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NEIGHBORLY POLYTOPES

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ABSTRACT
A $2m$-polytope Q is *neighborly* if each m vertices of Q determine a face. It is shown that the combinatorial structure of a neighborly $2m$-polytope determines the combinatorial structure of every subpolytope. We develop a construction of "sewing a vertex onto a polytope", which, when applied to a neighborly $2m$-polytope, yields a neighborly $2m$-polytope with one more vertex. Using this construction, we show that the number $g(2m+\beta, 2m)$ of combinatorial types of neighborly $2m$-polytopes with $2m+\beta$ vertices grows superexponentially as $\beta \to \infty$ ($m \geqq 2$ fixed) and as $m \to \infty$ ($\beta \geqq 4$ fixed).

1. Introduction

In this paper we deal with simplicial k-neighborly d-polytopes in general, and in particular with neighborly (i.e. m-neighborly) $2m$-polytopes.

In sections 2 and 3 we establish general combinatorial properties of such polytopes.

In sections 4 and 5 we present and investigate a construction of "sewing" an additional vertex to a neighborly $2m$-polytope.

By repeated use of this construction we obtain in section 6 lower bounds for the number $g(v, 2m)$ of combinatorial types of neighborly $2m$-polytopes with v vertices. $g(2m+4, 2m)$ increases superexponentially with m, and $g(v, m)$ tends to infinity superexponentially with v for each fixed $m \geqq 2$.

The techniques developed in sections 2–5 are used in [11] to prove that a non-cyclic neighborly $2m$-polytope with $v \geqq 2m+5$ vertices has at most $2m$ cyclic subpolytopes with $v-1$ vertices.

The notation and conventions in this paper follow [6]. In addition, we denote by $[A_1, A_2, \cdots]$ the set $\mathrm{conv}(A_1 \cup A_2 \cup \cdots)$, where $A_1, A_2, \cdots \subset R^d$. If $a \in R^d$, then $[\cdots, a, \cdots]$ stands for $[\cdots, \{a\}, \cdots]$.

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All the polytopes in this paper are simplicial polytopes but not a simplex, unless otherwise specified.

Throughout the paper, the letter P denotes a d-polytope, and Q denotes a neighborly $2m$-polytope.

We shall use the following characterization of faces [6, sec. 2.1, th. 11]: If P is a polytope and $T \subset \operatorname{vert} P$, then $[T] \in \mathcal{F}(P)$ iff $\operatorname{aff} T \cap [\operatorname{vert} P \setminus T] = \emptyset$.

We say that a point $x \in R^d$ covers a face Φ of P if x lies beyond all the facets of P that include Φ (see [6, ch. 5]).

By a subpolytope of P we mean the convex hull of a subset of $\operatorname{vert} P$.

We suppose that the reader is familiar with the basic facts about neighborly and cyclic polytopes (see [6, sec. 4.7 and ch. 7], and [8, pp. 82–93]), in particular with Gale's Evenness Condition.

2. Neighborly polytopes

In this section we are concerned with properties of the boundary complex of neighborly polytopes. The main result is that the combinatorial structure of an m-neighborly $2m$-polytope determines the combinatorial structure of every subpolytope.

LEMMA 2.1. Intersection lemma. *Let $S_1, \cdots, S_k$ be subsets of R^d. If $\bigcap_{i=1}^k [S_i] \neq \emptyset$ and if $a = \sum_{j=1}^{r_1} \lambda_{j1} a_{j1} \in \bigcap_{i=1}^k [S_i]$ where $\lambda_{11} > 0, \lambda_{j1} \geqq 0, a_{j1} \in S_1$ $(1 \leqq j \leqq r_1)$, $\sum_{j=1}^{r_1} \lambda_{j1} = 1$, then there are subsets $T_i \subset S_i$ $(1 \leqq i \leqq k)$ such that*

(1) $\bigcap_{i=1}^{k} [T_i] \neq \emptyset$,

(2) $\sum_{i=1}^{k} | T_i | \leqq (k-1)d + k$,

(3) $a_{11} \in T_1$.

PROOF. Let

$$V = R^{(k-1)(d+1)} = \underbrace{(R \times R^d) \times \cdots \times (R \times R^d)}_{k-1 \text{ times}}$$

For $x \in R^d$ and $1 \leqq i \leqq k$ define a point $x^i \in V$ as follows:

$$x^1 = (-1, -x, 0, \cdots, 0); \quad \text{for } 2 \leqq i \leqq k-1, \quad x^i = (\underbrace{0, \cdots, 0}_{(i-2)(d+1)}, 1, x, -1, -x, \underbrace{0, \cdots, 0}_{(k-i-1)(d+1)});$$

finally, $x^k = (0, \cdots, 0, 1, x)$. Define also $A = \bigcup_{i=1}^k \{x^i : x \in S_i\}$.

For $1 \leqq i \leqq k$ there are points $a_{ji} \in S_i$ and numbers $\lambda_{ji} \geqq 0$, $1 \leqq j \leqq r_i$, such that $\sum_{j=1}^{r_i} \lambda_{ji} = 1$ and $a = \sum_{j=1}^{r_i} \lambda_{ji} a_{ji}$. Then

$$\sum_{i=1}^{k} \sum_{j=1}^{r_i} \frac{1}{k} \lambda_{ji} = 1, \quad \frac{1}{k} \lambda_{11} > 0 \quad \text{and} \quad \sum_{i=1}^{k} \sum_{j=1}^{r_i} \frac{1}{k} \lambda_{ji} a^i_{ji} = 0_V \in V.$$

Hence $0_V \in \operatorname{conv} A$. Applying Carathéodory's theorem to the set A in V we find subsets $T_i \subset S_i$, $1 \leqq i \leqq k$, such that $a_{11} \in T_1$, $\sum_{i=1}^k |T_i| \leqq (k-1)(d+1)+1$ and $0_V \in \operatorname{conv}(\bigcup_{i=1}^k \{x^i : x \in T_i\})$. A close look at the definition of x^i reveals that $0_V \in \operatorname{conv}(\bigcup_{i=1}^k \{x^i : x \in T_i\})$ implies $\bigcap_{i=1}^k \operatorname{conv} T_i \neq \emptyset$. □

We shall use this lemma later with $k = 2$.

Definition 2.2. Let S be a subset of vert P. $[S]$ is a *missing face* (m.f.) *of* P if $[S]$ is not a face of P but for every proper subset T of S, $[T]$ is a face of P. (Compare [3, def. 2.1].)

