ISRAEL JOURNAL OF MATHEMATICS, Vol. 40, Nos. 3–4, 1981

THE BANACH–SAKS PROPERTY IS NOT L^2-HEREDITARY

BY
WALTER SCHACHERMAYER

ABSTRACT

We construct a Banach space E, which has the Banach–Saks property and such that $L^2(E)$ does not have the Banach–Saks property. The construction is a somewhat tree-like modification of Baernstein's space.

1. Introduction

Recall that a Banach space E has the Banach–Saks property (abbreviated (BS)) if for every bounded sequence $\{x_n\}_{n=1}^{\infty}$ in E there is a subsequence $\{x'_n\}_{n=1}^{\infty}$ converging in Cesaro-mean (i.e., $\|n^{-1}\sum_1^n x'_k - x\| \to 0$ for some $x \in E$).

We construct an example of a Banach space E having (BS) such that $L^2_{[0,1]}(E)$ does not have (BS).

After constructing this example I have been informed that J. Bourgain has already constructed a Banach space with this property ([3], [6]). However, our construction is quite different and — as we believe — simpler and there might be some interest in the technique of the construction.

Our space E will be a somewhat tree-like modification of Baernstein's space [1] and is based on a very elementary probabilistic lemma. It will be convenient to use interpolation theory (following an idea of B. Beauzamy [2]) to avoid certain technical difficulties arising in Baernstein's construction. Let us note however that it is possible to construct our example following exactly the lines of [1].

2. An elementary probabilistic result

LEMMA 1. *Let $N \in \mathbf{N}$ and let $X_1, \cdots, X_N$ be independent random variables taking their values in the set $\{1, \cdots, N\}$ in a uniformly distributed way (i.e. for $1 \leqq i, j \leqq N$, $P\{X_i = j\} = N^{-1}$). Let*

Received January 26, 1981 and in revised form March 12, 1981

$$Y(\omega) = \operatorname{card}\{j : \text{there is an } i,\ 1 \leqq i \leqq N, \text{ such that } X_i(\omega) = j\}.$$

Then

$$P\{Y \geqq N/4\} \geqq 1/4. \tag{1}$$

PROOF. For $0 \leqq n \leqq N$ let $Y_n(\omega) = \operatorname{card}\{j : \text{there is an } i,\ 1 \leqq i \leqq n, \text{ such that } X_i(\omega) = j\}$. Clearly

$$0 = Y_0 \leqq Y_1 \leqq \cdots \leqq Y_N = Y.$$

Assume that (1) does not hold; then it fails for each Y_n. Fix $1 \leqq n < N$ and let A be an atom in the σ-algebra generated by $X_1, \cdots, X_n$ such that, for $\omega \in A$, $Y_n(\omega) < N/4$. As X_{n+1} is independent of Y_n and the law of X_{n+1} is uniformly distributed,

$$P\{\omega \in A : Y_{n+1}(\omega) = Y_n(\omega) + 1\} \geqq \tfrac{3}{4} \cdot P(A).$$

Summing over the atoms on which Y_n is less than $N/4$ we see that on a set of probability greater than or equal to $\frac{3}{4} \cdot \frac{3}{4}$ we have $Y_{n+1} = Y_n + 1$, hence

$$E(Y_{n+1}) \geqq E(Y_n) + 9/16 \geqq E(Y_n) + 1/2.$$

It follows that $E(Y) = \sum_{n=0}^{N-1} E(Y_{n+1} - Y_n) \geqq N/2$. On the other hand $Y \leqq N$; hence if (1) does not hold, then

$$E(Y) \leqq \tfrac{3}{4} \cdot N/4 + \tfrac{1}{4} \cdot N < N/2,$$

a contradiction. □

3. Construction of the space

Let Φ_0 be the space of finite sequences and let $\{e_n\}_{n=1}^{\infty}$ be its natural base. We write n (uniquely) as $2^u + v$ $(0 \leqq v \leqq 2^u - 1)$, and associate to n the number $t(n) = v/2^u \in [0, 1[$.

A finite subset $\gamma = \{n_1, \cdots, n_l\}$ of $\mathbf{N}$, where $n_1 < n_2 < \cdots < n_l$, will be called *admissible*, if:

(1) $l \leqq n_1$ (Baernstein's condition).

(2) Let p be defined by $2^{p-1} < n_1 \leqq 2^p$. For every $0 \leqq j < 2^p$ there is only one i so that $t(n_i) \in [j/2^p, (j+1)/2^p[$.

For example, for $u \in \mathbf{N}$ the set $\gamma = \{2^u, 2^u + 1, \cdots, 2^{u+1} - 1\}$ is admissible.

