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THE PROBLEM OF ENVELOPES FOR BANACH SPACES

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ABSTRACT

Let X be a Banach space. A Banach space Y is an envelope of X if (1) Y is finitely representable in X; (2) any Banach space Z finitely representable in X and of density character not exceeding that of Y is isometric to a subspace of Y. Lindenstrauss and Pelczynski have asked whether any separable Banach space has a separable envelope. We give a negative answer to this question by showing the existence of a Banach space isomorphic to l_2, which has no separable envelope. A weaker positive result holds: any separable Banach space has an envelope of density character $\leqq \aleph_1$ (assuming the continuum hypothesis).

The aim of this paper is to prove the following result (relevant definitions appear below):

THEOREM 1. *For any $\varepsilon > 0$, there exists a Banach space $1 + \varepsilon$-isomorphic to l_2 and which has no separable envelope.*

This theorem gives a negative answer to a question of Lindenstrauss and Pelczynski ([3, problem 8]). There is still a result in the positive direction:

THEOREM 2. *Assume the continuum hypothesis; then any Banach space of density character at most $\aleph_1$ has an envelope of density character at most $\aleph_1$.*

We first recall some definitions.

DEFINITION 1. Let E, F be Banach spaces. F is *finitely representable* in E if for any finite dimensional subspace A of F and any $\varepsilon > 0$, there is a subspace B of E which is $1 + \varepsilon$-isomorphic to A.

DEFINITION 2. Let E be a Banach space; the *density character* of E is the smallest cardinal κ such that there exists a dense subset of E of cardinality κ.

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DEFINITION 3. Let E, F be Banach spaces; F is an *envelope* of E if:

i) F is finitely representable in E,

ii) any Banach space finitely representable in E and whose density character does not exceed that of F is isometric to a subspace of F.

In [3], Lindenstrauss and Pelczynski have shown that $L_p(0,1)$ is an envelope of l_p and have asked whether a separable Banach space has a separable envelope.

The notion of ultrapower [2] provides a nice approach to study finite representability: the following result was already observed in [4] (see also [5]), and will be used throughout the paper:

PROPOSITION 1. *F is finitely representable in E if and only if F is isometric to a subspace of some ultrapower of E.*

1. A separable Banach space with no separable envelope

Before we start building the counter example we state two more results that can be proved by the same method. These results may give an indication on the ideas that have led us.

THEOREM 3. *For any $\varepsilon > 0$, there exists a Banach space E, $1+\varepsilon$-isomorphic to l_2, such that if $\mathcal{U}$ is an ultrafilter on a set Θ, i and $-i$ are the only isometric embedding from E into $E^\Theta/\mathcal{U}$ (where i denotes the canonical embedding).*

To state the other result, we need one more definition.

DEFINITION 4. Let $(x_n)_{n\in N}$ be a sequence of elements of a given Banach space E.

i) $(x_n)_{n\in N}$ is *norm-indiscernible* if for any finite set of real numbers $\lambda_1,\cdots,\lambda_k$ and for any increasing sequence of integers $n_1<\cdots<n_k$ the following equality holds: $\|\lambda_1x_1+\cdots+\lambda_kx_k\| = \|\lambda_1x_{n_1}+\cdots+\lambda_kx_{n_k}\|$.

ii) $(x_n)_{n\in N}$ is *norm-indiscernible and symmetric* if the above equality holds for any finite set of integers (not necessarily increasing).

D. Dacunha Castelle and J. L. Krivine have shown that in any L_p-space $(1\leq p<\infty)$ any norm-indiscernible sequence is symmetric.

THEOREM 4. *For any $\varepsilon>0$ there exists a Banach space E, $1+\varepsilon$-isomorphic to l_2, and a sequence $(x_n)_{n\in N}$ in E which is norm-indiscernible but not symmetric.*

We first give the proof of Theorem 1. We start with the Hilbert space l_2. l_2 is endowed with the usual inner product

$$(x, y) = \sum_{n=0}^{\infty} x_n y_n .$$

The euclidean norm is defined by

$$|x| = \sqrt{(x, x)}.$$

The unit ball of l_2 will be denoted by B and the unit sphere by S. We will use quite freely the terminology coming from elementary geometry. Thus, a line will be any one-dimensional subspace of l_2; a plane any two-dimensional subspace; $L(x)$ will denote the line spanned by a non zero element x, $P(u, v)$ the plane spanned by two independent elements u, v. The angle α of two elements x, y is defined by $\cos \alpha = (x, y)/|x||y|$, $0 \leqq \alpha \leqq \pi$. If e is an element of norm 1 and if γ is a real number, $0 < \gamma < 1$, the *facet* $F(e, \gamma)$ is the set of elements y in B such that $(e, y) = 1 - \gamma$. Finally if x and x' are two elements of a given space, we denote by $[x, x']$ the set $\{y : y = \lambda x + (1 - \lambda)x', 0 \leqq \lambda \leqq 1\}$.

We pick a sequence $\delta_0, \delta_1, \cdots, \delta_n, \cdots$ of positive real numbers such that

$\delta_n > \delta_{n+1} \qquad n \in \boldsymbol{N}$

$\lim_{n \to \infty} \delta_n = \delta > 0$

δ_0 is small enough; a precise version of this statement is $(1 - \delta_0)^{-1} \leqq 1 + \varepsilon$ and $((\sqrt{2}/2) + 4\sqrt{\delta_0})(1 - \delta_0)^{-2} \leqq \cos\left(\frac{\pi}{4} - \frac{\pi}{100}\right)$.

