \mid r(\operatorname{tr} B)^2 = 0\}$ then we deduce that $D \cdot \operatorname{tr} B = 0$.

We show now that $A_1B = 0$. To this end, we prove that $A_1\beta = 0$ for $\beta \in B$ by induction on $|\operatorname{Supp} \beta|$. Thus suppose A_1 annihilates all $\gamma \in B$ with $|\operatorname{Supp} \gamma| < m$ and let $\beta \in B$ with $|\operatorname{Supp} \beta| = m$. Write

$$\beta = b_1\bar{y}_1 + b_2\bar{y}_2 + \cdots + b_m\bar{y}_m$$

with $y_1 < y_2 < \cdots < y_m$ and $b_i \in R$. Then $\beta\bar{y}_1^{-1} \in B^+$ so $b_1 \in B_1$ and $A_1b_1 = 0$. Thus we see that $A_1\beta \subseteq B$ and every element of $A_1\beta$ has support size smaller than m. Hence by induction we have $A_1^2\beta = A_1(A_1\beta) = 0$ so $A_1^2b_i = 0$ for all i. The result of the preceding paragraph now shows that $A_1b_i = 0$ and hence that $A_1\beta = 0$. Thus we conclude that $A_1B = 0$. In particular, by applying the trace map we see that $A_1 \cdot \operatorname{tr} B = 0$.

Finally we reverse the roles and show that $A \cdot \operatorname{tr} B = 0$. Here we proceed by induction on $|\operatorname{Supp} \alpha|$ with $\alpha \in A$ to prove that $\alpha(\operatorname{tr} B) = 0$. Say $|\operatorname{Supp} \alpha| = n$ and assume that the result is known for elements of smaller support size. Write

$$\alpha = \bar{x}_1a_1 + \bar{x}_2a_2 + \cdots + \bar{x}_na_n$$

with $x_1 < x_2 < \cdots < x_n$ and $a_i \in R$. Then $\bar{x}_1^{-1}\alpha \in A^+$ so $a_1 \in A_1$ and hence, by the above, $a_1(\operatorname{tr} B) \subseteq A_1(\operatorname{tr} B) = 0$. Thus we see that $\alpha(\operatorname{tr} B) \subseteq A$ and every element of $\alpha(\operatorname{tr} B)$ has support size smaller than n. Induction therefore yields $\alpha(\operatorname{tr} B)^2 = (\alpha \cdot \operatorname{tr} B)(\operatorname{tr} B) = 0$ so $a_i(\operatorname{tr} B)^2 = 0$. The result of the second paragraph now shows that $a_i(\operatorname{tr} B) = 0$ and hence $\alpha(\operatorname{tr} B) = 0$. In other words, we have proved that $A(\operatorname{tr} B) = 0$ and therefore, by applying the trace map, we conclude that $(\operatorname{tr} A)(\operatorname{tr} B) = 0$.

We remark that, even in the case of ordinary group rings, the above is false without the semiprime assumption. See [6, page 145] for an example.

Now suppose $R * G$ is given. If $H \lhd G$, then $\mathfrak{G}$ acts on $R * H$ by conjugation and hence $\mathfrak{G}$ permutes the ideals of $R * H$. Recall that $U \lhd \mathfrak{G}$, where U is the group of units of R, and $\mathfrak{G}/U \simeq G$. Thus since each ideal of $R * H$ is clearly U-invariant, the group $\mathfrak{G}/U \simeq G$ in fact permutes the ideals of $R * H$. We say that $R * H$ is G-prime if for all G-invariant ideals $A, B \subseteq R * H$, $AB = 0$ implies $A = 0$ or $B = 0$. Similarly $R * H$ is G-semiprime if for all G-invariant ideals $A \subseteq R * H$, $A^2 = 0$ implies $A = 0$. It is clear that if $R * H$ is prime or semiprime, then it is also G-prime or G-semiprime respectively.

The following is the main result of this section.

THEOREM 1.9. *Let R be a prime ring. Then $R * G$ is prime or semiprime, respectively, if and only if for all finite normal subgroups H of G, $R * H$ is G-prime or G-semiprime, respectively.*

PROOF. If $H \lhd G$ and if A is a G-invariant ideal of $R * H$, then it is trivial to see that $A(R * G) = (R * G)A$ is a two-sided ideal of $R * G$. Suppose now that $R * H$ is not G-prime. Then we can find nonzero G-invariant ideals $A, B \subseteq R * H$ with $AB = 0$. It therefore follows that

$$A(R * G) \cdot B(R * G) = (R * G)A \cdot B(R * G) = 0$$

and $R * G$ is not prime. Similarly, if $R * H$ is not G-semiprime, then by taking $A = B$ in the above argument we see that $R * G$ is not semiprime. We now consider the converse.

Suppose first that $R * G$ is not prime and let A and B be nonzero ideals of $R * G$ with $AB = 0$. If $A_1 = \theta(A)$ and $B_1 = \theta(B)$, then A_1 and B_1 are nonzero G-invariant ideals of $R * \Delta$ with $A_1B_1 = 0$, by Lemma 1.6(ii). Hence we now know that $R * \Delta$ is not G-prime.

Set $S = R * \Delta^+$ and observe that $R * \Delta$ is a suitable crossed product of Δ/Δ^+ over the ring S. Say $R * \Delta = S * (\Delta/\Delta^+)$. By Lemma 1.2, Δ/Δ^+ is a torsion free abelian group and hence an ordered group. We can now apply Lemma 1.7 and its notation to $A_1, B_1 \subseteq S * (\Delta/\Delta^+)$ with $A_1B_1 = 0$. Thus if $A_2 = \operatorname{tr}(\tilde{A}_1)$ and $B_2 = \operatorname{tr}(\tilde{B}_1)$, then we conclude from that lemma that $A_2B_2 = 0$. Certainly A_2 and B_2 are nonzero ideals of $S = R * \Delta^+$. Moreover it is trivial to see that, in its action on $S * (\Delta/\Delta^+) = R * \Delta$, the group $\mathfrak{G}$ leaves invariant A_1, B_1, $\tilde{A}_1$, $\tilde{B}_1$ and then A_2 and B_2. Thus we see that $R * \Delta^+$ is not G-prime.

In view of Lemma 1.2, there exists a finite normal subgroup H of G with $H \subseteq \Delta^+$ and $A_3 = A_2 \cap (R * H)$, $B_3 = B_2 \cap (R * H)$ both not zero. Then A_3 and

B_3 are G-invariant ideals of $R*H$ with $A_3B_3=0$ and $R*H$ is not G-prime. This completes the proof of the theorem in the prime case.

The semiprime argument is similar. Suppose A is a nonzero ideal of $R*G$ with $A^2=0$. If $A_1=\theta(A)$, then Lemma 1.6(ii) yields $A_1\neq 0$ and $A_1^2=0$. Again we view $R*\Delta = S*(\Delta/\Delta^+)$ and setting $A_2=\mathrm{tr}(\tilde{A}_1)$ we have $A_2^2=0$ by Lemma 1.7. Moreover A_2 is a nonzero G-invariant ideal of $S=R*\Delta^+$ so there exists a finite normal subgroup H of G with $H\subseteq\Delta^+$ and $A_3=A_2\cap(R*H)\neq 0$. Then A_3 is a nonzero G-invariant ideal of $R*H$ with $A_3^2=0$. Therefore $R*H$ is not G-semiprime and the result follows.

Thus we have reduced the prime and semiprime considerations from $R*G$ to $R*H$ for H a finite normal subgroup of G. Another consequence of this Δ-reduction is the following generalization to crossed products of [6, theorems 4.3.17 and 8.1.9]. Recall that A is an annihilator ideal of a ring S if and only if $A=l(B)$, the left annihilator of B, for some ideal B of S. This clearly implies that A is also an ideal. Recall also that the nilpotent radical NS of S is defined to be the join of all nilpotent ideals of S.

THEOREM 1.10. *Let $R*G$ be a crossed product over the prime ring R.*

(i) *If A is an annihilator ideal of $R*G$, then*

$$A=(A\cap(R*\Delta))R*G.$$

Furthermore, if $R*\Delta$ is semiprime, then

$$A=(A\cap(R*\Delta^+))R*G.$$

(ii) *The nilpotent radical $N=N(R*G)$ satisfies*

$$N=(N\cap(R*\Delta))R*G.$$

PROOF. Observe that for any ideal I of $R*G$ and any normal subgroup H of G we have

$$(I\cap(R*H))R*G\subseteq I\subseteq \pi_H(I)\cdot R*G.$$

Thus if $\pi_H(I)\subseteq I$, then we conclude that equality must occur throughout and hence that $I=(I\cap(R*H))R*G$.

We first consider (i). Let $A=l(B)$ for some ideal B of $R*G$. Then $AB=0$ so Lemma 1.6(i) yields $\theta(A)B=0$ and hence $\theta(A)\subseteq l(B)=A$. By the above observation, $A=(A\cap(R*\Delta))R*G$.

Now suppose that $R*\Delta$ is semiprime. By Lemma 1.6(ii) we have $A_1B_1=0$ where $A_1=\theta(A)$ and $B_1=\theta(B)$. Again we view $R*\Delta=S*(\Delta/\Delta^+)$ where

$S = R * \Delta^+$. By Lemma 1.2, Δ/Δ^+ is torsion free abelian and hence ordered. Thus since $A_1B_1 = 0$ and $S * (\Delta/\Delta^+)$ is semiprime, Lemma 1.8 applied to $S * (\Delta/\Delta^+)$ yields $(\operatorname{tr} A_1)(\operatorname{tr} B_1) = 0$. But it is clear that the trace map in $S * (\Delta/\Delta^+)$ corresponds to the map θ^+ in $R * \Delta$ so we actually have $\theta^+(A_1)\cdot\theta^+(B_1) = 0$. Moreover, clearly $\theta^+(A_1) = \theta^+(\theta(A)) = \theta^+(A)$ and similarly $\theta^+(B_1) = \theta^+(B)$. Since $B \subseteq \theta^+(B)\cdot R * G$ and $\theta^+(A)\cdot\theta^+(B) = 0$, we conclude therefore that $\theta^+(A) \subseteq l(B) = A$ and hence that $A = (A \cap (R * \Delta^+))R * G$. This completes the proof of (i).