We say that $[S]$ is a missing k-face (k-m.f.) of P if $[S]$ is a m.f. of P and $\dim [S] = k$.

Remark. Definition 2.2 remains meaningful even if we drop the assumption that the polytope P is simplicial. Also, Theorems 2.3, 2.4, 2.5 and their proofs hold for general polytopes.

Theorem 2.3. *If* $S \subset V = \operatorname{vert} P$, *then* $[S]$ *is a* m.f. *of* P *iff*

(1) $P \cap \operatorname{aff} S = [S]$,

(2) $\emptyset \neq [S] \cap [V \setminus S] \subset \operatorname{relint} [S]$,

(3) $|S| = \dim [S] + 1$.

Proof. Suppose $[S]$ is a m.f. of P. Clearly $[S] \subset P \cap \operatorname{aff} S$. If $x \in P \cap \operatorname{aff} S \setminus [S]$, then there are points $y \in \operatorname{relint} [S]$, $z \in \operatorname{relbd} [S]$ such that $z \in (x, y)$. There is a proper face F of $[S]$ such that $z \in F$. F is a face of P. Then $y \in P \setminus F$, $z \in F$, hence $x \notin P$, a contradiction. Hence $P \cap \operatorname{aff} S = [S]$. $[S]$ is not a face of P, hence $\operatorname{aff} S \cap [V \setminus S] \neq \emptyset$. So $[S] \cap [V \setminus S] = P \cap \operatorname{aff} S \cap [V \setminus S] = \operatorname{aff} S \cap [V \setminus S] \neq \emptyset$.

If $T \subsetneqq S$ then $[T]$ is a face of P, hence a face of $[S]$. So $[S]$ is a simplex and $\dim [S] = |S| - 1$.

If $x \in \operatorname{relbd} [S]$, then $x \in [T]$ for some $T \subsetneqq S$. $[T] \in \mathcal{F}(P)$, hence $[T] \cap [V \setminus T] = \emptyset$. So $x \notin [V \setminus T]$ and *a fortiori* $x \notin [V \setminus S]$. Therefore $[V \setminus S] \cap \operatorname{relbd} [S] = \emptyset$, hence $[V \setminus S] \cap [S] \subset \operatorname{relint}[S]$.

Now we turn to the converse part of the theorem. Suppose $S \subset V$ satisfies (1), (2), (3). To prove that $[S]$ is a m.f. of P it clearly suffices to show that if $T \subsetneqq S$ and $|T| = |S| - 1$, then $[T] \in \mathcal{F}(P)$. Let $x \in S$, $T = S \setminus \{x\}$.

(A) $P \cap \operatorname{aff} T = P \cap \operatorname{aff} S \cap \operatorname{aff} T = [S] \cap \operatorname{aff} T = [T]$.

(B) We claim that $[T] \cap [V \setminus T] = \varnothing$. If not, then there is a point $a \in [T] \cap [V \setminus S, x]$. There is a point $b \in [V \setminus S]$ such that $a \in [b, x]$. $b \in \operatorname{aff}\{a, x\}$ since $a \neq x$. It follows that $b \in \operatorname{aff} S \cap [V \setminus S] = [S] \cap [V \setminus S] \subset \operatorname{relint}[S]$. Since $a \in [b, x]$, $a \neq x$, we obtain $a \in \operatorname{relint}[S]$, a contradiction. From (A) and (B) we get

$$\operatorname{aff} T \cap [V \setminus T] = P \cap \operatorname{aff} T \cap [V \setminus T] = [T] \cap [V \setminus T] = \varnothing. \qquad \square$$

THEOREM 2.4. *A k-neighborly d-polytope P has no j-m.f. with $j < k$ or $j > d - k$.*

PROOF. It is obvious that P has no j-m.f. with $j < k$. Suppose $S \subset \operatorname{vert} P = V$ and $[S]$ is a j-m.f. of P. Then $[S] \cap [V \setminus S] \neq \varnothing$ (Theorem 2.3). Applying the intersection Lemma 2.1 we find subsets $T_1 \subset S$, $T_2 \subset V \setminus S$ such that $[T_1] \cap [T_2] \neq \varnothing$ and $|T_1| + |T_2| \leqq d + 2$. $T_1 = S$, because otherwise $[T_1]$ is a face of P, and then

$$[T_1] \cap [T_2] \subset [T_1] \cap [V \setminus T_1] = \varnothing.$$

$|T_2| \geqq k + 1$, because P is k-neighborly. Therefore $j + 1 = |S| = |T_1| \leqq (d+2) - |T_2| \leqq (d+2) - (k+1)$, $j \leqq d - k$. $\square$

An immediate consequence of Theorem 2.4 is the well-known fact:

THEOREM 2.5. *If P is a k-neighborly d-polytope and $k > \frac{1}{2}d$, then P is a simplex.*

There are many problems concerning the determination of $\mathscr{F}(P)$ from some partial information about $\mathscr{F}(P)$. A natural question is: under what circumstances does $\operatorname{skel}_j \mathscr{F}(P)$ determine $\mathscr{F}(P)$?

THEOREM 2.6. *If P is a simplicial k-neighborly d-polytope, then $\mathscr{F}(P)$ is determined by its $(d-k)$-skeleton; moreover, if $S \subsetneqq \operatorname{vert} P$ and $|S| > d - k + 1$, then $[S] \in \mathscr{F}(P)$ iff $[T] \in \mathscr{F}(P)$ for every $T \subset S$ with $|T| \leqq d - k + 1$.*

PROOF. Assume $S \subsetneqq \operatorname{vert} P$, $|S| \geqq d - k + 2$. If $[S] \in \mathscr{F}(P)$, then $T \subset S$ implies $[T] \in \mathscr{F}(P)$, since P is simplicial. If $[S] \notin \mathscr{F}(P)$, then there is a set $T \subset S$, such that $[T]$ is m.f. By Theorem 2.4, $|T| \leqq d - k + 1$. $\square$

An immediate consequence is

THEOREM 2.7. *If Q is an m-neighborly $2m$-polytope then $\operatorname{skel}_m Q$ determines $\mathscr{F}(Q)$.*

It is known, though much harder to prove, that the combinatorial structure of every simplicial polytope P of dimension $2m$ or $2m+1$ is determined by $\mathrm{skel}_m \mathcal{F}(P)$.