Let Δ be the set of admissible γ's. For $\gamma \in \Delta$ and $x = \{x_i\}_{i=1}^{\infty} \in \Phi_0$ define $\sigma(x, \gamma) = \sum_{i \in \gamma} |x_i|$ and $\|x\|_F = \sup\{\sigma(x, \gamma) : \gamma \in \Delta\}$.

Clearly $\| \ \|_F$ defines a norm on Φ_0, $\| \ \|_1 \geqq \| \ \|_F \geqq \| \ \|_{c_0}$. Let F denote the completion of $(\Phi_0, \| \ \|_F)$ and $i : l^1 \to F$ the canonical injection.

Recall [2] that an operator T from a Banach space A to a Banach space A_1 is said to have (BS) if any bounded sequence $\{x_n\}_{n=1}^\infty$ in A contains a subsequence $\{x'_n\}_{n=1}^\infty$ such that the Cesaro averages $n^{-1}\Sigma_1^n Tx'_n$ converge in A_1. It is shown in [2] that an operator T has (BS) iff the Lions–Peetre interpolation spaces $(A/\mathrm{Ker}\, T, A_1)_{q,p}$, $1 < p < \infty$, $0 < q < 1$ (or equivalently the Davis–Figiel–Johnson–Pelczynski factorisation space [4]) have (BS).

PROPOSITION 2. *The map* $i : l^1 \to F$ *has* (BS).

PROOF. Let $\{x_n\}_{n=1}^\infty$ be a bounded sequence in l^1. We may suppose that x_n is bounded in norm by 1 and converges coordinatewise to zero and by a standard perturbation argument we may assume that

$$x_n = \sum_{i=r(n-1)+1}^{r(n)} \lambda_i^{(n)} e_i$$

where $r(n)$ is an increasing sequence.

We now choose inductively a subsequence $\{n_k\}_{k=1}^\infty$ and infinite subsets M_k of N. Let $M_0 = \mathrm{N}$ and $n_1 = 1$ and suppose M_{k-1} and n_k are defined. Let p be such that $2^{p-1} < r(n_k) \leqq 2^p$ and consider the partition of $[0,1[$ into $[j/2^p, (j+1)/2^p[$, $j = 0, \cdots, 2^p - 1$. For $n \geqq n_k$ define

$$\mu_j^{(n)} = \max\{|\lambda_i^{(n)}| : t(i) \in [j/2^p, (j+1)/2^p[\}.$$

Note that, for every n, $\Sigma_{j=0}^{2^p-1} \mu_j^{(n)} \leqq 1$ as the x_n are bounded by 1 in the l^1-norm. Find a subsequence $\bar{M}_k$ of $M_{k-1} \cap [n_k + 1, \cdots, \infty[$ such that, for every $j = 0, \cdots, 2^{p-1}$, the sequence $\{\mu_j^{(n)}\}_{n \in \bar{M}_k}$ converges, to μ_j say. Clearly $\Sigma_{j=0}^{2^p-1} \mu_j \leqq 1$. Finally let M_k be the subset of $\bar{M}_k$ consisting of those n for which, for every $j = 0, \cdots, 2^p - 1$, $\mu_j^{(n)} \leqq \mu_j + 2^{-p}$ and define n_{k+1} by picking an arbitrary member of M_k. This completes the induction.

Note that for $\gamma \in \Delta$ and k such that $\inf(\gamma) \leqq r(n_k)$ and for every $l \in \mathrm{N}$

$$\sigma(x_{n_{k+1}} + \cdots + x_{n_{k+l}}, \gamma) \leqq 2. \tag{2}$$

Indeed, γ may pick for every $j = 0, \cdots, 2^p - 1$ at most one index i with $t(i) \in [j/2^p, (j+1)/2^p[$ (p defined as above), hence the contribution of this index is at most $\mu_j + 2^{-p}$. Summing over j we obtain (2).

Hence for n, l as above and γ such that $r(n_{k-1}) < \inf(\gamma) \leqq r(n_k)$

$$\sigma(x_{n_1} + \cdots + x_{n_{k+l}}, \gamma) \leqq \sigma(x_{n_k}, \gamma) + \sigma(x_{n_{k+1}} + \cdots + x_{n_{k+l}}, \gamma) \leqq 3.$$

It follows readily that for every $K \in \mathbf{N}$

$$\| K^{-1}(x_{n_1} + \cdots + x_{n_k}) \|_F \leqq 3K^{-1}$$

from which the proposition follows. □

Let $(E, \| \ \|_E)$ be the Davis–Figiel–Johnson–Pelczynski factorisation space of the injection $i : l^1 \to F$. As mentioned above, Proposition 2 implies that E has (BS). We may and do consider E as a space of sequences, containing l^1 and contained in F.