We let $e_0, e_1, \cdots, e_n, \cdots$ denote the unit vector basis of l_2.

We let $(q_i)_{i \in \boldsymbol{N}}$ be an enumeration (without repetition) of the set of rational numbers. We let H be the set of pairs (i, j) such that $i < j$ and $q_i < q_j$.
We let

$$c_{ij} = \frac{e_i + e_j}{\sqrt{2}}$$

$$d_{ij} = \frac{e_i - e_j}{\sqrt{2}}.$$

We let K be the set of elements x of l_2 such that

1) $x \in B$
2) $-1 + \delta_n \leqq (x, e_n) \leqq 1 - \delta_n \quad n \in \boldsymbol{N}$
3) $-1 + \delta \leqq (x, c_{ij}) \leqq 1 - \delta \quad (i, j) \in H$
4) $-1 + \delta \leqq (x, d_{ij}) \leqq 1 - \delta \quad (i, j) \in H.$

Clearly $(1 - \delta_0)B \subseteq K \subseteq B$; also K is an intersection of convex sets, therefore K is convex; finally $-K = K$ i.e. K is symmetric. Therefore K is the unit ball of a new norm defined by

$$\|x\| = (\sup\{\lambda : \lambda x \in K\})^{-1}.$$

The new norm is equivalent to the other one; more precisely we have

$$|x| \leqq \|x\| \leqq (1-\delta_0)^{-1}|x|.$$

We claim the space E equal to l_2 endowed with the new norm $\|x\|$ has no separable envelope and is $1+\varepsilon$-isomorphic to l_2.

We first make some remarks.

1. The unit sphere Σ of E is the union of the facets $F(e_i, \delta_i)$, $F(-e_i, \delta_i)$, $i \in \mathbf{N}$, of the facets $F(c_{ij}, \delta)$, $F(-c_{ij}, \delta)$, $F(d_{ij}, \delta)$, $F(-d_{ij}, \delta)$, $(i,j) \in H$ and of the elements x of S such that the line $L(x)$ does not meet any of those facets.

2. If P is a plane and F is a facet $F = F(e, \gamma)$, then either $P \cap F$ is empty or there exist two elements of S, x and x' such that $P \cap F$ is the set of elements $[x, x']$; it is easy to see that in this case $|x - x'| \leqq 2\sqrt{2\gamma - \gamma^2}$ (this is the euclidean diameter of F).

Lemma 1. *A plane P meets at most 8 of the facets of Σ.*

Let F and F' be two distinct facets of Σ, $F = F(u, \gamma)$, $F' = F(u', \gamma')$. Assume $x \in F$, $y \in F'$. We have $(x, y) - (u, u') = ((x, y) - (x, u')) + ((x, u') - (u, u'))$ so that $(x, y) \leqq (u, u') + |y - u'| + |x - u|$ but it is easy to see that

$$|y - u'| \leqq \sqrt{2\gamma} < 2\sqrt{\delta_0}, \quad \text{similarly} \quad |x - u| < 2\sqrt{\delta_0}$$

so that $(x, y) < (u, u') + 4\sqrt{\delta_0}$.

But (u, u') is at most $\sqrt{2}/2$ so that

$$(x, y) < \frac{\sqrt{2}}{2} + 2\sqrt{\delta_0}.$$

If we let $(x, y)/|x| \cdot |y| = \cos\alpha$, $0 \leqq \alpha \leqq \pi$, then α is not much smaller than $\pi/4$; more precisely

$$\alpha \geqq \frac{\pi}{4} - \frac{\pi}{100}.$$

Now assume P is a plane which contains nine elements belonging to nine distinct facets of Σ; two of these elements are such that their angle is at most $2\pi/9$; we get $(\pi/4) - (\pi/100) \leqq 2\pi/9$; contradiction.

In order to describe the intersection of Σ with a plane, we introduce some notation: assume u, v are elements of P such that $|u| = |v| = 1$, $(u, v) = 0$; assume $\alpha_0, \alpha'_0, \alpha_1, \alpha'_1, \cdots, \alpha_7, \alpha'_7$ are real numbers such that

i) $\alpha_0 = 0$

ii) $\alpha_k \leqq \alpha'_k$ $\qquad k = 0, \cdots, 7$

iii) $(\pi/8)+\alpha'_k \leqq \alpha_{k+1}$ $\quad k=0,\cdots,7$

iv) $0 \leqq \alpha_k < 2\pi$ $\quad k=0,\cdots,7$

v) $0 \leqq \alpha'_k < 2\pi$ $\quad k=0,\cdots,7$

(by α_8 we mean 2π).