For (ii), let A be a nilpotent ideal of $R * G$. Then $A^n = 0$ for some n so Lemma 1.6(ii) yields $\theta(A)^n = 0$. But $\theta(A)\cdot(R * G) = (R * G)\cdot\theta(A)$ so we see that $B = \theta(A)\cdot R * G$ also satisfies $B^n = 0$. Hence, if $N = N(R * G)$, then $B \subseteq N$ and hence $\theta(A) \subseteq N$. Since N is the join of all such A, we conclude therefore that $\theta(N) \subseteq N$. Thus $N = (N \cap (R * \Delta))R * G$ and the theorem is proved.

We close this section with an example to show that, even if R is a prime ring, the conjugating elements $y \in T$ are required in the conclusion of Lemma 1.3.

EXAMPLE. Let K be a field and consider the vector space M of all matrices over K whose rows and columns are subscripted by the integers Z. In other words, these are the doubly infinite matrices since $Z = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$. Now let R be the set of all row and column finite matrices in M. Then it is trivial to see that R is a prime ring with 1. We define the automorphism σ on R to be a positive shift by 2 in both the row and column directions. Thus for example, if $e_{ij} \in R$ denotes the element with 1 in the (i,j)th position and 0 elsewhere, then $e_{ij}^\sigma = e_{i+2,j+2}$. It is clear that σ is indeed an automorphism of R.

Now let $G = A\langle x\rangle$ be a group with A a normal abelian subgroup and with $G/A \simeq \langle x\rangle$ infinite cyclic. Furthermore, we assume the action of x on A is so chosen that $\Delta(G) = \langle 1\rangle$. It is a trivial matter to construct such examples. Finally we define the action of G on R by the condition that A acts trivially and that x acts like σ. This is clearly well defined and we form the skew group ring RG.

Fix $a \in A$, $a \neq 1$ and define the elements $\alpha = e_{00} - e_{01}a$, $\beta = e_{01} + e_{11}a^{-1} \in RG$. We claim that $\alpha g\beta = 0$ for all $g \in G$, but that $\theta(\alpha)\theta(\beta) \neq 0$. The latter is of course trivial to see. Namely, since $\Delta = \langle 1\rangle$, we have $\theta(\alpha) = e_{00}$, $\theta(\beta) = e_{01}$ and hence $\theta(\alpha)\theta(\beta) = e_{00}e_{01} = e_{01} \neq 0$. For the former, it suffices to show that $(g^{-1}\alpha g)\beta = 0$ for all $g \in G$. Write $g = bx^i$ with $b \in A$. Then since A is abelian and acts trivially on R we have $g^{-1}\alpha g = x^{-i}\alpha x^i$ and thus we may assume that $g = x^i$. If $i = 0$, then we have easily $(x^{-i}\alpha x^i)\beta = \alpha\beta = 0$. On the other hand, if $i \neq 0$, then since

$$x^{-i}\alpha x^{i} = e_{2i,2i} - e_{2i,2i+1}a^{x^i}$$

and since $\{2i, 2i+1\}$ is disjoint from $\{0, 1\}$, we again have $(x^{-i}\alpha x^{i})\beta = 0$. Thus the linear identity $\alpha g \beta = 0$ for all $g \in G$ yields a suitable counterexample and the conjugating elements are definitely needed in Lemma 1.3. Observe that here, for all $y \in G \setminus A$ we have $\theta(\alpha)^y \beta = 0$.

§2. *X*-inner automorphisms

In the last section we reduced the questions of the primeness and semiprimeness of $R * G$ from G itself to the finite normal subgroups of G. The goal here is to further study these finite normal subgroups. Actually this problem has already been effectively handled in [2] using ideas from [3], at least in the case of skew group rings. As we will see, crossed products cause no additional difficulty, so much of this section is essentially an amplification of the work in [2]. For the sake of completeness, we include the proofs of several requisite ring theoretic facts found in [4] and [3]. In this section, R will always denote a prime ring with 1.

We begin by briefly discussing a certain ring of quotients $S = Q_0(R)$ which is defined in [4] essentially as follows. Consider the set of all left R-module homomorphisms $f : {}_RA \to {}_RR$ where A ranges over all nonzero two-sided ideals of R. Two such functions are said to be equivalent if they agree on their common domain, which is a nonzero ideal since R is prime. It is easy to see that this is an equivalence relation. Indeed, what is needed here is the observation that if $f : {}_RA \to {}_RR$ with $Af = 0$ and if f is defined on $b \in R$, then $bf = 0$. This follows since $Ab \subseteq A$ so $0 = (Ab)f = A(bf)$ and hence $bf = 0$ in this prime ring. We let $\hat{f}$ denote the equivalence class of f and we let $S = Q_0(R)$ be the set of all such equivalence classes.

The arithmetic in S is defined in a fairly obvious manner. Suppose $f : {}_RA \to {}_RR$ and $g : {}_RB \to {}_RR$. Then $\hat{f} + \hat{g}$ is the class of $f + g : {}_R(A \cap B) \to {}_RR$ and $\hat{f}\hat{g}$ is the class of the composite function $fg : {}_R(BA) \to {}_RR$. It is easy to see that these definitions make sense and that they respect the equivalence relation. Furthermore the ring axioms are surely satisfied so S is in fact a ring with 1. Finally let $a_r : {}_RR \to {}_RR$ denote right multiplication by $a \in R$. Then the map $a \to \hat{a}_r$ is easily seen to be a ring homomorphism from R into S. Moreover, if $a \neq 0$, then $Ra_r \neq 0$ and hence $\hat{a}_r \neq 0$ by the observation of the preceding paragraph. We conclude therefore that R is embedded isomorphically in S and hence we will view R as a subring of S with the same 1.

As we see below, the reason for studying S is that it is close to R and yet large enough to contain certain needed additional units. The following lemma, essentially contained in [4], lists the closeness properties we require. Additional facts concerning $Q_0(R)$ can be found in [1].

LEMMA 2.1. *Let $S = Q_0(R)$ be as above.*

(i) *If $s \in S$ and $As = 0$ for some nonzero ideal A of R, then $s = 0$.*

(ii) *If $s_1, s_2, \cdots, s_n \in S$, then there exists a nonzero ideal A of R with $As_1, As_2, \cdots, As_n \subseteq R$.*

(iii) *S is a prime ring.*

(iv) *If σ is an automorphism of R, then σ extends uniquely to an automorphism of S.*

(v) *If $C = \mathbf{C}_S(R)$, then C is a field and the center of S.*

PROOF. Suppose $f : {}_RA \to {}_RR$ and $a \in A$. Then $a_r f$ is defined on ${}_RR$ and for all $b \in R$ we have

$$b(a_r f) = (ba)f = b(af) = b(af)_r.$$

Hence $\hat{a}_r \hat{f} = (\widehat{af})_r$ and the map f translates in S to right multiplication by $\hat{f}$. We will use both of these interpretations in the proof with hopefully no confusion.

(i) Let $s \in S$ with $As = 0$. If $s = \hat{f}$, then the above shows that f vanishes on an ideal in its domain and hence $s = \hat{f} = 0$.

(ii) Let $s_1, s_2, \cdots, s_n \in S$ with $s_i = \hat{f}_i$. Then we can surely assume that all f_i are defined on the common domain A and from this we have $As_i = Af_i \subseteq R$ for all i.

(iii) It follows from (i) and (ii) above that every nonzero ideal of S meets R nontrivially. Thus since R is prime, so is S.

(iv) Now let σ be an automorphism of R and let f: ${}_RA \to {}_RR$. Then the map $f^\sigma : {}_RA^\sigma \to {}_RR$ given by $a^\sigma f^\sigma = (af)^\sigma$ is surely a left R-module homomorphism and from this it follows easily that the map $\hat{f} \to \widehat{f^\sigma}$ gives rise to an automorphism of S extending σ. To prove uniqueness of extension, it suffices to show that if τ is an automorphism of S fixing R elementwise, then $\tau = 1$. To this end, let $s \in S$ and let A be a nonzero ideal of R with $As \subseteq R$. Then for all $a \in A$ we have

$$as = (as)^\tau = a^\tau s^\tau = as^\tau.$$

Thus $A(s - s^\tau) = 0$ and (i) implies that $s^\tau = s$.

(v) Finally let $C = \mathbf{C}_S(R)$ and suppose $s \in C$, $s \neq 0$. Since s commutes with R it is clear that $T = \{t \in R \mid ts = 0\}$ is a two sided ideal of R and hence we must have $T = 0$ by (i). Let $A \neq 0$ be an ideal of R with $As \subseteq R$. Then the map $f : {}_RA \to {}_RR$ given by $af = as \in R$ is a left R-module homomorphism which is one to one and onto $Af = B$. Hence there exists an inverse map $g : {}_RB \to {}_RR$ so that $fg = 1$ on A. But observe that $B = Af = As = sA$ is a nonzero two sided ideal of R so $\hat{g} \in S$ is now an inverse of $\hat{f} = s$. Moreover, since s commutes with R, we must have $s^{-1} \in C$ and therefore C is at least a skew field. Finally, conjugation by s induces an automophism of S which is trivial on R. Hence by (iv) the automorphism must also be trivial on S. Thus s is central and the lemma is proved.