THEOREM 2.8. *Let P, P^+ be k-neighborly d-polytopes, not necessarily simplicial. Suppose* $\mathrm{vert}\, P = V$, $\mathrm{vert}\, P^+ = V \cup \{x\}$, $x \notin P$, $T \subset V$, $\dim [T] \geqq d - k$. *If for every $S \subsetneqq T$, $[S, x] \in \mathcal{F}(P^+)$, then $[T] \subset \mathrm{bd}\, P$.*

REMARK. The assumption, that $[S, x] \in \mathcal{F}(P^+)$ for all $S \subsetneqq T$, implies that $T \cup \{x\}$ is affinely independent and therefore $\dim [T] = |T| - 1$.

PROOF. If $S \subsetneqq T$ then $[S, x] \in \mathcal{F}(P^+)$, hence $[S] \in \mathcal{F}(P)$. Therefore $[T]$ is either a face or a m.f. of P. Assume $[T]$ is a m.f. of P. Hence $\dim [T] \leqq d - k$ (Theorem 2.4). But $\dim [T] \geqq d - k$, hence $\dim [T] = d - k$ and $|T| = d - k + 1$. If $[V \setminus T] \cap \mathrm{relint}\, [x, T] \neq \emptyset$ then from the intersection Lemma 2.1 we obtain subsets $S \subset T$ and $R \subset V \setminus T$ that $[R] \cap [x, S] \neq \emptyset$ and $|R| + |S| + 1 \leqq d + 2$. $[R] \notin \mathcal{F}(P^+)$, hence $|R| \geqq k + 1$. $|S| \leqq d + 1 - |R| \leqq d - k$. Hence $S \subsetneqq T$. $[x, S] \in \mathcal{F}(P^+)$ contradicts $[x, S] \cap [R] \neq \emptyset$. Therefore $[V \setminus T] \cap \mathrm{relint}\, [x, T] = \emptyset$.

Let H be a hyperplane that separates $V \setminus T$ from $[x, T]$, $V \setminus T \subset H^-$, $T \cup \{x\} \subset H^+$. $[T]$ is a m.f. of P, hence there is a point $a \in \mathrm{relint}\, [T] \cap [V \setminus T]$ (Theorem 2.3). It follows that $a \in H$, hence $T \subset H$. So $V \subset H^-$, $[T] \subset \mathrm{bd}\, P$. □

COROLLARY 2.9. *If P is a simplicial polytope then the hypotheses of Theorem* 2.8 *imply $[T] \in \mathcal{B}(P) = \mathcal{F}(P) \setminus \{P\}$.*

We conclude this section with the following result: The combinatorial structure of an m-neighborly $2m$-polytope determines the combinatorial structure of every subpolytope. A somewhat different proof of this result appears in [10].

THEOREM 2.10. *Let Q, Q^+ be m-neighborly $2m$-polytopes,* $\mathrm{vert}\, Q^+ = \mathrm{vert}\, Q \cup \{x\}$, $x \notin Q$. *Then $\mathcal{F}(Q^+)$ determines $\mathcal{F}(Q)$.*

Theorem 2.10 is an immediate consequence of Theorem 2.7 and the following lemma:

LEMMA 2.11. *Under the assumption of Theorem* 2.10, $\mathrm{skel}_m \mathcal{F}(Q^+)$ *determines* $\mathrm{skel}_m \mathcal{F}(Q)$; *moreover, if $T \subset \mathrm{vert}\, Q$, $|T| = m + 1$, then $[T] \in \mathcal{F}(Q)$ iff either*

(1) $[T] \in \mathcal{F}(Q^+)$, *or*

(2) $[T] \notin \mathcal{F}(Q^+)$, *but $[x, S] \in \mathcal{F}(Q^+)$ for every $S \subsetneqq T$.*

PROOF. Let $T \subset \operatorname{vert} Q$, $|T| = m+1$. If (1) holds then obviously $[T] \in \mathcal{F}(Q)$. If (2) holds then Corollary 2.9 implies $[T] \in \mathcal{F}(Q)$. Conversely, if $[T] \in \mathcal{F}(Q)$ but $[T] \notin \mathcal{F}(Q^+)$ then x lies beyond all the facets of Q that include T. Therefore, if $S \subset T$, then x lies beyond some facet of P that includes S. But if $S \subsetneqq T$, then $[S] \in \mathcal{F}(Q^+)$. Therefore x lies beneath at least one facet of P that includes S, hence $[S, x] \in \mathcal{F}(Q^+)$ (see [6, section 5.2]). □

The following alternative formulation of Theorem 2.10 will be useful in the sequel:

THEOREM 2.12. *Let Q_1, Q_2 be m-neighborly $2m$-polytopes and let the bijection $\varphi : \operatorname{vert} Q_1 \to \operatorname{vert} Q_2$ be a combinatorial equivalence between Q_1 and Q_2.*

If $A \subset \operatorname{vert} Q_1$, then the restriction of φ to A is a combinatorial equivalence between $[A]$ and $[\varphi(A)]$.

3. Universal faces

DEFINITION 3.1. Suppose $\Phi \in \mathcal{B}(P)(=\mathcal{F}(P)\setminus\{P\})$. Φ is a *u-universal face* (u-u.f.) *of P* if $[\Phi, S] \in \mathcal{B}(P)$ for every $S \subset \operatorname{vert} P$ with $|S| \leqq u$.

Define $\mathcal{B}(P, u) = \{\Phi \in \mathcal{B}(P) : \Phi$ is a u-u.f. of $P\}$. Note that $\mathcal{B}(P, 0) = \mathcal{B}(P)$, and $\varnothing \in \mathcal{B}(P, k)$ iff P is k-neighborly.

DEFINITION 3.2. Φ is a *universal face* (u.f.) *of P* if $\Phi \in \mathcal{B}(P, u)$ with $u = [\frac{1}{2}(d\text{-}\dim \Phi - 1)] = [\frac{1}{2}(d - |\operatorname{vert} \Phi|)]$. A 1-dimensional u.f. is called a *universal edge* (u.e.).

Definitions 3.1 and 3.2 can be reformulated using the notion of a quotient polytope P/Φ introduced in [8, ch. 2, th. 16]. (A quotient polytope P/Φ is a polytope K whose face lattice $\mathcal{F}(K)$ is isomorphic to the upper segment $[\Phi, P]$ of $\mathcal{F}(P)$.)

DEFINITION 3.1*. $\Phi \in \mathcal{B}(P, u)$ iff either (a) $u = 0$ and $\Phi \in \mathcal{B}(P)$, or (b) $u > 0$ and the quotient polytope P/Φ is u-neighborly, with $|\operatorname{vert} P| - |\operatorname{vert} \Phi|$ vertices.