Let $x_k=(\cos\alpha_k)u+(\sin\alpha_k)v$; $\Sigma(u,v;\alpha_0,\alpha'_0,\cdots,\alpha_7,\alpha'_7)$ will denote the union of the following sets:

$$X_k=[x_k,x'_k] \qquad k=0,\cdots,7$$
$$Y_k=\{y: y=(\cos\beta)u+(\sin\beta)v, \alpha'_k \leqq \beta \leqq \alpha_{k+1}\} \qquad k=0,\cdots,7.$$

(Recall that $[x_k,x'_k]=\{x: x=\lambda x_k+(1-\lambda)x'_k, 0\leqq\lambda\leqq 1\}$.)
Clearly, if P is a plane, $\Sigma\cap P$ is of the form $\Sigma(u,v;\alpha_0,\alpha'_0,\cdots,\alpha_7,\alpha'_7)$ for some elements u,v and some real numbers $\alpha_0,\alpha'_0,\cdots,\alpha_7,\alpha'_7$, satisfying conditions i) to v).

We now turn to ultrapowers of E. Let $\mathcal{U}$ be an ultrafilter on a set Θ. We recall that the ultrapower $E^\Theta/\mathcal{U}$ is the quotient space Π_0/N where $\Pi_0=\{(x_\theta)_{\theta\in\Theta}: x_\theta\in E$ and for some λ, $\|x_\theta\|\leqq\lambda\}$ and $N=\{(x_\theta)_{\theta\in\Theta}: \lim_{\mathcal{U}}\|x_\theta\|=0\}$. The norm on $E^\Theta/\mathcal{U}$ is computed via the formula

$$\|(x_\theta)_{\theta\in\Theta}\|=\lim_{\mathcal{U}}\|x_\theta\|.$$

Therefore $E^\Theta/\mathcal{U}$ is $l_2^\Theta/\mathcal{U}$ endowed with a new norm equivalent ot the euclidean norm and satisfying more precisely for $x=(x_\theta)_{\theta\in\Theta}$

$$|x|\leqq\|x\|\leqq(1-\delta_0)^{-1}|x|.$$

This shows that it makes sense to speak of orthogonal vectors, length, angle $\cdots$ in $E^\Theta/\mathcal{U}$.

If $(C_\theta)_{\theta\in\Theta}$ are subsets of E then $\Pi_{\theta\in\Theta}C_\theta/\mathcal{U}$ is the set of elements x which can be written $(x_\theta)_{\theta\in\Theta}$ with $x_\theta\in C_\theta$.

LEMMA 2. *Let P be a plane in $l_2^\Theta/\mathcal{U}$; the intersection of P with the unit sphere of $E^\Theta/\mathcal{U}$ is of the form $\Sigma(u,v;\alpha_0,\alpha'_0,\cdots,\alpha_7,\alpha'_7)$.*

Observe first that P is the ultraproduct of $\Pi_{\theta\in\Theta}P_\theta/\mathcal{U}$ for a family of planes P_θ, $P_\theta\subseteq l_2$. If x is an element of P such that $\|x\|=1$ then we may replace x_θ by $x_\theta/\|x_\theta\|$ or by a fixed element of norm 1 if $x_\theta=0$ so that if Σ_P is the unit sphere of P, then $\Sigma_P=\Pi_{\theta\in\Theta}\Sigma\cap P_\theta/\mathcal{U}$. $P_\theta\cap\Sigma$ is $\Sigma(u_\theta,v_\theta;\alpha_{\theta 0},\alpha'_{\theta 0},\cdots,\alpha_{\theta 7},\alpha'_{\theta 7})$. For any element $x=(x_\theta)_{\theta\in\Theta}$ such that

$$x_\theta=\lambda_\theta u_\theta+\mu_\theta v_\theta,$$

we have

$$x = \left(\lim_{\mathcal{U}} \lambda_\theta\right) u + \left(\lim_{\mathcal{U}} \mu_\theta\right) v$$

where $u = (u_\theta)_{\theta \in \Theta}$, $v = (v_\theta)_{\theta \in \Theta}$.
It follows easily that

$$\Sigma_P = \Sigma(u, v\,; \alpha_0, \alpha'_0, \cdots, \alpha_7, \alpha'_7)$$

where

$$\alpha_k = \lim_{\mathcal{U}} \alpha_{\theta k}, \quad \alpha'_k = \lim_{\mathcal{U}} \alpha'_{\theta k}.$$

Furthermore if

$$x_{\theta k} = (\cos \alpha_{\theta k}) u_\theta + (\sin \alpha_{\theta k}) v_\theta$$

and

$$x'_{\theta k} = (\cos \alpha'_{\theta k}) u_\theta + (\sin \alpha'_{\theta k}) v_\theta$$

as above, then

$$x_k = (x_{\theta k})_{\theta \in \Theta} = (\cos \alpha_k) u + (\sin \alpha_k) v$$

$$x'_k = (x'_{\theta k})_{\theta \in \Theta} = (\cos \alpha'_k) u + (\sin \alpha'_k) v$$

and

(*)
$$|x_k - x'_k| = \lim_{\mathcal{U}} |x_{\theta k} - x'_{\theta k}|.$$

Before we state the next lemma, we recall that there is a canonical embeddin from E into $E^\Theta/\mathcal{U}$ (which maps x onto the element $i(x)$ of $E^\Theta/\mathcal{U}$ given by th constant function equal to x). We still write $e_0, e_1, \cdots, e_n, \cdots$ instead o $i(e_0), i(e_1), \cdots, i(e_n)$. We let $\tilde{e}_n = e_n(1 - \delta_n)$; similarly, we let, for $(i, j) \in H$ $\tilde{c}_{ij} = c_{ij}(1 - \delta)$, $\tilde{d}_{ij} = d_{ij}(1 - \delta)$.