The next result shows that S contains the units we need. It is a very special case of the work in [3] and its proof is a minor modification of the proof of [4, theorem 1].

LEMMA 2.2. *Let σ be an automorphism of R and let $a, b \in R$ be fixed nonzero elements. If*

$$arb = br^{\sigma}a^{\sigma}$$

for all $r \in R$, then there exists a unit $s \in S = Q_0(R)$ such that $b = as$ and such that conjugation by s induces the automorphism σ on R.

PROOF. Let $A = RaR$, $B = RbR$ and define the maps $f : A \to B$ and $g : B \to A$ by

$$f : \sum_i x_i a y_i \to \sum_i x_i b y_i^{\sigma}$$

$$g : \sum_i x_i b y_i \to \sum_i x_i a y_i^{\sigma^{-1}}.$$

To see that f is well defined, it suffices to show that $0 = \Sigma_i x_i a y_i$ implies that $0 = \Sigma_i x_i b y_i^{\sigma}$. To this end, suppose $0 = \Sigma_i x_i a y_i$. Then for all $r \in R$ the formula $atb = bt^{\sigma}a^{\sigma}$ yields

$$0 = \left(\sum x_i a y_i\right) rb = \left(\sum x_i b y_i^{\sigma}\right) r^{\sigma}a^{\sigma}$$

and hence $0 = \Sigma\, x_i b y_i^{\sigma}$ since $a^{\sigma} \neq 0$ and R is prime. Similarly if $0 = \Sigma_i x_i b y_i$, then for all $r \in R$ we have

$$0 = \left(\sum x_i b y_i\right) ra^{\sigma} = \left(\sum x_i a y_i^{\sigma^{-1}}\right) r^{\sigma^{-1}}b$$

and we deduce that $0 = \Sigma\, x_i a y_i^{\sigma^{-1}}$. Thus both f and g are well defined and since they are clearly left R-module homomorphisms, we have $\hat{f} = s \in S$ and $\hat{g} \in S$. Furthermore $fg = 1$ on A and $gf = 1$ on B so $\hat{g} = s^{-1}$ and s is a unit in S.

Observe that $a_r f$ is defined on R and for all $x \in R$ we have

$$x(a_r f) = (xa)f = xb = xb_r.$$

Thus $\hat{a}_r\hat{f} = \hat{b}_r$ or equivalently $as = b$. Finally let $c \in R$. Then gc_rf is defined on B and for all $xby \in B$ we have

$$(xby)(gc_rf) = (xay^{\sigma^{-1}})(c_rf) = (xay^{\sigma^{-1}}c)f$$
$$= xbyc^{\sigma} = (xby)c_r^{\sigma}.$$

Thus $\hat{g}\hat{c}_r\hat{f} = \widehat{c_r^{\sigma}}$ so $s^{-1}cs = c^{\sigma}$ and the result follows.

This lemma motivates the following definition due to Kharchenko [3]. An automorphism σ of R is said to be X-inner if and only if it is induced by conjugation by a unit of $S = Q_0(R)$. In other words, these automorphisms arise from those units $s \in S$ with $s^{-1}Rs = R$. If s and t are two such units, then clearly so is st. Thus we see immediately from Lemma 2.1(iv) that the set of all X-inner automorphisms of R is in fact a normal subgroup of the group of all automorphisms of R.

Now let $R * G$ be given. Then $\mathfrak{G}$ acts on R and the elements of U surely act as X-inner automorphisms. Thus since $\mathfrak{G}/U \simeq G$ we see that

$$G_{\mathrm{inn}} = \{x \in G \mid {}^{\bar{x}} \text{ is an } X\text{-inner automorphism of } R\}$$

is a normal subgroup of G.

By Lemma 2.1(iv) the automorphism ${}^{\bar{x}}$ of R extends to a unique automorphism of S which we denote by the same symbol. It then seems reasonable to extend $R * G$ to a crossed product $S * G$ of G over S using the multiplication formula

$$(a\bar{x})(b\bar{y}) = (ab^{\bar{x}^{-1}}t(x, y))\overline{xy}$$

for $a, b \in S$ and $x, y \in G$. Here of course $t : G \times G \to U$ is the given map for $R * G$. In the case of skew group rings it is fairly obvious that this gives rise to an associative multiplication. However for crossed products a certain amount of checking is necessary and this will be done in the first part of the lemma below.

The next two results are applications of Lemma 2.2. They indicate why these X-inner automorphisms are important here and why the structure of $S * G$ is somewhat nicer than that of $R * G$. We fix notation so that $R * G$ is given, $S = Q_0(R)$ and C is the center of S.

LEMMA 2.3. *There exists a unique crossed product $S * G$ extending $R * G$. Let E be the centralizer of S in $S * G$. Then $E \subseteq S * G_{\mathrm{inn}}$, $S * G_{\mathrm{inn}} = S \otimes_C E$ and $E = C^t[G_{\mathrm{inn}}]$, some twisted group algebra of G_{inn} over the field C. Furthermore if H is a subgroup of G_{inn}, then $S * H = S \otimes_C (E \cap (S * H))$ and $E \cap (S * H) = C^t[H] \subseteq C^t[G_{\mathrm{inn}}]$, where the latter is the natural inclusion.*

PROOF. Since automorphisms of R extend uniquely to automorphisms of S by Lemma 2.1(iv), it is clear that the above definition for $S * G$ is the only possible extension of $R * G$. We need only verify the associativity of multiplication. Let us observe first that $\mathfrak{G}$ acts on R and hence the uniqueness of extension implies that we obtain a group action of $\mathfrak{G}$ on S. We use this implictly in the computations below. Let $a, b, c \in S$ and $x, y, z \in G$. Then by definition

$$[(a\bar{x})(b\bar{y})](c\bar{z}) = s_1\overline{xyz}$$

and

$$(a\bar{x})[(b\bar{y})(c\bar{z})] = s_2\overline{xyz}$$

for suitable $s_1, s_2 \in S$. In fact we see easily that

$$s_1 = ab^{\bar{x}^{-1}}c^{\overline{xy}^{-1}t(x,y)^{-1}}r_1$$

and

$$s_2 = ab^{\bar{x}^{-1}}c^{\bar{y}^{-1}\bar{x}^{-1}}r_2$$

where r_1 and r_2 are elements of R independent of a, b and c. Since $\bar{x}\bar{y} = t(x, y)\overline{xy}$, the a, b and c terms in the two expressions are equal. But in the special case when $a = b = c = 1$ the above products occur in the associative ring $R * G$ so surely $r_1 = r_2$. Hence $s_1 = s_2$ and $S * G$ is associative.

Now let E be the centralizer of S in $S * G$. We show first that $E \subseteq S * G_{\mathrm{inn}}$. To this end let $\alpha \in E$ and let $x \in \operatorname{Supp}\alpha$. Say $\alpha = s\bar{x} + \cdots$. Then by Lemma 2.1(i)(ii) there exists $a \in R$ with $a\alpha \in R * G$ and with $x \in \operatorname{Supp} a\alpha$. Since α commutes with ra we have $ara\alpha = a\alpha ra$ for all $r \in R$. Hence the above implies that

$$aras\bar{x} = as\bar{x}ra = asr^{\bar{x}^{-1}}a^{\bar{x}^{-1}}\bar{x}.$$

Thus since $b = as$ is a nonzero element of R we see that the identity

$$arb = br^{\bar{x}^{-1}}a^{\bar{x}^{-1}}$$

holds for all $r \in R$. Therefore Lemma 2.2 implies that conjugation by $\bar{x}^{-1}$ is an X-inner automorphism and thus $x \in G_{\mathrm{inn}}$.

For each $x \in G_{\mathrm{inn}}$ choose a unit $s_x \in S$ inducing the automorphism $^{\bar{x}}$ on R and let $\tilde{x} = s_x^{-1}\bar{x}$. We claim that the elements $\tilde{x}$ for all $x \in G_{\mathrm{inn}}$ form an S-basis for $S * G_{\mathrm{inn}}$ and a C-basis for E. The former is obvious and for the latter we know at least that the $\tilde{x}$'s are C-linearly independent. Observe that each $\tilde{x}$ is a unit in $S * G$ which acts by conjgation on S centralizing all of R. Hence by Lemma

2.1(iv), $\bar{x}$ must centralize all of S and we have $\bar{x} \in E$. Finally suppose $\alpha \in E \subseteq S * G_{\text{inn}}$. Then we can write $\alpha = \Sigma a_x \bar{x}$ with $a_x \in S$ and it is clear that each $a_x \bar{x}$ centralizes S. Thus since $\bar{x}$ is a unit in E we have $a_x \in S \cap E = C$ and we conclude that the elements $\bar{x}$ do indeed form a C-basis for E. Since C is surely central in E we now know that $S * G_{\text{inn}} = S \otimes_C E$.

Note that E is an associative C-algebra with basis $\{\bar{x} \mid x \in G_{\text{inn}}\}$. Furthermore for $x, y \in G_{\text{inn}}$, $\bar{x}\bar{y} \in E$ and $\bar{x}\bar{y} = s\widetilde{xy}$ for some $s \in S$. Thus clearly s must be a nonzero element of C and we deduce that E is isomorphic to $C^t[G_{\text{inn}}]$, some twisted group algebra of G_{inn} over the field C. Moreover by the way this algebra is constructed, the remaining observations on $S * H$ for H a subgroup of G_{inn} are obvious, so the lemma is proved.