REMARK. $\dim P/\Phi = \dim P - \dim \Phi - 1 = \dim P - |\operatorname{vert} \Phi|$. Since we assume that P is not a simplex, $|\operatorname{vert} P| > \dim P + 1$. Therefore, if $\Phi \in \mathcal{B}(P, u)$, $u > 0$, then $|\operatorname{vert} P/\Phi| = |\operatorname{vert} P| - |\operatorname{vert} \Phi| > \dim P/\Phi + 1$, i.e., P/Φ is not a simplex, and therefore $u \leqq \frac{1}{2}\dim P/\Phi \leqq \frac{1}{2}\dim P$.

DEFINITION 3.2*. Φ is a u.f. of P iff either (a) Φ is a facet of P, or (b) $\Phi \in \mathcal{B}(P)$ and P/Φ is a neighborly (i.e., $[\frac{1}{2}\dim P/\Phi]$-neighborly) polytope with $|\operatorname{vert} P| - |\operatorname{vert} \Phi|$ vertices.

One can use Definitions 3.1 and 3.2 in the case where P is a simplex. The various remarks that follow these definitions remain valid if we adhere to the convention that a simplex Δ is u-neighborly for $0 \leqq u \leqq \dim \Delta$ only.

THEOREM 3.3. *Let P be a k-neighborly d-polytope. If $\Psi_i \in \mathcal{B}(P, u_i)$ for $i = 1, 2$ and $t = u_1 + u_2 + k - d \geqq 0$, then $\Psi = [\Psi_1, \Psi_2] \in \mathcal{B}(P, t)$.*

PROOF. Assume $A \subset \operatorname{vert} P \setminus \Psi$ and $[\Psi, A] \notin \mathcal{B}(P)$. It suffices to show that $|A| > t$.

There are three pairwise disjoint sets $B \subset A$, $T_1 \subset \operatorname{vert} \Psi_1$, $T_2 \subset \operatorname{vert} \Psi_2$ such that $[T_1, T_2, B]$ is a m.f. of P. By Theorem 2.4, $|T_1| + |T_2| + |B| \leqq d - k + 1$. $[T_1, \Psi_2, B] \notin B(P)$, hence $|T_1| + |B| \geqq u_2 + 1$. By the same reasoning $|T_2| + |B| \geqq u_1 + 1$. Therefore $d - k + 1 + |B| \geqq |T_1| + |T_2| + 2|B| \geqq u_1 + u_2 + 2$, hence $|A| \geqq |B| \geqq u_1 + u_2 + k - d + 1 = t + 1$. □

The next theorem is a very useful special case of Theorem 3.3. From this point onward, the letter Q will always denote an m-neighborly $2m$-polytope.

THEOREM 3.4. *Let $\Psi_1, \cdots, \Psi_l$ be pairwise disjoint universal faces of Q. If Ψ_1 has $2v_i$ vertices $(1 \leqq i \leqq l)$ and $v_1 + \cdots + v_l \leqq m$, then $\Psi = [\Psi_1, \cdots, \Psi_l]$ is a* u.f. *of Q.*

REMARK. If Φ is a face of Q with $2j$ vertices, then Φ is a u.f. iff Φ is an $(m - j)$-u.f.

PROOF. Theorem 3.4 in its full generality follows from the case $l = 2$ by induction on l. For $l = 2$, apply Theorem 3.3 with $d = 2m$, $k = m$, $u_1 = m - v_1$, $u_2 = m - v_2$. □

Let $C = C(v, 2m)$ be a cyclic $2m$-polytope with v vertices. There is a natural cyclic order on vert C. Assume that $a_1, a_2, \cdots, a_v, a_1$ are the vertices of C in this order. From Gale's Evenness Condition it follows that $[a_i, a_{i+1}]$ $(1 \leqq i < v)$ and $[a_v, a_1]$ are u.e.s of C, and if $v \geqq 2m + 3$ then $C(v, 2m)$ has no other u.e.s. See also [3, section 3]. The existence of a hamiltonian circuit of u.e.s characterizes cyclic polytopes:

THEOREM 3.5. *Assume $|\operatorname{vert} Q| \geqq 2m + 3$. If Q has a simple circuit of length v consisting of universal edges, then $Q \cong C(v, 2m)$.*

PROOF. *Case I*: $v < 2m + 3$.

Choose $2m + 3 - v$ vertices of Q which are not in the given circuit, and add them to the vertices of the circuit. The resulting set spans a subpolytope Q' of Q. Q' is neighborly, hence $Q' \cong C(2m + 3, 2m)$ (see [6, th. 7.2.3]). The edges of the given circuit are u.e.s in Q', but Q', being a cyclic polytope with $2m+3$ vertices, has only one simple circuit of u.e.s of length $2m+3$, a contradiction.

Case II: $v \geqq 2m+3$.

Suppose that the vertices of $C(v,2m)$, in their natural cyclic order, are $a_1, \cdots, a_v, a_1$. Let φ be a $1:1$ mapping of vert $C(v,2m)$ into vert Q, which maps the cycle $a_1, \cdots, a_v, a_1$ onto the given circuit of universal edges of Q. Gale's Evenness Condition and Theorem 3.4 imply that φ induces a $1:1$ mapping of the set of facets of $C(v,2m)$ into the set of facets of Q. By Lemma 3.6 below, φ is a combinatorial equivalence between $C(v,2m)$ and Q. □

LEMMA 3.6. *Suppose $\varphi : \text{vert}\, P \to \text{vert}\, P'$ is a bijection. If for every facet F of P, $F' = [\varphi(\text{vert}\, F)]$ is a facet of P', then φ is a combinatorial equivalence between P and P'.*

PROOF. Since the incidence graph of facets and subfacets of the polytope P' is connected, it is enough to verify the following assertion:

If F', G' are two adjacent facets of P' (i.e., $F' \cap G'$ is a subfacet of P') and if $F' = \varphi(F)$ for some facet F of P, then $G' = \varphi(G)$ for some facet G of P.

Indeed, consider the set $\varphi^{-1}(\text{vert}\,(F' \cap G'))$. Since P and P' are simplicial, this is the set of vertices of a subfacet H of P. H is included in precisely two different facets F, G of P. Since $\varphi(F) = F'$, $\varphi(G)$ must be G'. □

REMARK. Lemma 3.6 holds even without assuming that P and P' are simplicial. The proof in the general case is slightly more involved, and uses induction on $\dim P$.