We also recall that if x is an element of the facet $F(e_n, \delta_n)$ of E, the $|x - \tilde{e}_n| \leqq \sqrt{2\delta_n - \delta_n^2}$; we let $\gamma_n = \sqrt{2\delta_n - \delta_n^2}$; γ_n is a strictly decreasing sequ ence; finally, we observe that for any n, there exists x_n, x'_n in $F(e_n, \delta_n)$ such tha $[x_n, x'_n] \subseteq F(e_n, \delta_n)$, $|x_n - x'_n| = 2\gamma_n$, and $(x_n + x'_n)/2 = \tilde{e}_n$.

LEMMA 3. *Let x and x' be two elements of $l_2^\Theta/\mathcal{U}$ such that $[x, x']$ is included i the unit sphere of $E^\Theta/\mathcal{U}$ and $|x - x'| \geqq 2\gamma_n$; then, there exists an integer m $0 \leqq m \leqq n$, such that*

$$\sup(|x - \tilde{e}_m|, |x' - \tilde{e}_m|) \leqq \gamma_m;$$

or

$$\sup(|x + \tilde{e}_m|, |x' + \tilde{e}_m|) \leqq \gamma_m.$$

Let P be the plane spanned by x and x'; if we keep the notations of the proof of the preceding lemma, the hypotheses mean that

$$[x, x'] \subseteq \Sigma_P; \quad |x - x'| \geqq 2\gamma_n.$$

The inclusion $[x, x'] \subseteq \Sigma_P$ shows that for some integer k, $[x, x'] \subseteq [x_k, x'_k]$, $0 \leqq k \leqq 7$. Now equality $(*)$ tells us that

$$2\gamma_n \leqq |x_k - x'_k| = \lim_{\mathcal{U}} |x_{\theta k} - x'_{\theta k}|$$

so that $X = \{\theta : |x_{\theta k} - x'_{\theta k}| > 2\gamma_{n+1}\} \in \mathcal{U}$.
Now if $\theta \in X$, $[x_{\theta k}, x'_{\theta k}]$ can only be included in one of the facets $F(\pm e_m, \delta_m)$, $0 \leqq m \leqq n$ (otherwise, we would have $|x_{\theta k} - x'_{\theta k}| \leqq 2\gamma_{n+1}$). It follows that there exists an integer $m, 0 \leqq m \leqq n$, such that

$$\{\theta : [x_{\theta k}, x'_{\theta k}] \subseteq F(e_m, \delta_m)\} \in \mathcal{U},$$

or

$$\{\theta : [x_{\theta k}, x'_{\theta k}] \subseteq F(-e_m, \delta_m)\} \in \mathcal{U}.$$

We only deal with the first case; we get:

$$\{\theta : |x_{\theta k} - \tilde{e}_m| \leqq \gamma_m\} \in \mathcal{U}$$

$$\{\theta : |x'_{\theta k} - \tilde{e}_m| \leqq \gamma_m\} \in \mathcal{U}.$$

In the ultrapower this gives

$$|x_k - \tilde{e}_m| \leqq \gamma_m$$

$$|x'_k - \tilde{e}_m| \leqq \gamma_m.$$

But for some λ, $0 < \lambda < 1$, $x = \lambda x_k + (1 - \lambda)x'_k$; therefore

$$|x - \tilde{e}_m| = |\lambda(x_k - \tilde{e}_m) + (1 - \lambda)(x'_k - \tilde{e}_m)| \leqq \gamma_m,$$

similarly $|x' - \tilde{e}_m| \leqq \gamma_m$.

We now study isometric embeddings from E into $E^\Theta / \mathcal{U}$; this means that we try to forget the euclidean structure; the next lemma shows that we are actually able to recapture it.

LEMMA 4. *Let ϕ be an isometry from E into $E^\Theta / \mathcal{U}$, then, ϕ is an isometry from l_2 into $l_2^\Theta / \mathcal{U}$.*

PROOF. Let P be a plane in $l_2^{\Theta}/\mathcal{U}$. The unit sphere of P (for the norm of $E^{\Theta}/\mathcal{U}$) is $\Sigma_P = \Sigma(u, v; \alpha_0, \alpha'_0, \cdots, \alpha_7, \alpha'_7)$. Clearly, from the fact that $\alpha_{k+1} \geqq \alpha_k + (\pi/8)$, it follows that Σ_P has infinitely many elements x such that $|x| = \lim_{\mathcal{U}} |x_\theta| = 1$. We claim that the restriction to P of the norm of $l_2^{\Theta}/\mathcal{U}$ is the unique inner-product norm on P whose unit sphere has at least 21 elements in common with Σ_P. Indeed, the unit sphere of any other inner product norm is an ellipse and therefore can have at most 20 points in common with Σ_P (at most 4 with the circle and two with any facet).

Now if $|\phi(x)| \neq |x|$, then if Q is any plane in E such that $x \in Q$, two distinct inner-product norms can be defined on $P = \phi(Q)$:

$$N_1(y): y \to |y| = N_1(y)$$

$$N_2(y): y \to |\phi^{-1}(y)| = N_2(y).$$

Let S_1 and S_2 be the unit spheres of N_1, N_2 respectively. Clearly, S_1 has infinitely many points in common with Σ_P. $\phi^{-1}(S_2)$ has infinitely many points in common with the unit sphere of Q (for the norm of E), therefore as ϕ is an isometry, S_2 has infinitely many points in common with Σ_P; contradiction.