We now fix the notation G_{inn}, E and $C^t[G_{\text{inn}}]$ as given in the previous lemma for use in the remainder of this section. The second application of Lemma 2.2 is the key ingredient in the work of [2] on skew group rings.

LEMMA 2.4. *Let H be a subgroup of G and let A be a nonzero ideal of $R * H$. Then there exists a nonzero element $\alpha \in A$ such that $\alpha = a\beta$ for some $a \in R$ and $\beta \in E \cap (S * H)$.*

PROOF. Let $\alpha \neq 0$ be an element of minimal nonzero support size in A. Since we can multiply α by any $\bar{y}$ with $y \in H$ without changing the support size, we may clearly assume that $1 \in \operatorname{Supp} \alpha$. Write $\alpha = \Sigma a_x \bar{x}$ and let $a = a_1$. Then for any $r \in R$, $\gamma = ar\alpha - \alpha ra \in A$ and $|\operatorname{Supp} \gamma| < |\operatorname{Supp} \alpha|$. Thus $\gamma = 0$ and $ar\alpha = \alpha ra$ for all $r \in R$. In particular for all $x \in \operatorname{Supp} \alpha$ we have

$$ara_x \bar{x} = a_x \bar{x} ra = a_x r^{\bar{x}^{-1}} a^{\bar{x}^{-1}} \bar{x}$$

and hence

$$ara_x = a_x r^{\bar{x}^{-1}} a^{\bar{x}^{-1}}.$$

Lemma 2.2 now applies and we conclude that there exists a unit $b_x \in S$ such that $a_x = ab_x$ and with conjugation by b_x inducing the automorphism ${}^{\bar{x}^{-1}}$ on R. In particular, if $\beta = \Sigma b_x \bar{x}$ then $\alpha = a\beta$. Moreover, conjugation by $b_x \bar{x}$ yields an automorphism of S trivial on R so $b_x \bar{x} \in E$ and hence $\beta \in E$. Thus $\beta \in E \cap (S * H)$ and the lemma is proved.

It is perhaps worth noting, as we see from the proof, that any element $\alpha \in A$ with $\operatorname{tr} \alpha \neq 0$ and with α of minimal support size satisfies the above condition. We now wish to exploit the relationship between $R * H$ and $C^t[H]$ for $H \subseteq G_{\text{inn}}$. Let $H \lhd G$ with $H \subseteq G_{\text{inn}}$. Then $\mathfrak{G}$ acts on $R * H$, $S * H$ and S so $\mathfrak{G}$ acts on $E \cap (S * H) = C^t[H]$, the centralizer of S in $S * H$. But observe that U

acts trivially on $C'[H]$ so we see that $G \simeq \mathfrak{G}/U$ in fact acts on $C'[H]$. Thus we can speak of $C'[H]$ as being G-prime or G-semiprime.

LEMMA 2.5. *Let H be a subgroup of G_{inn}.*

(i) *If $H \lhd G$, then $R * H$ is G-prime or G-semiprime, respectively, if and only if $C'[H]$ is G-prime or G-semiprime, respectively.*

(ii) *$R * H$ is prime or semiprime, respectively, if and only if $C'[H]$ is prime or semiprime, respectively.*

PROOF. The proof of (ii) is precisely the same as that of (i) but ignoring all reference to ideals being G-invariant. Thus we consider only (i).

Let A and B be nonzero G-invariant ideals of $C'[H]$ with $AB = 0$. Then since $S * H = S \otimes_C C'[H]$, by Lemma 2.3, we see that $A_1 = SA$ and $B_1 = SB$ are nonzero G-invariant ideals of $S * H$ with $A_1B_1 = 0$. Furthermore, it is apparent from Lemma 2.1(i) (ii) that $A_2 = A_1 \cap (R * H)$ and $B_2 = B_1 \cap (R * H)$ are nonzero G-invariant ideals of $R * H$ with $A_2B_2 = 0$. Thus if $C'[H]$ is not G-prime, then neither is $R * H$. Moreover, by taking $A = B$, we see that if $C'[H]$ is not G-semiprime, then neither is $R * H$.

Now let A and B be nonzero G-invariant ideals of $R * H$ with $AB = 0$. Define

$$\tilde{A} = \{\gamma \in C'[H] \mid I\gamma \subseteq A \text{ for some nonzero ideal } I \subseteq R\}$$

and

$$\tilde{B} = \{\gamma \in C'[H] \mid I\gamma \subseteq B \text{ for some nonzero ideal } I \subseteq R\}.$$

We claim that $\tilde{A}$ and $\tilde{B}$ are nonzero G-invariant ideals of $C'[H]$ with $\tilde{A}\tilde{B} = 0$. We first consider $\tilde{A}$. Suppose $\gamma_1, \gamma_2 \in \tilde{A}$ with $I_1\gamma_1, I_2\gamma_2 \subseteq A$ and let $\delta \in C'[H]$. By Lemma 2.1(ii) there exists a nonzero ideal J of R with $J\delta \subseteq R * H$. Hence since γ_1, γ_2 and δ commute with R we have

$$(I_1 \cap I_2)(\gamma_1 + \gamma_2) \subseteq A,$$

$$I_1J(\gamma_1\delta) = (I_1\gamma_1)(J\delta) \subseteq A$$

and

$$JI_1(\delta\gamma_1) = (J\delta)(I_1\gamma_1) \subseteq A.$$

Thus $\gamma_1 + \gamma_2$, $\gamma_1\delta$, $\delta\gamma_1 \in \tilde{A}$ so $\tilde{A}$ is an ideal of $C'[H]$ which is clearly G-invariant. Moreover, by Lemma 2.4, there exists $\alpha = a\beta \in A$ with $\alpha \neq 0$, $a \in R$ and $\beta \in C'[H]$. Thus since β commutes with R we have

$$(RaR)\beta = R(a\beta)R \subseteq A$$

so $\beta \in \tilde{A}$ and $\tilde{A} \neq 0$. The same is of course true for $\tilde{B}$.

Now let $\gamma_1 \in \tilde{A}$, $\gamma_2 \in \tilde{B}$ with $I_1\gamma_1 \subseteq A$ and $I_2\gamma_2 \subseteq B$. Then

$$I_1I_2(\gamma_1\gamma_2) = (I_1\gamma_1)(I_2\gamma_2) \subseteq AB = 0$$

so since I_1I_2 is a nonzero ideal of R, Lemma 2.1(i) implies that $\gamma_1\gamma_2 = 0$. Thus $\tilde{A}\tilde{B} = 0$ and we see that if $R * H$ is not G-prime then neither is $C^t[H]$. Moreover, by taking $A = B$, we conclude that if $R * H$ is not G-semiprime, then neither is $C^t[H]$. This completes the proof.

The next result is a useful consequence of the above work.

PROPOSITION 2.6. *Let R be a prime ring.*

(i) *If A is a nonzero ideal of $R * G$, then $A \cap (R * G_{\text{inn}}) \neq 0$.*

(ii) *$R * G$ is prime or semiprime, respectively, if and only if $R * G_{\text{inn}}$ (or equivalently $C^t[G_{\text{inn}}]$) is G-prime or G-semiprime, respectively.*

PROOF. Part (i) is an immediate consequence of Lemma 2.4, with $H = G$, and Lemma 2.3.

Suppose A and B are nonzero ideals of $R * G$ with $AB = 0$. Then by (i), $A_1 = A \cap (R * G_{\text{inn}})$ and $B_1 = B \cap (R * G_{\text{inn}})$ are nonzero G-invariant ideals of $R * G_{\text{inn}}$ with $A_1B_1 = 0$. Conversely if A and B are nonzero G-invariant ideals of $R * G_{\text{inn}}$ with $AB = 0$, then $A_1 = (R * G)A$ and $B_1 = (R * G)B$ are nonzero ideals of $R * G$ with $A_1B_1 = 0$. Thus we see that $R * G$ is prime if and only if $R * G_{\text{inn}}$ is G-prime. Moreover, by taking $A = B$ we conclude that $R * G$ is semiprime if and only if $R * G_{\text{inn}}$ is G-semiprime. Finally the equivalence of the above properties of $R * G_{\text{inn}}$ with the corresponding ones of $C^t[G_{\text{inn}}]$ follows from Lemma 2.5(i) with $H = G_{\text{inn}}$.

We now combine the work of this section with the earlier Δ-method results to obtain the following two theorems which we prove together. Theorem 2.8 and its corollary are the main results of this paper on the question of the primeness of $R * G$. Theorem 2.7, which concerns semiprimeness, will be further sharpened in the next section. Note also that in the latter result, the expected condition of G-semiprimeness of $R * H$ is replaced by the simpler condition of ordinary semiprimeness.

THEOREM 2.7. *Let $R * G$ be a crossed product of G over the prime ring R. Then $R * G$ is semiprime if and only if for all finite normal subgroups H of G with $H \subseteq G_{\text{inn}}$ we have $R * H$ (or equivalently $C^t[H]$) semiprime.*

THEOREM 2.8. *Let $R * G$ be a crossed product of G over the prime ring R. Then*

$R * G$ is prime if and only if for all finite normal subgroups H of G with $H \subseteq G_{\mathrm{inn}}$ we have $R * H$ (or equivalently $C^t[H]$) G-prime. In particular, if $\Delta^+(G) \cap G_{\mathrm{inn}} = \langle 1 \rangle$, then $R * G$ is prime.