As we mentioned above, every proper face is a 0-u.f. The next two theorems can be considered as generalizations of Theorem 2.6, and Corollary 2.9 to Theorem 2.8.

THEOREM 3.7. *Assume P is a k-neighborly d-polytope, $S \subsetneqq \text{vert}\, P$. $[S]$ is a u-u.f. of P iff $[T]$ is a u-u.f. of P for every $T \subset S$ with $|T| \leqq d-k-u+1$.*

PROOF. If $[S] \notin \mathscr{B}(P,u)$ then there is a set $A \subset \text{vert}\, P \setminus S$, $|A| \leqq u$ with $[S,A] \notin \mathscr{B}(P)$. There are sets $T \subset S$, $B \subset A$ such that $[T,B]$ is a m.f. of P, hence $|T| + |B| \leqq d-k+1$ (Theorem 2.4).

Case I: $|T| \leqq d-k-u+1$.

Then $[T] \notin \mathscr{B}(P,u)$ a contradiction.

Case II: $|T| > d-k-u+1$.

Suppose $R \subset T$, $|R| = |T| - (d - k - u + 1)$. Then $|T \setminus R| = d - k - u + 1$, $T \setminus R \subset S$ and $[T \setminus R] \notin \mathscr{B}(P, u)$ since $|B \cup R| = |B| + |R| = |B| + |T| - (d - k - u + 1) \leqq (d - k + 1) - (d - k - u + 1) = u$. □

THEOREM 3.8. *Assume P, P^+ are k-neighborly d-polytopes,* $\operatorname{vert} P^+ = \operatorname{vert} P \cup \{x\}$, *$x \notin P$, $\emptyset \neq A \subset \operatorname{vert} P$ and $|A| > d - k - u$.*

If $[A \setminus \{q\}, x] \in \mathscr{B}(P^+, u)$ for every $q \in A$ then $[A] \in \mathscr{B}(P, u)$.

PROOF. We have to show that if $W \subset \operatorname{vert} P \setminus A$, $|W| \leqq u$, then $[A, W] \in \mathscr{B}(P)$.

If $|W| < u$, choose any $q \in A$; then $[A, W, x] = [A \setminus \{q\}, x, W \cup \{q\}] \in \mathscr{B}(P^+)$, hence $[A, W] \in \mathscr{B}(P)$. Now assume that $|W| = u$. Then $|A \cup W| = |A| + u > d - k$. By Corollary 2.9 it suffices to prove that $[S, x] \in \mathscr{B}(P^+)$ for all $S \subsetneqq A \cup W$.

Assume $S \subsetneqq A \cup W$. If $S \not\supset A$ choose a point $q \in A \setminus S$; then $[A \setminus \{q\}, x, W] \in \mathscr{B}(P^+)$, $S \subset (A \setminus \{q\}) \cup W$, hence $[S, x] \in \mathscr{B}(P^+)$. If $S \supset A$, then $W \cap S \subsetneqq W$. Choose any $q \in A$; then $[S, x] = [A \setminus \{q\}, x, (W \cap S) \cup \{q\}] \in \mathscr{B}(P^+)$. □

COROLLARY 3.9. *Let Q, Q^+ be neighborly $2m$-polytopes,* $\operatorname{vert} Q^+ = \operatorname{vert} Q \cup \{x\}$, *$x \notin Q$. If $\emptyset \neq A \subset \operatorname{vert} Q$, $|A| \geqq 2$ and if $[A \setminus \{q\}, x]$ is a* u.f. *of Q^+ for every $q \in A$, then $[A]$ is a* u.f. *of Q.*

PROOF. $|A| = 2j$ with $j \geqq 1$ or $|A| = 2j - 1$ with $j \geqq 2$. In both cases apply Theorem 3.8 with $d = 2m$, $k = m$, $u = m - j$. □

We shall use this corollary later with $|A| = 2$.

THEOREM 3.10. *If $|\operatorname{vert} Q| \geqq 2m + 3$, then the graph of the universal edges of Q is either a hamiltonian circuit, or a union of disjoint simple paths.*

PROOF. In view of Theorem 3.5, it is sufficient to prove that no vertex of Q is included in three u.e.s of Q.

Suppose x, a, b, c are distinct vertices of Q, and $[x, a]$, $[x, b]$, $[x, c]$ are u.e.s of Q. Choose a set A of $2m - 1$ vertices of Q other than x, a, b, c, and define $Q' = [x, a, b, c, A]$. Q' is a neighborly $2m$-polytope with $2m + 3$ vertices, hence $Q' \cong C(2m + 3, 2m)$. $[x, a]$, $[x, b]$, $[x, c]$ are u.e.s of Q', but no vertex of $C(2m + 3, 2m)$ is included in three u.e.s, a contradiction. □

4. The sewing construction

In this section we describe a construction, called sewing, and some related notions. This construction, first introduced by the author in [9], will play a central role in the sequel.

The "facet-splitting" operation of Barnette [5] is, in a sense, dual to our sewing construction. We shall discuss the relationship between these two constructions in section 7.4.

DEFINITION 4.1. If $\Psi \in \mathcal{F}(P)$ and $M \subset \operatorname{vert} P \setminus \Psi$, then we say that M is a *missing face of P relative to Ψ* if $[M, \Psi] \notin \mathcal{B}(P)$, but $[M', \Psi] \in \mathcal{B}(P)$ for every $M' \subsetneqq M$.

Define: $\mathcal{M}(P/\Psi) = \{M : M$ is a m.f. of P relative to $\Psi\}$. Finally define: $\mathcal{M}(P) = \mathcal{M}(P/\emptyset)$.

PROPOSITION 4.2. (1) $\mathcal{M}(P/P) = \{\emptyset\}$.