LEMMA 5. *Let ϕ be an isometry from E into $E^{\Theta}/\mathcal{U}$; then for every integer n, $\phi(e_n) = \pm e_n$.*

PROOF. Assume the lemma is false and let n be the first integer such that $\phi(e_n) \neq \pm e_n$. Let x_n and x'_n be such that

$$[x_n, x'_n] \subseteq F(e_n, \delta_n) \subseteq \Sigma$$

$$|x_n - x'_n| = 2\gamma_n; \quad \frac{x_n + x'_n}{2} = \tilde{e}_n.$$

The set $[\phi(x_n), \phi(x'_n)]$ is included in the unit sphere of Σ. By Lemma 4,

$$|\phi(x_n) - \phi(x'_n)| = 2\gamma_n.$$

We now may apply Lemma 3; it follows that for some m, $0 \leqq m \leqq n$, either

$$|\phi(x_n) - \tilde{e}_m| \leqq \gamma_m \quad \text{and} \quad |\phi(x'_n) - \tilde{e}_m| \leqq \gamma_m$$

or

$$|\phi(x_n) + \tilde{e}_m| \leqq \gamma_m \quad \text{and} \quad |\phi(x'_n) + \tilde{e}_m| \leqq \gamma_m.$$

If $m < n$, then by our choice of n we get

$$\phi(\tilde{e}_m) = \pm \tilde{e}_m$$

so that either

$$|\phi(x_n) - \phi(\tilde{e}_m)| \leqq \gamma_m$$

or

$$|\phi(x_n) + \phi(\tilde{e}_m)| \leqq \gamma_m;$$

this gives

$$|x_n - \tilde{e}_m| \leqq \gamma_m$$

or

$$|x_n + \tilde{e}_m| \leqq \gamma_m,$$

hence

$$|\tilde{e}_n - \tilde{e}_m| \leqq \gamma_m + \gamma_n \quad \text{i.e.} \quad |e_n - e_m| \leqq \gamma_m + \gamma_n + \delta_m + \delta_n$$

or

$$|e_n + e_m| \leqq \gamma_m + \gamma_n \quad \text{i.e.} \quad |e_n + e_m| \leqq \gamma_m + \gamma_n + \delta_m + \delta_n.$$

But $\delta_m + \delta_n + \gamma_m + \gamma_n \leqq 6\sqrt{\delta_0} \leqq 1$ and $|e_n \pm e_m| = \sqrt{2}$; contradiction. So, for example,

$$|\phi(x_n) - \tilde{e}_n| \leqq \gamma_n$$

and also

$$|\phi(x'_n) - \tilde{e}_n| \leqq \gamma_n.$$

The following equality is well known and holds in any Hilbert space:

$$|\phi(x_n) + \phi(x'_n) - 2\tilde{e}_n|^2 + |\phi(x_n) - \phi(x'_n)|^2 = 2(|\phi(x_n) - \tilde{e}_n|^2 + |\phi(x'_n) - \tilde{e}_n|^2)$$

if

$$|\phi(x_n) - \phi(x'_n)| = 2\gamma_n$$

$$|\phi(x_n) - \tilde{e}_n| \leqq \gamma_n$$

$$|\phi(x'_n) - \tilde{e}_n| \leqq \gamma_n.$$

It implies

$$|\phi(x_n) + \phi(x'_n) - 2\tilde{e}_n| = 0$$

so that

$$\phi(\tilde{e}_n) = \frac{\phi(x_n + x'_n)}{2} = \tilde{e}_n,$$

which implies,

$$\phi(e_n) = e_n;$$

in the other case, a similar argument yields $\phi(e_n) = -e_n$.

We are now going to define many distinct spaces finitely representable in E; we first state an easy lemma.

LEMMA 6. *Let $i < j$. Then either $(i,j) \in H$ and $\| c_{ij} \| = \| d_{ij} \| = (1-\delta)^{-1}$ or $(i,j) \notin H$ and $\| c_{ij} \| = \| d_{ij} \| = 1$.*

LEMMA 7. *For any $\xi \in \mathbf{R} - \mathbf{Q}$ there exists an ultrafilter $\mathcal{U}_\xi$ on $\mathbf{N}$, and an element a_ξ of $E^{\mathbf{N}} / \mathcal{U}_\xi$ such that*

$$\| a_\xi \pm e_i \| = \sqrt{2}(1-\delta)^{-1} \quad \text{if} \quad q_i < \xi$$