PROOF. We first consider the prime case. If H is a finite normal subgroup of G with $R * H$ not G-prime, then by Theorem 1.9 $R * G$ is not prime. Conversely if $R * G$ is not prime, then by Theorem 1.9 there exists such a finite normal subgroup H with $R * H$ not G-prime. Suppose in fact that A and B are nonzero G-invariant ideals of $R * H$ with $AB = 0$ and set $H_{\mathrm{inn}} = H \cap G_{\mathrm{inn}} \lhd G$. Then by Lemmas 2.4 and 2.3 we see that $A_1 = A \cap (R * H_{\mathrm{inn}})$ and $B_1 = B \cap (R * H_{\mathrm{inn}})$ are nonzero G-invariant ideals of $R * H_{\mathrm{inn}}$ with $A_1 B_1 = 0$. Thus $H_{\mathrm{inn}} \subseteq G_{\mathrm{inn}}$ and $R * H_{\mathrm{inn}}$ is not G-prime. Lemma 2.5(i) now yields the first part of Theorem 2.8.

Note furthermore that if H is a finite normal subgroup of G with $H \subseteq G_{\mathrm{inn}}$, then $H \subseteq \Delta^+(G) \cap G_{\mathrm{inn}}$. Thus if $\Delta^+(G) \cap G_{\mathrm{inn}} = \langle 1 \rangle$, then $H = \langle 1 \rangle$ and $R * H = R$ is prime. We therefore conclude from the above that $R * G$ is prime in this case.

It is clear that the argument of the first paragraph also shows that $R * G$ is semiprime if and only if for all finite normal subgroups H of G with $H \subseteq G_{\mathrm{inn}}$ we have $R * H$ (or equivalently $C^t[H]$) G-semiprime. But observe that $C^t[H]$ is a finite dimensional C-algebra and hence if it has a nonzero nilpotent ideal, it has a characteristic such ideal, namely its Jacobson radical. Thus $C^t[H]$ is semiprime if and only if it is G-semiprime and hence, by Lemmas 2.5(i) (ii), the same is true for $R * H$. This completes the proof of Theorem 2.7.

COROLLARY 2.9. *Let R be a prime ring with $S = Q_0(R)$ and assume that the crossed product $R * G$ is semiprime. Then $R * G$ is prime if and only if $S * G$ contains no nontrivial central idempotent.*

PROOF. Suppose $e \neq 0, 1$ is a central idempotent in $S * G$. Then $A = e(S * G)$ and $B = (1 - e)(S * G)$ are nonzero ideals of $S * G$ with $AB = 0$. Furthermore, by Lemma 2.1(i) (ii), $A_1 = A \cap (R * G)$ and $B_1 = B \cap (R * G)$ are nonzero ideals of $R * G$ with $A_1 B_1 = 0$. Thus $R * G$ is not prime.

Conversely suppose $R * G$ is not prime. Then by Theorem 2.8 there exists a finite normal subgroup H of G with $H \subseteq G_{\mathrm{inn}}$ such that $C^t[H]$ is not G-prime. But, by Theorem 2.7, since $R * G$ is semiprime, we know that $C^t[H]$ is semiprime. Thus $C^t[H]$ is a finite dimensional semisimple C-algebra. Let A and B be nonzero G-invariant ideals of $C^t[H]$ with $AB = 0$. Since $C^t[H]$ is semisimple, we know that, as a ring, A has an identity element e which is of course a central idempotent in $C^t[H]$. Certainly $e \neq 0, 1$ since $A, B \neq 0$ and

$AB = 0$. Furthermore, e is a characteristic element of A and A is $\mathfrak{G}$-invariant so e commutes with $\mathfrak{G}$. But $e \in C^t[H] \subseteq E$, the centralizer of S in $S * G$, so we conclude that e is a nontrivial central idempotent in $S * G$. The result follows.

We remark that if $G \neq \langle 1 \rangle$ is a finite group, then the integral group ring $Z[G]$ is never prime. Furthermore, by [6, theorem 2.1.8], $Z[G]$ contains no nontrivial idempotents. Of course the rational group ring $Q[G]$ does contain the nontrivial central idempotent $e = 1/|G|\Sigma_{x \in G} x$. We now offer two more examples. The first shows that the G-prime condition in Theorem 2.8 cannot be replaced by ordinary primeness.

EXAMPLE. Let F be a field containing an element ε of prime order p and form the ordinary group ring $F[\langle x \rangle]$ where $\langle x \rangle$ is cyclic of order p. Then $F[\langle x \rangle]$ certainly admits an automorphism σ fixing F with $x^\sigma = \varepsilon x$ and we form the skew group ring $F[\langle x \rangle]\langle y \rangle$ where $\langle y \rangle$ is infinite cyclic and y acts like σ. It is trivial to see that this ring is in fact $F^t[G]$, a twisted group algebra of the group $G = \langle x \rangle \times \langle y \rangle$ over F. Since p is prime, it is clear that $H = \langle x \rangle$ is the unique nonidentity finite normal subgroup of G. Furthermore, it is immediate that $F^t[H] = F[H]$ is G-prime, since it contains the p primitive idempotents

$$e_i = (1/p) \sum_{j=0}^{p-1} (\varepsilon^i x)^j$$

for $i = 0, 1, \cdots, p-1$ and these are permuted transitively by $\mathfrak{G}$. Thus by Theorem 2.8, $F^t[G]$ is prime. On the other hand, $F^t[H] = F[H]$ is certainly not prime.

The second example shows that the G-prime or G-semiprime assumption in Proposition 2.6 cannot be replaced by just ordinary primeness or semiprimeness. It is fairly easy to construct an appropriate skew group ring counterexample for this with G not acting faithfully. However to get a faithful action requires a little more work and we first isolate two necessary facts in the following lemma.

LEMMA 2.10. *Let R be a ring and let W be a group of units in R. Then W acts on R as inner automorphisms by conjugation.*

(i) *Suppose H is a group acting on R and normalizing W. Then the semidirect product WH acts on R with H acting as given and with W acting by conjugation.*

(ii) *Let $\bar{W}$ be an isomorphic copy of W and form the skew group ring $R\bar{W}$ where $\bar{W}$ acts on R as W does. If W is abelian, then $R\bar{W} \simeq R[\bar{W}]$, where the latter is of course the ordinary group ring of $\bar{W}$ over R.*

PROOF. (i) This is fairly obvious and a simple argument is as follows. Form the skew group ring RH. Then H and W are contained in the group of units of

RH and H normalizes W. Thus since $H \cap W = \langle 1 \rangle$, the semidirect product WH exists in RH and the action of WH on RH by conjugation yields the appropriate action on R.

(ii) Let $^{-}: W \to \bar{W}$ denote the isomorphism. For each $x \in W$ define $x^{\sigma} = x^{-1}\bar{x} \in R\bar{W}$. Since W is abelian, $\bar{W}$ centralizes W and hence σ is an isomorphism of W with a group of units in $R\bar{W}$. Indeed for $x, y \in W$ we have

$$x^{\sigma}y^{\sigma} = (x^{-1}\bar{x})(y^{-1}\bar{y}) = (x^{-1}y^{-1})(\bar{x}\bar{y}) = (y^{-1}x^{-1})(\overline{xy}) = (xy)^{\sigma}.$$

Furthermore the elements of W^{σ} are clearly an R-basis for $R\bar{W}$ centralizing the ring R. Thus $R\bar{W} = RW^{\sigma} = R[W^{\sigma}] \simeq R[\bar{W}]$.

Example. Let F be a field of characteristic $p > 0$ containing an element λ of infinite multiplicative order and set $K = F(\zeta)$, the rational function field over F in the indeterminate ζ. Let $R = M_2(K)$ be the simple ring of 2×2 matrices over K and set

$$W = \left\{ \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \,\middle|\, \alpha \in I \right\}$$

where $I = \zeta F[\zeta] \subseteq K$. Then W is an elementary abelian p-group of units of R and since $W \cap \mathbf{Z}(R) = \langle 1 \rangle$ we see that W acts faithfully on R by conjugation.

Let σ be the automorphism of K fixing F and defined by $\zeta^{\sigma} = \lambda\zeta$. Then σ extends in an obvious manner to an automorphism of $R = M_2(K)$ and, since σ normalizes the set I, we see that σ normalizes the group W. By Lemma 2.10(i), the semidirect product $W\langle\sigma\rangle$ acts on R and we consider the skew group ring RG where $G = \bar{W}\langle\bar{\sigma}\rangle$ is an isomorphic copy of $W\langle\sigma\rangle$ which acts in the same way. Observe that R is simple so $Q_0(R) = R$. Thus it follows that every X-inner automorphism of R is an ordinary inner automorphism which must therefore centralize the scalar matrices in R. We conclude from this that $G_{\text{inn}} = \bar{W}$, since $\zeta^{\sigma^n} = \lambda^n\zeta$ implies that every element $x \in G \setminus \bar{W}$ acts nontrivially on K. Moreover since W acts faithfully on R we see that G does also.

Now W is abelian, so it follows from Lemma 2.10 (ii) that $RG_{\text{inn}} = R\bar{W} \simeq R[\bar{W}]$. Hence, since $R = M_2(K)$ we have $RG_{\text{inn}} \simeq M_2(K[\bar{W}])$. But K is a field of characteristic p and $\bar{W}$ is a nonidentity elementary abelian p-group. Thus $K[\bar{W}]$ is not semiprime and therefore neither is RG_{inn}. Finally we explain the choice of I. If $f(\zeta) \in I$ and if $f(\zeta)^{\sigma^n} = f(\zeta)$ for some $n \neq 0$, then since $f(\zeta)$ is a polynomial with zero constant term and since λ has infinite multiplicative order, we see that $f(\zeta) = 0$. Thus every nonzero element of I has infinitely many $\langle\sigma\rangle$-conjugates. It therefore follows easily that $\Delta^{+}(G) = \langle 1 \rangle$ so RG is prime, by Theorem 2.8, even though RG_{inn} is not even semiprime.