(2) *If F is a facet of P, then $\mathcal{M}(P/F) = \{\{q\} : q \in \operatorname{vert} P \setminus F\}$.*

(3) *$M \in \mathcal{M}(P)$ iff $[M]$ is a* m.f. *of P.*

(4) *If $M \in \mathcal{M}(P/\Psi)$, then there is a set S in $\mathcal{M}(P)$ such that $M \cup \Psi \supset S \supset M$.*

(5) *If $\Psi \in \mathcal{F}(P)$, then for every vertex b in* $\operatorname{vert} P \setminus \Psi$ *there are sets M and N in $\mathcal{M}(P/\Psi)$ such that $b \in M$ and $b \notin N$.*

If Q is a neighborly $2m$-polytope then the following properties hold too:

(6) *If $M \in \mathcal{M}(Q)$ then $|M| = m + 1$.*

(7) *If Ψ is a* u.f. *of Q, $|\operatorname{vert} \Psi| = 2j$ and $M \in \mathcal{M}(Q/\Psi)$, then $|M| = m - j + 1$.*

(8) *Ψ is a u-*u.f. *of Q iff $|M \cap \Psi| \leqq m - u$ for every $M \in \mathcal{M}(Q)$, or equivalently, if $|M \setminus \Psi| \geqq u + 1$ for every $M \in \mathcal{M}(Q)$.*

In particular

(9) *E is a* u.e. *of Q iff no element of $\mathcal{M}(Q)$ includes* $\operatorname{vert} E$.

PROOF. (1)–(4) follow immediately from Definitions 4.1 and 2.2. In order to establish (5), take $M = \{b\} \cup C$, where C is a minimal subset (with respect to inclusion) of $\operatorname{vert} P \setminus (\Psi \cup \{b\})$ such that $[b, C, \Psi] \notin \mathcal{B}(P)$ but $[C, \Psi] \in \mathcal{B}(P)$, and take N to be a minimal subset of $\operatorname{vert} P \setminus (\Psi \cup \{b\})$ such that $[N, \Psi] \notin \mathcal{B}(P)$.

(6) and (7) follow from Theorem 2.4, and (6) implies (8) and (9). □

DEFINITION 4.3. A *tower in P* is a strictly increasing sequence $\mathcal{T} = \{\Phi_j\}_{j=1}^{k}$ of non-empty proper faces of P. Sometimes we shall adjoin the empty face as a first element Φ_0 of $\mathcal{T}$. If $\Phi \in \mathcal{F}(P)$, denote by $\mathcal{F}_\Phi$ the set of all facets of P which include Φ. We denote $\mathcal{F}_{\Phi_j}$ by $\mathcal{F}_j$ (in order to avoid double subscripts). Note that $\mathcal{F}_1 \supset \mathcal{F}_2 \supset \cdots \mathcal{F}_k$. Define $\mathcal{C} = \mathcal{C}(P, \mathcal{T}) = \mathcal{F}_1 \setminus (\mathcal{F}_2 \setminus (\cdots \setminus \mathcal{F}_k) \cdots)$. It is easy to see that $\mathcal{C} = (\mathcal{F}_1 \setminus \mathcal{F}_2) \cup (\mathcal{F}_3 \setminus \mathcal{F}_4) \cup \cdots$ where the last term in the union is $\mathcal{F}_{k-1} \setminus \mathcal{F}_k$ if k is even or $\mathcal{F}_k$ if k is odd. With the convention that $\mathcal{F}_j = \emptyset$ and $\Phi_j = P$ for $j > k$ we can simply write $\mathcal{C} = \bigcup_{i=1}^{\infty} (\mathcal{F}_{2i-1} \setminus \mathcal{F}_{2i})$ and similarly $\mathcal{F}_0 \setminus \mathcal{C} = \bigcup_{i=0}^{\infty} (\mathcal{F}_{2i} \setminus \mathcal{F}_{2i+1})$, where $\mathcal{F}_0$ is the set of all facets of P.

We say that $\mathcal{T} = \{\Phi_j\}_{j=1}^{m}$ is a *universal tower* (u.t.) *in Q* if

(1) Q is a neighborly $2m$-polytope,

(2) Φ_j is a u.f. of Q for $1 \leqq j \leqq m$,

(3) $|\operatorname{vert} \Phi_j| = 2j$ for $1 \leqq j \leqq m$.

Let $\mathscr{D}$ be a set of facets of P. We say that a point $x \in R^d$ *lies exactly beyond* $\mathscr{D}$ *with respect to* P if x lies beyond every facet of P that is in $\mathscr{D}$ and beneath every other facet of P. If it is clear from the context what is the polytope P, we omit the phrase "with respect to P".

LEMMA 4.4. *Let $\mathscr{T}$ be a tower in P, $\mathscr{C} = \mathscr{C}(P, \mathscr{T})$. Then there is a point $x \in R^d$ which lies exactly beyond $\mathscr{C}$.*

PROOF. By induction on the height k of $\mathscr{T}$. If $k = 0$, define $\mathscr{C} = \emptyset$. In that case every point $x \in \operatorname{int} P$ lies exactly beyond $\mathscr{C}$. If $k \geqq 1$, let $\mathscr{T}' = \mathscr{T} \setminus \{\Phi_1\}$ and $\mathscr{C}' = \mathscr{C}(P, \mathscr{T}')$. By the induction hypothesis, there is a point $x' \in R^d$, which lies exactly beyond $\mathscr{C}'$. Note that $\mathscr{C}' \subset \mathscr{F}_1$ and $\mathscr{C} = \mathscr{F}_1 \setminus \mathscr{C}'$. Choose a point $p \in \operatorname{relint} \Phi_1$ and let $x = (1+\varepsilon)p - \varepsilon x'$. If ε is positive and sufficiently small, then x lies exactly beyond $\mathscr{C}$. □

The construction which we have just described will enable us to construct a large variety of neighborly polytopes by adding new vertices to existing neighborly polytopes.

From here until the end of section 4 we adhere to the following convention:

CONVENTION 4.5. Q is a neighborly $2m$-polytope, Q is not simplex, $\mathscr{T} = \{\Phi_j\}_{j=1}^m$ is a u.t. in Q, $\mathscr{C} = \mathscr{C}(Q, \mathscr{T})$, x lies exactly beyond $\mathscr{C}$ with respect to Q, and $Q^+ = [Q, x]$.

Define $S_j = \operatorname{vert} \Phi_j \setminus \Phi_{j-1}$ for $j = 1, 2, \cdots, m+1$ (recall that $\Phi_0 = \emptyset$ and $\Phi_j = Q$ for $j > m$).

THEOREM 4.6. (1) *Q^+ is a simplicial $2m$-polytope and* $\operatorname{vert} Q^+ = \operatorname{vert} Q \cup \{x\}$.

(2) *Q^+ is neighborly.*

(3) *If $0 < j \leqq m$ is even, then Φ_j is a* u.f. *of Q^+.*

(4) *If $0 < j \leqq m$ is odd, then Φ_j is not a* u.f. *of Q^+, but if $j < m$ then Φ_j is still a face of Q^+.*

(5) *If $a \in S_j$ for some $1 \leqq j \leqq m$, then $[\Phi_{j-1}, a, x]$ is a* u.f. *of Q^+.*

LEMMA 4.7. (1) $\operatorname{vert} Q^+ = \operatorname{vert} Q \cup \{x\}$.