$$\| a_\xi \pm e_j \| = \sqrt{2} \qquad \text{if} \quad q_j > \xi.$$

PROOF. Let $\mathcal{U}_\xi$ be an ultrafilter on $\mathbf{N}$ such that

$$\lim_{\mathcal{U}_\xi} q_i = \xi.$$

In the space $E^{\mathbf{N}} / \mathcal{U}_\xi$ we let a_ξ be the element given by $(e_n)_{n \in \mathbf{N}}$. Let i be a fixed integer and assume $q_i < \xi$ then $\{ j : q_i < q_j \text{ and } i < j \} \in \mathcal{U}_\xi$ so that by Lemma 6, $\| a_\xi + e_i \| = \lim_{\mathcal{U}_\xi} (\| e_j + e_i \|)_{j \in \mathbf{N}} = \lim_{\mathcal{U}_\xi} (\sqrt{2} \| c_{ij} \|)_{j \in \mathbf{N}} = \sqrt{2}(1-\delta)^{-1}$.
Similarly, $\| a_\xi - e_i \| = \sqrt{2}(1-\delta)^{-1}$. On the other hand if j is a fixed integer and $q_j > \xi$, then $\{ i : q_i < q_j \text{ and } i > j \} \in \mathcal{U}_\xi$ so that by Lemma 6: $\| a_\xi + e_j \| = \lim_{\mathcal{U}_\xi} (\| e_i + e_j \|)_{i \in \mathbf{N}} = \lim_{\mathcal{U}_\xi} (\sqrt{2} \| c_{ij} \|)_{i \in \mathbf{N}} = \sqrt{2}$; similarly $\| a_\xi - e_j \| = \sqrt{2}$.

We are now able to prove Theorem 1:

CLAIM. *E is $1+\varepsilon$-isomorphic to l_2 and has no separable envelope.*

PROOF OF CLAIM. We have for any x in E

$$| x | \leqq \| x \| \leqq (1-\delta_0)^{-1} | x |,$$

so that E is $(1-\delta_0)^{-1}$ isomorphic to l_2; if $(1-\delta_0)^{-1} \leqq 1+\varepsilon$, E is $1+\varepsilon$-isomorphic to l_2.

Assume E has a separable envelope F. F is a subspace of an ultrapower of E, $E^{\Theta} / \mathcal{U}$. Let $(\mathcal{U}_\xi)_{\xi \in \mathbf{R}-\mathbf{Q}}$ be ultrafilters on $\mathbf{N}$, a_ξ be distinguished elements in $E^{\mathbf{N}} / \mathcal{U}_\xi$ respectively, as in Lemma 7. Let E_ξ be the subspace of $E^{\mathbf{N}} / \mathcal{U}_\xi$ spanned by a_ξ and the elements of E. Let ϕ_ξ be an isometric embedding from E_ξ into F. $\phi_\xi \restriction E$ is an isometric embedding from E into $E^{\mathbf{e}}/\mathcal{U}$; therefore, by Lemma 5,

$$\phi_\xi(e_i) = \pm e_i.$$

Let b_ξ be $\phi_\xi(a_\xi)$; by Lemma 7 and the above we get

$$\| b_\xi \pm e_i \| = \sqrt{2}(1-\delta)^{-1} \quad \text{if} \quad q_i < \xi.$$

$$\| b_\xi \pm e_j \| = \sqrt{2} \quad \text{if} \quad q_j > \xi.$$

If $\xi \neq \xi'$ there exists an integer k such that $\xi < q_k < \xi'$; we get

$$\| b_\xi \pm e_k \| = \sqrt{2}$$

$$\| b_{\xi'} \pm e_k \| = \sqrt{2}(1-\delta)^{-1}$$

so that

$$\| b_\xi - b_{\xi'} \| \geqq \sqrt{2}((1-\delta)^{-1} - 1) = \rho.$$

Finally the $(b_\xi)_{\xi \in \mathbf{R}-\mathbf{Q}}$ are an uncountable family of elements with mutual distance $\geqq \rho > 0$. This cannot happen in a separable Banach space.

We now briefly indicate how to modify the above construction in order to prove Theorems 3 and 4.

Proof of Theorem 3. We let $(\delta_n)_{n \in \mathbf{N}}$ and δ as above; we also keep the other notations.

We let K_1 be the set of elements x in l_2 such that:

i) $x \in B$ (B is the unit ball of l_2)

ii) $-1+\delta_n \leqq (x, e_n) \leqq 1-\delta_n$, $n \in \mathbf{N}$

iii) $-1+\delta \leqq (x, c_{ij}) \leqq 1-\delta$, $j = i+1$; $i, j \in \mathbf{N}$.

K_1 is the unit ball of E_1; E_1 is $1+\varepsilon$-isomorphic to l_2 provided δ_0 is small enough. If ϕ is an isometry from E_1 into $E_1^{\mathbf{e}}/\mathcal{U}$, then, it is shown as above that $\phi(e_n) = \pm e_n$.

Now

$$\| e_i + e_{i+1} \| / \sqrt{2} = (1-\delta)^{-1}$$

whereas

$$\| e_i - e_{i+1} \| / \sqrt{2} = 1$$

so that for any i, $\phi(e_i)=e_i$ implies $\phi(e_{i+1})=e_{i+1}$ and $\phi(e_i)=-e_i$ implies $\phi(e_{i+1})=-\phi(e_{i+1})$. It follows that either $\phi(e_0)=e_0$ and ϕ is the canonical embedding i from E_1 into $E_1^{\mathfrak{P}}/\mathcal{U}$ or $\phi(e_0)=-e_0$ and ϕ is equal to $-i$. **Q.E.D.**

Proof of Theorem 4. We keep the same notations as above. We let K_2 be the set of elements x in l_2 such that

i) $x \in B$

ii) $-1+\delta \leqq (x,(2e_i+e_j)/\sqrt{5}) \leqq 1-\delta,\ i<j.$

K_2 is the unit ball of a Banach space E_2; E_2 is $1+\varepsilon$-isomorphic to l_2 provided δ is small enough.