We close this section with a remark on the intersection of ideals of $R * G$ with R. Note that G acts on the ring $C'[G_{\text{inn}}]$ and we say that the latter ring is G-simple if and only if it contains no proper G-invariant ideal. Part (ii) of the following of course applies when G is finite.

PROPOSITION 2.11. *Let $R * G$ be a crossed product of G over the prime ring R.*

(i) *Every nonzero ideal of $R * G$ has a nonzero intersection with R if and only if $C'[G_{\text{inn}}]$ is G-simple.*

(ii) *Suppose that $|G_{\text{inn}}| < \infty$. Then every nonzero ideal of $R * G$ has a nonzero intersection with R if and only if $R * G$ is prime.*

PROOF. We first consider (i). Suppose $C'[G_{\text{inn}}]$ is not G-simple and let I be a nonzero G-invariant ideal of that ring. Then, by Lemma 2.3, $S \otimes I$ is a nonzero G-invariant ideal of $S * G_{\text{inn}} = S \otimes_C C'[G_{\text{inn}}]$, where $S = Q_0(R)$. Moreover suppose I' is a complementary C-subspace for I in $C'[G_{\text{inn}}]$ with $1 \in I'$. Then we have the direct sum $S \otimes C'[G_{\text{inn}}] = (S \otimes I) + (S \otimes I')$ and hence $(S \otimes I) \cap S = 0$. Now set $B = (S \otimes I)(S * G)$. Since $S \otimes I$ is G-invariant, B is a two-sided ideal of $S * G$ with $B \cap (S * G_{\text{inn}}) = S \otimes I$. It follows from the latter that $B \neq 0$ and $B \cap S = 0$. Finally if $A = B \cap (R * G)$, then, by Lemma 2.1(i) (ii), A is a nonzero ideal of $R * G$ with $A \cap R = 0$.

Conversely suppose that $C'[G_{\text{inn}}]$ is G-simple and let A be a nonzero ideal of $R * G$. Then, by Lemma 2.4, there exists a nonzero element $\alpha = a\beta \in A$ with $a \in R$ and $\beta \in E \cap (S * G) = C'[G_{\text{inn}}]$. Certainly $(RaR)\beta \subseteq A$ since $R\beta = \beta R$. Let

$$\tilde{A} = \{\gamma \in C'[G_{\text{inn}}] \mid I\gamma \subseteq A \text{ for some nonzero ideal } I \subseteq R\}.$$

Then, as in Lemma 2.5, $\tilde{A}$ is a nonzero G-invariant ideal of $C'[G_{\text{inn}}]$. But $C'[G_{\text{inn}}]$ is G-simple, so we conclude that $\tilde{A} = C'[G_{\text{inn}}]$. Since $1 \in \tilde{A}$ we then have $A \cap R \neq 0$ and (i) is proved.

Finally we consider (ii). Since G_{inn} is a finite group, it is clear from the Wedderburn theorems that $C'[G_{\text{inn}}]$ is G-simple if and only if it is G-prime. The result now follows from (i) above and Proposition 2.6 (ii).

§3. Twisted group rings

In this final section we study twisted group rings with two goals in mind. In the first part, we consider when twisted group algebras of finite groups are semiprime. In conjunction with Theorem 2.7, this then yields our main results on the semiprimeness of $R * G$. In the second part, we sharpen the Δ-lemmas of Section 1 in the case of twisted group rings over prime rings. As an application we determine the center of the classical ring of quotients of a twisted group algebra.

We start by considering finite groups G. Observe that if F is a field then $F^t[G]$ is a finite dimensional F-algebra. In particular, $F^t[G]$ is semiprime if and only if it is semisimple. The following result from [7] is a combination of Maschke's theorem and an observation on von Neumann regularity.

LEMMA 3.1. *Let $F^t[G]$ be a twisted group algebra of the finite group G over the field F and let H be a subgroup of G.*

(i) *If $F^t[G]$ is semisimple, then so is $F^t[H]$.*

(ii) *If $[G:H]\neq 0$ in F and if $F^t[H]$ is semisimple, then so is $F^t[G]$.*

PROOF. (i) Suppose that $F^t[G]$ is semisimple. Then $F^t[G]$ is certainly also von Neumann regular. Hence if $\alpha \in F^t[H]$, then there exists $\beta \in F^t[G]$ with $\alpha\beta\alpha = \alpha$. Thus by applying the projection map π_H we have $\alpha\gamma\alpha = \alpha$ where $\gamma = \pi_H(\beta) \in F^t[H]$. We conclude therefore that $F^t[H]$ is also von Neumann regular and hence also semisimple.

(ii) Suppose now that $F^t[H]$ is semisimple and that $n = [G:H]$ is not zero in F. We show that all $F^t[G]$-modules are completely reducible by the usual averaging process. Let V be a right $F^t[G]$-module and W a submodule. Since $F^t[H]$ is semisimple, there exists an $F^t[H]$-complement for W in V and hence there is an $F^t[H]$-projection map $f: V \to W$.

Let $x_1, x_2, \cdots, x_n$ be a right transversal for H in G. Since $1/n \in F$ we can define $g: V \to W$ by

$$g(v) = (1/n)\sum_1^n f(v\bar{x}_i^{-1})\bar{x}_i.$$

This does indeed map to W since $f(v\bar{x}_i^{-1}) \in W$ and W is an $F^t[G]$-submodule. Let $x \in G$. Then x permutes the right cosets $Hx_1, Hx_2, \cdots, Hx_n$ of H by right multiplication and thus $x_i x \in Hx_{i'}$, where $i \to i'$ is a permutation of the subscripts $\{1, 2, \cdots, n\}$. This implies that $\bar{x}_i\bar{x} = \alpha_i\bar{x}_{i'}$ for some unit $\alpha_i \in F^t[H]$ and since $\bar{x}_i^{-1}\alpha_i = \bar{x}\bar{x}_{i'}^{-1}$ we have

$$g(v)\bar{x} = (1/n)\sum f(v\bar{x}_i^{-1})\bar{x}_i\bar{x} = (1/n)\sum f(v\bar{x}_i^{-1})\alpha_i\bar{x}_{i'}$$

$$= (1/n)\sum f(v\bar{x}_i^{-1}\alpha_i)\bar{x}_{i'} = (1/n)\sum f(v\bar{x}\bar{x}_{i'}^{-1})\bar{x}_{i'} = g(v\bar{x}).$$

Thus g is clearly an $F^t[G]$-module homomorphism. Moreover, if $w \in W$, then $w\bar{x}_i^{-1} \in W$ so $f(w\bar{x}_i^{-1}) = w\bar{x}_i^{-1}$ and $g(w) = w$. We conclude that g is an $F^t[G]$-

projection of V onto W and the kernel of g is an appropriate $F'[G]$-complement for W in V. Thus $F'[G]$ is a completely reducible ring and therefore semisimple.

In particular if F is a field of characteristic 0, then the above lemma with $H = \langle 1 \rangle$ shows that $F'[G]$ is semisimple. This then combines with Theorem 2.7 to yield

THEOREM 3.2. *Let R be a prime ring of characteristic* 0. *Then the crossed product $R * G$ is semiprime.*

Suppose now that F has characteristic $p > 0$ and that G is finite. If P is a Sylow p-subgroup of G then, by Lemma 3.1, $F'[G]$ is semisimple if and only if $F'[P]$ is. Thus we are reduced to considering finite p-groups in characteristic p and we let $JF'[P]$ denote the Jacobson radical of such a twisted group algebra. The next two lemmas are motivated in part by the work of [7].

LEMMA 3.3. *Let $F'[P]$ be a twisted group algebra of a finite p-group P over a field of characteristic p. Then*

(i) *$F'[P]/JF'[P]$ is a purely inseparable field extension of F of finite degree.*

(ii) *$F'[P]$ is commutative if and only if P is abelian.*

(iii) *If $P \subseteq G'$, the commutator subgroup of the finite group G, and if $F'[P] \subseteq F'[G]$, then $F'[P] \simeq F[P]$.*

PROOF. Let $\bar{F}$ denote the algebraic closure of F. Then

$$F'[P] \subseteq \bar{F} \otimes_F F'[P] = \bar{F}'[P]$$

and, by [6, lemma 1.2.10], $\bar{F}'[P] \simeq \bar{F}[P]$.

(i) By [6, lemma 3.1.6], $J\bar{F}[P]$ is a nilpotent ideal with $\bar{F}[P]/J\bar{F}[P] \simeq \bar{F}$. By the isomorphism, the same is true of $\bar{F}'[P]$. Thus if $I = J\bar{F}'[P] \cap F'[P]$, then I is a nilpotent ideal of $F'[P]$ with $F'[P]/I$ an F-subalgebra of $\bar{F}$. It follows from this that $F'[P]/I$ is a finite dimensional field extension of F and hence also that $I = JF'[P]$. Furthermore the field extension is purely inseparable since it is generated by the images of the elements $\bar{x}$ with $x \in P$ and these satisfy $\bar{x}^{p^n} \in F$ for some n.

(ii) It is clear that $F'[P]$ is commutative if and only if $\bar{F} \otimes_F F'[P] \simeq \bar{F}[P]$ is commutative and surely the latter occurs if and only if P is abelian.