(2) *If $M \subset \operatorname{vert} Q \cup \{x\}$, then $M \in \mathscr{M}(Q^+)$ iff either* (a) $M = \bigcup_{\nu=1}^{j} S_{2\nu-1} \cup A$ *for some integer $0 \leqq j \leqq (m+1)/2$ and some $A \in \mathscr{M}(Q/\Phi_{2j})$, or* (b) $M =$

$\bigcup_{\nu=1}^{j} S_{2\nu} \cup A \cup \{x\}$ *for some integer* $0 \leqq j \leqq m/2$ *and some set* $A \in \mathcal{M}(Q/\Phi_{2j+1})$.

(Note that $\Phi_{m+1} = Q$ and $\mathcal{M}(Q, Q) = \{\emptyset\}$.)

PROOF. *Step* 1: We show that if M is of type (a), then $[M] \not\in \mathcal{B}(Q^+)$.

Assume $M = \bigcup_{\nu=1}^{j} S_{2\nu-1} \cup A$, $A \in \mathcal{M}(Q/\Phi_{2j})$, $0 \leqq j \leqq (m+1)/2$. If $j = 0$, then $[M] = [A] \not\in \mathcal{B}(Q)$. Assume $j > 0$, hence $S_1 \subset M$.

Let F be a facet in $\mathcal{F}_{[M]}$, and let μ be the maximal integer ν such that $1 \leqq \nu \leqq j$ and $F \in \mathcal{F}_{2\nu-1}$. Then $F \not\in \mathcal{F}_{2\mu}$, since otherwise, if $\mu < j$ then $F \in \mathcal{F}_{2\mu+1}$, because $S_{2\mu+1} \subset M \subset F$, and if $\mu = j$ then $F \supset A \cup \Phi_{2j}$, in contradiction to $A \in \mathcal{M}(Q/\Phi_{2j})$. Therefore $\mathcal{F}_{[M]} \subset \bigcup_{\nu=1}^{j} (\mathcal{F}_{2\nu-1} \setminus \mathcal{F}_{2\nu}) \subset \mathcal{C}$. It follows that if M is of type (a), then $[M] \not\in \mathcal{B}(Q^+)$.

Step 2: We show that if M is of type (b), then $[M] \not\in \mathcal{B}(Q^+)$. Define $M^- = M \setminus \{x\}$. The rest of step 2 is similar to step 1: we prove that $\mathcal{F}_{[M^-]} \subset \bigcup_{\nu=0}^{j} (\mathcal{F}_{2\nu} \setminus \mathcal{F}_{2\nu+1}) \subset \mathcal{F}_0 \setminus \mathcal{C}$ and conclude that $[M] \not\in \mathcal{B}(Q^+)$.

Step 3: Now we show that if $S \subset \operatorname{vert} Q$ and $[S] \not\in \mathcal{B}(Q^+)$, then S includes a set M of type (a). Since $[S] \not\in \mathcal{B}(Q^+)$, it follows that if $F \in \mathcal{F}_0$ and $F \supset S$, then $F \in \mathcal{C}$.

Let j be the first nonnegative integer such that $[S, \Phi_{2j}] \not\in \mathcal{B}(Q)$. Clearly $2j \leqq m+2$. Since $\Phi_m \in \mathcal{B}(Q^+)$ for even m, it follows that $2j \leqq m+1$.

We proceed to show that $S_{2\nu-1} \subset S$ for $1 \leqq \nu \leqq j$. Since $[S, \Phi_{2\nu-2}] \in \mathcal{B}(Q)$, if $S_{2\nu-1} \not\subset S$ then Q has a facet F such that $F \supset S \cup \Phi_{2\nu-2}$, $F \not\supset \Phi_{2\nu-1}$, hence $F \not\in \mathcal{C}$ — a contradiction. Since $[S, \Phi_{2j}] \not\in \mathcal{B}(Q)$, S includes a set $A \in \mathcal{M}(Q/\Phi_{2j})$. $S \supset M = \bigcup_{\nu=1}^{j} S_{2\nu-1} \cup A$, M is of type (a). From this it follows that $\operatorname{vert} Q \subset \operatorname{vert} Q^+$. $x \in \operatorname{vert} Q^+$, by the definition of x, hence assertion (1) follows.

Step 4: Now prove that if $x \in S \subset \operatorname{vert} Q^+$ and $[S] \not\in \mathcal{B}(Q^+)$, then S includes a set M of type (a) or (b). Denote $S^- = S \setminus \{x\}$. If $[S^-] \not\in \mathcal{B}(Q^+)$, then S^- includes a set of type (a), by step 3. If $[S^-] \in \mathcal{B}(Q^+)$, then one can show, as in step 3, that S includes a set of type (b).

Step 5: Note that all the sets of type (a) or (b) have $m+1$ elements (see 4.2(7)).

Step 6: Let S be an element of $\mathcal{M}(Q^+)$. S includes a set M of type (a) or (b) (by steps 3, 4). But $[M] \not\in \mathcal{B}(Q^+)$ (by steps 1, 2). Hence $M = S$, and S is of type (a) or (b).

Step 7: Conversely, assume M is of type (a) or (b). $[M] \not\in \mathcal{B}(Q^+)$, hence M includes a set $S \in \mathcal{M}(Q^+)$. By step 6, S is of type (a) or (b), hence $|S| = |M| = m+1$. It follows that $M = S \in \mathcal{M}(Q^+)$.

PROOF OF THEOREM 4.6. (1) and (2) follow immediately from Lemma 4.7.

(3) Assume $0 < 2p \leqq m$. By 4.2(8), in order to prove that Φ_{2p} is a u.f. of Q^+, it suffices to show that $|M \setminus \Phi_{2p}| \geqq m - 2p + 1$ or, equivalently, $|M \cap \Phi_{2p}| \leqq 2p$ for every $M \in \mathcal{M}(Q^+)$.

Case I: Assume $M = S_1 \cup \cdots \cup S_{2j-1} \cup A$, $A \in \mathcal{M}(Q/\Phi_{2j})$. If $j \leqq p$, then $|M \setminus \Phi_{2p}| = |A \setminus \Phi_{2p}| \geqq m - 2p + 1$, since $[A, \Phi_{2p}] \notin \mathcal{B}(Q)$. If $j \geqq p$, then $|M \cap \Phi_{2p}| = |S_1 \cup \cdots \cup S_{2p-1}| = 2p$.