Assume $k_1,\cdots,k_n,\cdots$ is an increasing subsequence of the integers; let σ be the isometry from l_2 into l_2 defined by $\sigma(e_n)=e_{k_n}$. It is easy to see that σ is an isometry from E_2 into E_2 so that $(e_n)_{n\in N}$ is a norm-indiscernible sequence. It is not symmetric because

$$\sqrt{5}=\| e_1+2e_2\| \neq \| e_2+2e_1\| = (1-\delta)^{-1}\sqrt{5}\,.$$

2. Envelopes of density $\aleph_1$

We now prove Theorem 2. Actually, this result can be derived from a result in model-theory; thus what we have to do is to describe a proper setting to transfer this result. We assume the reader is familiar with model theory and we refer to [1] for all the definitions and the results we need.

We are interested in the following type of structures:

$$\mathfrak{A}=(|\mathfrak{A}|,+^{\mathfrak{A}},(\cdot q^{\mathfrak{A}})_{q\in Q},B^{\mathfrak{A}})$$

where

$|\mathfrak{A}|$ is a set (the domain of $\mathfrak{A}$)

$+^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|^2$ to $|\mathfrak{A}|$

$\cdot q^{\mathfrak{A}}$ is a function from $|\mathfrak{A}|$ to $|\mathfrak{A}|$ for any $q\in \boldsymbol{Q}$

$B^{\mathfrak{A}}$ is a subset of $|\mathfrak{A}|$.

The appropriate language L to discuss such structures includes, besides variables $(v_i)_{i\in N}$:

a binary function symbol $+$

for each q in $\boldsymbol{Q}$ a unary function symbol q

a unary predicate symbol B.

To any Banach space E, we can associate a structure of the above type $\mathfrak{A}(E)$, the interpretation of B being the unit ball of E. Conversely, if $\mathfrak{H}$ is a structure elementarily equivalent to $\mathfrak{A}(E)$ it is possible to associate to $\mathfrak{H}$ a normed-$\boldsymbol{Q}$-space $[\mathfrak{H}]$ by the following procedure:

first let

$$\Pi_{\mathfrak{H}} = \{x : x \in |\mathfrak{H}| \quad \text{and for some} \quad q > 0,\ q^{\mathfrak{H}}x \in B^{\mathfrak{H}}\}.$$

$\Pi_{\mathfrak{H}}$ is closed under $+^{\mathfrak{H}}$ because $\mathfrak{H}$ is elementarily equivalent to $\mathfrak{A}(E)$; this shows that for any two elements x, y in $|\mathfrak{H}|$ and for $q,\ q' > 0$,

$$(q^{\mathfrak{H}}x \in B^{\mathfrak{H}} \text{ and } q'^{\mathfrak{H}}y \in B^{\mathfrak{H}}) \rightarrow \left(\frac{qq'}{q+q'}\right)^{\mathfrak{H}}(x +^{\mathfrak{H}} y) \in B^{\mathfrak{H}}.$$

$\Pi_{\mathfrak{H}}$ is also closed under $\cdot\, q^{\mathfrak{H}}$, $q \in \boldsymbol{Q}$. For any element x in $\Pi_{\mathfrak{H}}$ let

$$\| x \| = (\sup\{q : q^{\mathfrak{H}}x \in B^{\mathfrak{H}}\})^{-1}.$$

Clearly

$$\| x \| \geqq 0$$

$$\| \cdot q^{\mathfrak{H}} x \| = | q | \| x \|.$$

Also

$$\| x +^{\mathfrak{H}} y \| \leqq \| x \| + \| y \|.$$

To prove this, let $\varepsilon > 0$ and let q, q' be such that

$$1/q \leqq \| x \| + \varepsilon,\ \ q^{\mathfrak{H}}x \in B^{\mathfrak{H}}$$

$$1/q' \leqq \| y \| + \varepsilon,\ \ q'^{\mathfrak{H}}y \in B^{\mathfrak{H}}.$$

As before, we can infer that $((qq')/(q+q'))^{\mathfrak{H}}(x +^{\mathfrak{H}} y) \in B^{\mathfrak{H}}$, so that

$$\| x +^{\mathfrak{H}} y \| \leqq (q+q')/(qq') \leqq \| x \| + \| y \| + 2\varepsilon.$$

As ε is arbitrary, the result follows.

Let $N_{\mathfrak{H}} = \{x : x \in \Pi_{\mathfrak{H}} \text{ and } \| x \| = 0\}$; $\Pi_{\mathfrak{H}}/N_{\mathfrak{H}}$ is a $\boldsymbol{Q}$-vector-space; the mapping: $x \rightarrow \| x \|$ is a $\boldsymbol{Q}$-norm on $\Pi_{\mathfrak{H}}/N_{\mathfrak{H}}$ i.e. satisfies

$$\forall x \forall y (\| x + y \| \leqslant \| x \| + \| y \|)$$

$$\forall x (\| qx \| = | q | \| x \|)$$

$$\forall x (\| x \| = 0 \leftrightarrow x = 0).$$

We let $[\mathfrak{H}]$ be $\Pi_{\mathfrak{H}}/N_{\mathfrak{H}}$.