(iii) Again we let

$$\mathfrak{G} = \{u\bar{x} \mid u \in U = F \setminus \{0\},\ x \in G\}.$$

Then $U = F \setminus \{0\}$ is central in $\mathfrak{G}$, so $\mathfrak{G}$ is center-by-finite and hence $\mathfrak{G}'$ is finite by

[6, lemma 4.1.4]. Furthermore, since $\mathfrak{G}/U \simeq G$ we have $\mathfrak{G}'/V \simeq G'$ where $V = U \cap \mathfrak{G}'$. Now $P \subseteq G'$ and $\mathfrak{G}'$ is finite, so there exists a finite p-subgroup $Q \subseteq \mathfrak{G}'$ with $QV/V \simeq P$. But $Q \cap V = \langle 1 \rangle$ since F, being a field of characteristic p, has no elements of order p. Thus $Q \simeq P$ and we conclude that for each $x \in P$ there exists a unique $f_x \in V \subseteq F$ with $f_x \bar{x} = \tilde{x} \in Q$. Since $\tilde{x}\tilde{y} = \widetilde{xy}$ and since the elements $\{\tilde{x} \mid x \in P\}$ form an F-basis of $F'[P]$, we deduce that $F'[P] \simeq F[P]$.

Lemma 3.4. *Let $F'[P]$ be a twisted group algebra of a finite p-group P over a field F of characteristic p. Then $F'[P]$ is semisimple if and only if for all elementary abelian central subgroups P_0 of P we have $F'[P_0]$ semisimple.*

Proof. If $F'[P]$ is semisimple, then by Lemma 3.1(i) so is $F'[P_0]$ for all subgroups P_0 of P.

Conversely assume that $F'[P]$ is not semisimple. If P is nonabelian, then we can take P_0 to be a subgroup of order p in $\mathbf{Z}(P) \cap P'$. It follows from Lemma 3.3(iii) that $F'[P_0] \simeq F[P_0]$ and the latter is not semisimple. On the other hand, suppose that P is abelian so that, by Lemma 3.3(ii), $F'[P]$ is commutative. Let $P_0 = \{x \in P \mid x^p = 1\}$ so that P_0 is a central elementary abelian subgroup of P. Since $JF'[P]$ is a nonzero nilpotent ideal, we can choose $\alpha \in JF'[P]$, $\alpha \neq 0$ with $\alpha^p = 0$. Furthermore since $F'[P]$ is commutative, we can multiply α by some $\bar{y}$ with $y \in P$ to guarantee that $\operatorname{tr} \alpha \neq 0$ while still preserving the fact that $\alpha^p = 0$. Say $\alpha = \sum a_x \bar{x}$ with $a_x \in F$, $a_1 \neq 0$. Then, again using the commutativity of the ring, we have

$$0 = \alpha^p = \sum_x a_x^p \bar{x}^p.$$

Hence if $\beta = \pi_{P_0}(\alpha) = \sum_{x^p=1} a_x \bar{x}$, then $\beta \neq 0$ and $\beta^p = 0$. Thus β generates a nontrivial nilpotent ideal in $F'[P_0]$ and $F'[P_0]$ is not semisimple.

It is unfortunately not true that $F'[P]$ is semisimple if and only if for all subgroups P_0 of order p we have $F'[P_0]$ semisimple. An easy example motivated by unpublished work of R. Snider is as follows. Let $F = K(\zeta)$ where $\operatorname{char} K = p$ and let $F'[P]$ be the twisted group algebra of the group $P = \langle x \rangle \times \langle y \rangle$ of order p^2 given by $\bar{x}\bar{y} = \bar{y}\bar{x}$ and $\bar{x}^p = \zeta$, $\bar{y}^p = 1 + \zeta$. If z is a nonidentity element of P then $z = x^i y^j$ for some i, j with $0 \leqq i, j < p$ and $\bar{z}^p = \bar{x}^{pi}\bar{y}^{pj} = \zeta^i(1+\zeta)^j = b$ is not a pth power in F. Hence since $F'[\langle z \rangle]$ is isomorphic to the polynomial ring $F[\eta]$ modulo the principal ideal $(\eta^p - b)$, we see that $F'[\langle \bar{z} \rangle]$ is a field and thus semisimple. On the other hand $\alpha = 1 + \bar{x} - \bar{y}$ satisfies $\alpha^p = 0$ so $F'[P]$ is not semisimple.

Theorem 3.2 is of course our main result on the semiprimeness question if R

has characteristic 0. The following is now our main result on this same question in characteristic p.

THEOREM 3.5. *Let $R * G$ be a crossed product of the group G over the prime ring R of characteristic $p > 0$.*

(i) *If $\Delta^+(G) \cap G_{\text{inn}}$ contains no elements of order p, then $R * G$ is semiprime.*

(ii) *If the commutator subgroup of $\Delta^+(G) \cap G_{\text{inn}}$ contains an element of order p, then $R * G$ is not semiprime.*

(iii) *$R * G$ is semiprime if and only if $R * P$ (or equivalently $C^t[P]$) is semiprime for all finite elementary abelian p-subgroups P of $\Delta^+(G) \cap G_{\text{inn}}$.*

PROOF. Suppose first that $R * G$ is not semiprime. Then by Theorem 2.7 there exists a finite normal subgroup H of G with $H \subseteq \Delta^+(G) \cap G_{\text{inn}}$ and with $R * H$ (or equivalently $C^t[H]$) not semiprime. If P is a Sylow p-subgroup of H, then since C is a field of characteristic p, Lemma 3.1(ii) implies that $C^t[P]$ is not semiprime. Thus, by Lemmas 3.4 and 2.5(ii), P has an elementary abelian subgroup P_0 with $C^t[P_0]$ (or equivalently $R * P_0$) not semiprime. Since $P_0 \subseteq \Delta^+(G) \cap G_{\text{inn}}$ this clearly proves (i) and one direction of (iii).

Now suppose there is a finite p-subgroup P of $\Delta^+(G) \cap G_{\text{inn}}$ with $C^t[P]$ (or equivalently $R * P$) not semiprime. This equivalence is of course a consequence of Lemma 2.5(ii). Since $\Delta^+(G) \cap G_{\text{inn}} \lhd G$, it follows from Lemma 1.2(iii) that there exists a finite normal subgroup H of G with $P \subseteq H \subseteq \Delta^+(G) \cap G_{\text{inn}}$. By Lemma 3.1(i), $C^t[H]$ is not semiprime. Thus by Theorem 2.7, $R * G$ is not semiprime. This clearly yields the opposite direction of (iii).

Finally suppose $(\Delta^+(G) \cap G_{\text{inn}})'$ contains a subgroup P of order p. Then since $\Delta^+(G)$ is locally finite, there exists a finite subgroup H of $\Delta^+(G) \cap G_{\text{inn}}$ with $P \subseteq H'$. But $C^t[P] \subseteq C^t[H]$, so we conclude from Lemma 3.3(iii) that $C^t[P] \simeq C[P]$ is not semisimple. The above therefore implies that $R * G$ is not semiprime, so (ii) follows and the theorem is proved.

Thus we see that the question of the semiprimeness of $R * G$ is answered unambiguously if $\Delta^+(G) \cap G_{\text{inn}}$ is either a p'-group or it contains a nonabelian p-subgroup. In the remaining case, when $\Delta^+(G) \cap G_{\text{inn}}$ is not a p'-group but has all p-subgroups abelian, the result can of course go either way. In view of Lemma 3.3(i), this problem really occurs because C, the center of $Q_0(R)$, can be a nonperfect field. Indeed we have

COROLLARY 3.6. *Let R be a prime ring of characteristic $p > 0$ and assume that the center of $Q_0(R)$ is a perfect field. Then a crossed product $R * G$ is semiprime if and only if $\Delta^+(G) \cap G_{\text{inn}}$ has no elements of order p.*

PROOF. If $\Delta^+(G)\cap G_{\rm inn}$ is a p'-group, then Therorem 3.5(i) implies that $R*G$ is semiprime. On the other hand, suppose $\Delta^+(G)\cap G_{\rm inn}$ contains a subgroup P of order p. Then since C is perfect, Lemma 3.3(i) implies that $JC^t[P]\neq 0$ and Theorem 3.5(iii) yields the result.

This completes our work on the semiprimeness problem. In the remainder of this section we study twisted group rings of infinite groups and sharpen the Δ-lemmas of the first section. This extends some work of [5].

Let $R^t[G]$ be a twisted group ring of G over R. Then we let U^t denote the group of central units of R and we define

$$\mathfrak{G}^t=\{u\bar{x}\mid u\in U^t,\ x\in G\}.$$

Since $t(x,y)\in U^t$ for all $x,y\in G$, we see that $\mathfrak{G}^t$ is a multiplicative subgroup of units of $R^t[G]$. Observe that $\mathfrak{G}^t$ acts by conjugation on $R^t[G]$ and that U^t acts trivially. Thus we have a well defined action of $G\simeq \mathfrak{G}^t/U^t$ on $R^t[G]$.

If $x\in G$, we let $\mathbf{C}^t(x)=\{g\in G\mid \bar{g}\bar{x}=\bar{x}\bar{g}\}$. Then $\mathbf{C}^t(x)$ is a subgroup of G contained in $\mathbf{C}(x)$. Furthermore, if $g\in\mathbf{C}(x)$, then $\bar{g}^{-1}\bar{x}\bar{g}=\lambda_x(g)\cdot\bar{x}$ and $\lambda_x:\mathbf{C}(x)\to U^t$ is a linear character into the center Z of R with kernel precisely $\mathbf{C}^t(x)$.