PROPOSITION 3. *$L^2([0,1]; E)$ does not have* (BS).

PROOF. Let $\{f_u\}_{u=1}^{\infty}$ be a sequence of independent random variables such that f_u takes the value e_{2^u+v} (i.e. the $2^u + v$th unit vector) with probability 2^{-u} $(v = 0, \cdots, 2^u - 1)$.

$\{f_u\}_{u=1}^{\infty}$ is a bounded sequence in $L^\infty([0,1]; l^1)$ hence in particular it is bounded in $L^2([0,1]; E)$.

Also for almost every $\omega \in [0,1]$, $\{f_u(\omega)\}_{u=1}^{\infty}$ converges weakly to zero in F. (Indeed, it is shown in the proof of Proposition 2 that for any sequence of unit vectors there is a subsequence converging strongly to zero in Cesaro-mean.) It follows from [4], that $\{f_u(\omega)\}_{u=1}^{\infty}$ converges weakly to zero in E. By [5], theorem IV.1.1, we conclude that $\{f_u\}_{u=1}^{\infty}$ tends weakly to zero in $L^2([0,1]; E)$.

Now fix any subsequence $\{f_{u_k}\}_{k=1}^{\infty}$. As the norm of $L^2(E)$ is stronger than that of $L^1(F)$ the following assertion will prove Proposition 3:

$$K^{-1} \| f_{u_1} + \cdots + f_{u_K} \|_{L^1(F)} \geqq 1/32, \qquad K = 2, 4, \cdots, 2^p, \cdots. \tag{3}$$

Indeed, assume $K = 2^{p+1}$ and for $k = 2^p + 1, \cdots, 2^{p+1}$ define the random variables X_k with values in $\{0, \cdots, 2^p - 1\}$ by

$$X_k(\omega) = j \qquad \text{if } f_{u_k}(\omega) = e_n \quad \text{and} \quad t(n) \in [j/2^p, (j+1)/2^p[.$$

The random variables $\{X_k\}_{k=2^p+1}^{2^{p+1}}$ satisfy the assumptions of Lemma 1, hence on a set B of probability $\geqq 1/4$ the sequence $\{X_k(\omega)\}_{k=2^p+1}^{2^{p+1}}$ hits at least 2^{p-2} different j's. Fix such an ω and find a set $\gamma = \{n_1, \cdots, n_l\} = \{u^{u_{k_1}} + v_1, \cdots, 2^{u_{k_l}} + v_l\}$ such that

(i) $n_1 < \cdots < n_l$,

(ii) $l = 2^{p-2}$,

(iii) $f_{u_{k_i}}(\omega) = e_{2^{u_{k_i}} + v_i}$ for some $k_i \in \{2^p + 1, \cdots, 2^{p+1}\}$,

(iv) the $v_i / 2^{u_{k_i}}$ lie in different $[j/2^p, (j+1)/2^p[$.

Then it is easy to check that γ is admissible and therefore

$$K^{-1}\|f_{u_1}(\omega)+\cdots+f_{u_K}(\omega)\|_F \geqq 2^{-(p+1)}\sigma(f_{u_1}(\omega)+\cdots+f_{u_K}(\omega),\gamma)$$
$$\geqq 2^{-(p+1)}\cdot 2^{p-2}=1/8.$$

Integrating over B we obtain (3). □

REMARK. The proof actually shows that for $1<p<\infty$, $L^p(E)$ does not have (BS) and that $L^1(E)$ does not have the weak Banach–Saks property (called (BSR) in [2]).

REFERENCES

1. A. Baernstein, *On reflexivity and summability*, Studia Math. **42** (1972), 91–94.
2. B. Beauzamy, *Banach Saks properties and spreading models*, Math. Scand. **44** (1979), 357–384.
3. J. Bourgain, *On the Banach–Saks property in Lebesgue spaces*, Vrije Universiteit Brussel, preprint, 1979.
4. W. J. Davis, T. Figiel, W. B. Johnson and A. Pelczynski, *Factoring weakly compact operators*, J. Functional Analysis **17** (1974), 311–327.
5. J. Diestel and J. J. Uhl, Jr., *Vector Measures*, Surveys of the Amer. Math. Soc. 15, Rhode Island, 1977.
6. S. Guerre, *La propriété de Banach–Saks ne passe pas de E à $L^2(E)$*, d'après J. Bourgain, Séminaire d'Analyse Fonctionelle, Ecole Polytechnique Paris, 1979/80.

DEPARTMENT OF MATHEMATICS
UNIVERSITÄT LINZ
A-4040 LINZ, AUSTRIA