The same procedure can be carried through for substructures of $\mathfrak{H}$ and the following is clear.

LEMMA 8. *Assume* $\mathfrak{H}$ *is elementarily equivalent to* $\mathfrak{A}(E)$. *Let* $\mathfrak{H}'$ *be a substructure of* $\mathfrak{H}$, *then* $[\mathfrak{H}']$ *is isometric to a subspace of* $[\mathfrak{H}]$.

(This means that there exists a $\boldsymbol{Q}$-linear, $\boldsymbol{Q}$-norm preserving injection from $[\mathfrak{H}']$ into $[\mathfrak{H}]$).

When $\mathfrak{H}$ is the ultrapower of $\mathfrak{A}(E)$ in the classical sense, the above construction coincides with the ultrapower construction for Banach spaces.

LEMMA 9. $[\mathfrak{A}(E)^I/\mathcal{U}] = E^I/\mathcal{U}$.

PROOF. We let $\mathfrak{H}$ stand for $\mathfrak{A}(E)^I/\mathcal{U}$ and we define a mapping $\psi : \Pi_{\mathfrak{H}} \to E^I/\mathcal{U}$; if x is the element $(x_i)_{i\in I}$ of $\Pi_{\mathfrak{H}}$ we let

$$\psi(x) = (x_i)_{i\in I}.$$

This definition makes sense because for some $q > 0$:

$$q^{\mathfrak{H}} x \in B^{\mathfrak{H}} \quad \text{i.e.}$$

$$\{i : q^{\mathfrak{A}(E)} x_i \in B^{\mathfrak{A}(E)}\} \in \mathcal{U} \quad \text{i.e.}$$

$$\{i : \| x_i \| \leqq 1/q\} \in \mathcal{U}.$$

Clearly, ψ is linear and onto. ψ is norm preserving because

$$\sup\{q : q^{\mathfrak{H}} x \in B^{\mathfrak{H}}\} = \sup\{q : \{i : \| q x_i \| \leqq 1\} \in \mathcal{U}\}$$

$$= \sup\left\{ q : q \lim_{\mathcal{U}} \| x_i \| \leqq 1 \right\} = \left(\lim_{\mathcal{U}} \| x_i \| \right)^{-1}.$$

Finally, the kernel of ψ is $N_{\mathfrak{H}}$ so that ψ induces an isometry from $\Pi_{\mathfrak{H}}/N_{\mathfrak{H}}$ onto $E^I/\mathcal{U}$.

We now quote a result from model theory which is part of a deep theorem of Keisler (cf. [1]).

THEOREM 5. (Keisler.) *Assume the continuum hypothesis*; *let* $\mathfrak{A}$ *be a given structure of cardinality* $\leqq \aleph_1$; *there exists an ultrafilter* $\mathcal{U}$ *on* $\boldsymbol{N}$ *such that any structure* $\mathfrak{H}$ *elementarily equivalent to* $\mathfrak{A}$ *and of cardinality* $\leqq \aleph_1$ *is isomorphic to a substructure of* $\mathfrak{A}^{\boldsymbol{N}}/\mathcal{U}$.

To prove Theorem 4, we let E be a given Banach space of density character $\leqq \aleph_1$ and we apply the above theorem with $\mathfrak{A} = \mathfrak{A}(E)$. Clearly, $[\mathfrak{A}(E)^{\boldsymbol{N}}/\mathcal{U}]$ has cardinality $\leqq \aleph_1$ and therefore density character $\leqq \aleph_1$. We claim $[\mathfrak{A}(E)^{\boldsymbol{N}}/\mathcal{U}]$ is an envelope of E.

$[\mathfrak{A}(E)^{\boldsymbol{N}}/\mathcal{U}]$ is $E^{\boldsymbol{N}}/\mathcal{U}$ and therefore it is finitely representable in E.

Let F be a Banach space finitely representable in E and of density character $\leqq \aleph_1$; we may assume F is a subspace of some ultrapower $E^J/\mathfrak{D}$. $\mathfrak{A}(E)^J/\mathfrak{D}$ is elementarily equivalent to $\mathfrak{A}(E)$; therefore, by the Lowenheim-Skolem theorem, there exists a substructure $\mathfrak{H}$ of $\mathfrak{A}(E)^J/\mathfrak{D}$ such that

$\mathfrak{H}$ is elementarily equivalent to $\mathfrak{A}(E)$

$[\mathfrak{H}] \supseteq F$

the cardinality of $\mathfrak{H}$ is at most $\aleph_1$.

(Notice that the cardinality of F is at most $\aleph_1$ so that if ψ is the canonical mapping from $\Pi_{\mathfrak{A}(E)^J/\mathfrak{D}}$ onto $E^J/\mathfrak{D}$ there exists a set X of cardinality $\leqq \aleph_1$ such that $\psi(X) \supseteq F$.)

By Keisler's theorem $\mathfrak{H}$ is isomorphic to a substructure of $\mathfrak{A}(E)^N/\mathcal{U}$, therefore $[\mathfrak{H}]$ is isometric to a subspace of $E^N/\mathcal{U}$ so that F is isometric to a subspace of $E^N/\mathcal{U}$. **Q.E.D.**

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