We let $\Delta^t(G)=\{x\in G\mid [G:\mathbf{C}^t(x)]<\infty\}$. Then $\Delta^t(G)$ is clearly a normal subgroup of G with $\Delta^t(G)\subseteq\Delta(G)$. In fac. if R is prime it is easy to see that $\Delta^+(G)\subseteq\Delta^t(G)\subseteq\Delta(G)$ as follows. If $x\in\Delta^+(G)$, then $[G:\mathbf{C}(x)]<\infty$ and $\mathbf{C}(x)/\mathbf{C}^t(x)$ is isomorphic to the image in $Z\setminus\{0\}$ of λ_x. But if $x^n=1$, then it follows easily that (see for example [6, lemma 1.2.6])

$$\lambda_x(g)^n=\lambda_{x^n}(g)=\lambda_1(g)=1.$$

Since R is prime, Z is an integral domain and has at most n elements of order dividing n. Thus $|\mathbf{C}(x)/\mathbf{C}^t(x)|\leqq n$ and $x\in\Delta^t(G)$. We let $\theta^t:R^t[G]\to R^t[\Delta^t]$ denote the natural projection.

The following lemma explains why we need the additional properties of the set $\mathcal{T}(D)$ given in Lemma 1.3.

LEMMA 3.7. *Suppose Z is a commutative integral domain and let $\lambda_i:G\to Z\setminus\{0\}$ for $i=1,2,\cdots,n$ be n distinct Z-linear characters. Let $\alpha_1,\alpha_2,\cdots,\alpha_n$ belong to a torsion free Z-module V. Suppose that T is a subset of G and that for all $x\in T$*

$$\lambda_1(x)\alpha_1+\lambda_2(x)\alpha_2+\cdots+\lambda_n(x)\alpha_n=0.$$

Then either all $\alpha_i=0$ or there exist $g_1,g_2,\cdots,g_m\in G$ for some m with $\bigcap_i Tg_i=\varnothing$.

PROOF. We proceed by induction on n. If $n = 1$, then certainly either $\alpha_1 = 0$ or $T = \varnothing$. Suppose now that $\alpha_1 \neq 0$. Since $\lambda_1 \neq \lambda_n$ there exists $g \in G$ with $\lambda_1(g) \neq \lambda_n(g)$. Of course g need not belong to T. Set $S = T \cap Tg^{-1}$. If $x \in S$, then $x \in T$ and $xg \in T$ so we have

$$0 = \sum_1^n \lambda_i(xg)\alpha_i = \sum_1^n \lambda_i(x)\lambda_i(g)\alpha_i$$

and

$$0 = \lambda_n(g) \sum_1^n \lambda_i(x)\alpha_i = \sum_1^n \lambda_i(x)\lambda_n(g)\alpha_i.$$

Subtracting then yields

$$0 = \sum_1^{n-1} \lambda_i(x)\cdot(\lambda_i(g) - \lambda_n(g))\alpha_i$$

for all $x \in S$. Since $\alpha_1 \neq 0$, $\lambda_1(g) \neq \lambda_n(g)$ and V is a torsion free Z-module, we see that the above is a nontrivial dependence which holds for all $x \in S$. Hence, by induction, there exist $g_1, g_2, \cdots, g_m \in G$ with $\bigcap_j Sg_j = \varnothing$. Then certainly $(\bigcap_j Tg_j) \cap (\bigcap_j Tg^{-1}g_j) = \varnothing$ and the lemma is proved.

We now obtain two versions of the Δ-reduction which we prove simultaneously.

LEMMA 3.8. *Let R be a prime ring and let $\alpha_1, \alpha_2, \cdots, \alpha_n$, $\beta_1, \beta_2, \cdots, \beta_n \in R^t[G]$. Suppose that*

$$x_1\bar{x}\beta_1 + \alpha_2\bar{x}\beta_2 + \cdots + \alpha_n\bar{x}\beta_n = 0$$

for all $x \in G$. Then

$$\theta(\alpha_1)\beta_1 + \theta(\alpha_2)\beta_2 + \cdots + \theta(\alpha_n)\beta_n = 0,$$

$$\alpha_1\theta(\beta_1) + \alpha_2\theta(\beta_2) + \cdots + \alpha_n\theta(\beta_n) = 0$$

and

$$\theta(\alpha_1)\theta(\beta_1) + \theta(\alpha_2)\theta(\beta_2) + \cdots + \theta(\alpha_n)\theta(\beta_n) = 0.$$

LEMMA 3.9. *Let R be a prime ring and let $\alpha_1, \alpha_2, \cdots, \alpha_n$, $\beta_1, \beta_2, \cdots, \beta_n \in R^t[G]$. Suppose that*

$$x_1\bar{x}\beta_1 + \alpha_2\bar{x}\beta_2 + \cdots + \alpha_n\bar{x}\beta_n = 0$$

for all $x \in G$. Then

$$\theta'(\alpha_1)\beta_1 + \theta'(\alpha_2)\beta_2 + \cdots + \theta'(\alpha_n)\beta_n = 0,$$

$$\alpha_1\theta'(\beta_1) + \alpha_2\theta'(\beta_2) + \cdots + \alpha_n\theta'(\beta_n) = 0$$

and

$$\theta'(\alpha_1)\theta'(\beta_1) + \theta'(\alpha_2)\theta'(\beta_2) + \cdots + \theta'(\alpha_n)\theta'(\beta_n) = 0.$$

PROOF. The third equation in each follows from either of the first two by just applying the appropriate map θ or θ'. The first two equations are just right-left analogues of each other so we will only prove the first one.

Let D be the union of the supports of the finitely many α_i and β_i. Then D is finite, $H = \mathbf{C}_G(D \cap \Delta)$ has finite index in G and we consider the function defined for all $x \in H$ by

$$\tau(x) = \theta(\alpha_1)^{\bar{x}}\beta_1 + \theta(\alpha_2)^{\bar{x}}\beta_2 + \cdots + \theta(\alpha_n)^{\bar{x}}\beta_n.$$

By Lemma 1.3, τ vanishes on $T = \mathcal{T}(D)$.

Observe that H centralizes each $a \in \operatorname{Supp}\theta(\alpha_i)$ and hence for $x \in H$ we have $\bar{a}^{\bar{x}} = \lambda_a(x)\bar{a}$. Thus if we group terms in $\tau(x)$ with the same Z-linear character, then since G acts trivially on R we see immediately that

$$\tau(x) = \lambda_1(x)\gamma_1 + \lambda_2(x)\gamma_2 + \cdots + \lambda_m(x)\gamma_m$$

for suitable distinct linear characters $\lambda_j : H \to Z\setminus\{0\}$ and suitable elements $\gamma_j \in R^t[G]$. But τ vanishes on T and $\bigcap_k Th_k \neq \emptyset$ for all $h_1, h_2, \cdots, h_s \in H$ by Lemma 1.3. Furthermore, since R is prime, $R^t[G]$ is a torsion free Z-module. Thus we conclude from Lemma 3.7 that $\gamma_j = 0$ for all j. In particular

$$\sum_i \theta(\alpha_i)\beta_i = \tau(1) = \sum_j \gamma_j = 0$$

and Lemma 3.8 follows.

Finally observe that if $a \in D \cap \Delta$, then $[H : \mathbf{C}'_H(a)] = |\lambda_a(H)|$. Thus since $[G:H] < \infty$ we see that $a \in \Delta'(G)$ if and only if $|\lambda_a(H)| < \infty$. It therefore follows immediately, since $\Delta' \subseteq \Delta$, that

$$\sum_i \theta'(\alpha_i)\beta_i = \sum_j{}' \gamma_j = 0$$

where the second sum is over all those j with $|\lambda_j(H)| < \infty$. This completes the proof of Lemma 3.9.

As an application we consider the centrof the classical ring of quotients of a twisted group algebra $F^t[G]$ assuming this quotient ring exists. The argument is essentially the same as given in [6] for ordinary group algebras, so we merely sketch it here. For twisted group algebras, the role of Δ is played by Δ'.

Suppose S is a finite subset of $\Delta'(G)$. Then since $\Delta' \lhd G$ and $\Delta' \subseteq \Delta$, we know that $S \subseteq H \lhd G$ where H is a finitely generated subgroup of Δ'. It then follows that $[G : \mathbf{C}'_G(H)] < \infty$ so $[H : \mathbf{C}'_H(H)] < \infty$. Here of course

$$\mathbf{C}'_G(H) = \{g \in G \mid \bar{g}\bar{h} = \bar{h}\bar{g} \text{ for all } h \in H\}.$$

Now $\mathbf{C}'_H(H)$ has a characteristic torsion free abelian subgroup A of finite index. It then follows easily that $F^t[A]$ is a central integral domain in $F^t[H]$. Moreover the ring of fractions $F^t[A]^{-1}F^t[H]$ is clearly some twisted group algebra of the finite group H/A over the field $F^t[A]^{-1}F^t[A]$. Finally the action of G on $F^t[H]$ is given by the finite group $G/\mathbf{C}'_G(H)$. With these observations, the proof of [6, lemma 4.4.4] goes over to yield

LEMMA 3.10. *Let $\alpha \in F^t[G]$.*

(i) *If $\alpha \in \mathbf{Z}(F^t[G])$, then α is a zero divisor in $F^t[G]$ if and onlf ip is a zero divisor in $\mathbf{Z}(F^t[G])$.*

(ii) *If α is not a zero divisor in $F^t[G]$, then there exists $\gamma \in F^t[G]$ such that $\theta'(\gamma\alpha)$ is central in $F^t[G]$ and not a zero divisor.*

We should remark that $\mathbf{Z}(F^t[G]) \subseteq F^t[\Delta']$. Now given Lemmas 3.9 and 3.10, the proof of [6, theorem 4.4.5] goes over immediately to yield

THEOREM 3.11. *Let F be a field and let the twisted group algebra $F^t[G]$ be an Ore ring. Then the center of the classical ring of quotients $Q_c(F^t[G])$ is equal to $Q_c(\mathbf{Z}(F^t[G]))$.*

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