Case II: Assume $M = S_2 \cup \cdots \cup S_{2j} \cup A \cup \{x\}$, $A \in \mathcal{M}(Q/\Phi_{2j+1})$. If $j < p$, then $|M \setminus \Phi_{2p}| = 1 + |A \setminus \Phi_{2p}| \geqq m - 2p + 2$. If $j \geqq p$, then $|M \cap \Phi_{2p}| = 2p$.

(4) Assume $1 \leqq 2p + 1 \leqq m$. Take $M = S_1 \cup \cdots \cup S_{2p+1} \cup A$ for some $A \in \mathcal{M}(Q/\Phi_{2p+2})$. Then $M \in \mathcal{M}(Q^+)$, $|\Phi_{2p+1} \cap M| = 2p + 2$, hence Φ_{2p+1} is not a u.f. of Q^+. If j is odd and $< m$, then $\Phi_{j+1} \in \mathcal{B}(Q^+)$, hence $\Phi_j \in \mathcal{B}(Q^+)$. In fact, Φ_j is an $(m - j - 1)$-u.f. of Q^+.

(5) *Case I.* Assume $2 \leqq 2p \leqq m$, $a \in S_{2p}$ and consider $\Psi = [\Phi_{2p-1}, a, x]$.

Subcase Ia. $M = S_1 \cup \cdots \cup S_{2j-1} \cup A$, $A \in \mathcal{M}(Q/\Phi_{2j})$. If $j \leqq p$, then $|M \setminus \Psi| \geqq |A \setminus \Phi_{2p}| \geqq m - 2p + 1$. If $j \geqq p$, then $|M \cap \Psi| = |S_1 \cup \cdots \cup S_{2p-1}| = 2p$.

Subcase Ib. $M = S_2 \cup \cdots \cup S_{2j} \cup A \cup \{x\}$, $A \in \mathcal{M}(Q/\Phi_{2j+1})$. If $j < p$, then $|M \setminus \Psi| \geqq |A \setminus \Phi_{2p}| \geqq m - 2p + 1$. If $j \geqq p$, then $|M \cap \Psi| = |S_2 \cup \cdots \cup S_{2p-2} \cup \{a, x\}| = 2p$.

Therefore Ψ is $(m - 2p)$-universal in P^+.

Case II. Assume $1 \leqq 2p + 1 \leqq m$, $a \in S_{2p+1}$, and consider $\Psi = [\Phi_{2p}, a, x]$.

Subcase IIa. $M = S_1 \cup \cdots \cup S_{2j-1} \cup A$, $A \in \mathcal{M}(Q/\Phi_{2j})$. If $j \leqq p$, then $|M \setminus \Psi| \geqq |A \setminus \Phi_{2p+1}| \geqq m - (2p + 1) + 1$. If $j > p$, then $|M \cap \Psi| = |S_1 \cup \cdots \cup S_{2p-1} \cup \{a\}| = 2p + 1$.

Subcase IIb. $M = S_2 \cup \cdots \cup S_{2j} \cup A \cup \{x\}$, $A \in \mathcal{M}(Q/\Phi_{2j+1})$. If $j \leqq p$, then $|M \setminus \Psi| \geqq |A \setminus \Phi_{2p+1}| \geqq m - (2p + 1) + 1$. If $j \geqq p$, then $|M \cap \Psi| = 1 + |S_2 \cup \cdots \cup S_{2p}| = 2p + 1$.

Therefore Ψ is $(m - (2p + 1))$-universal in Q^+. □

THEOREM 4.8. *If* $\Psi \in \mathcal{B}(Q, u)$, $\Psi \cap \Phi_{m-1} = \emptyset$ *and* $|\Psi \cap S_m| \leqq 1$, *then* $\Psi \in \mathcal{B}(Q^+, u)$.

PROOF. Assume $M = \bigcup_{\nu=1}^{j} S_{2\nu-1} \cup A$, $A \in \mathcal{M}(Q/\Phi_{2j})$ or $M = \bigcup_{\nu=1}^{j} S_{2\nu} \cup A \cup \{x\}$, $A \in \mathcal{M}(Q/\Phi_{2j+1})$. Define $r = 2j - 1$ in the first case, $r = 2j$ in the second case. There is a set $\bar{M} \in \mathcal{M}(Q)$ such that $A \cup \Phi_{r+1} \supset \bar{M} \supset A$. If $r < m$, then $|M \cap \Psi| = |A \cap \Psi| \leqq |\bar{M} \cap \Psi| \leqq m - u$. If $r = m$, then $|M \cap \Psi| \leqq 1$. If $u = m$, then $\Psi = \emptyset$, hence $0 = |M \cap \Psi| \leqq m - u = 0$. If $u < m$, then $|M \cap \Psi| \leqq 1 \leqq m - u$. □

THEOREM 4.9. *If* $\Psi \in \mathcal{B}(Q,u)$, $\Psi \cap \Phi_m = \varnothing$, $a \in S_p$, $b \in S_{p+1}$, $1 \leqq p < m$ *and* $[\Psi, a, b] \in \mathcal{B}(Q, u-1)$, *then* $[\Psi, a, b] \in \mathcal{B}(Q^+, u-1)$.

PROOF. We use the same notations as in the proof of Theorem 4.8. If $r \geqq p$, then $|M \cap (\Psi \cup \{a,b\})| = |A \cap \Psi| + 1 \leqq |\bar{M} \cap \Psi| + 1 \leqq m - u + 1$. If $r < p$ then $|M \cap (\Psi \cup \{a,b\})| = |A \cap (\Psi \cup \{a,b\})| \leqq |\bar{M} \cap (\Psi \cup \{a,b\})| \leqq m - (u-1)$. □

THEOREM 4.10. *An edge E of Q^+ is a* u.e. *of Q^+ iff either*

(1) $E = [a,x]$ *and* $a \in S_1$, *or*

(2) *E is a* u.e. *of Q and either* (a) $E \cap \Phi_m = \varnothing$ *or* (b) $E = [a,b]$ *with* $a \in S_p$ *and* $b \in S_{p+1}$ *for some* $1 \leqq p \leqq m$.

PROOF. In view of Theorems 4.6(5), 4.8, 4.9 and 3.10 it suffices to prove that if $E = [a,b]$ is an edge of Q and either

(α) $a \in S_p$, $b \notin \Phi_{p+1}$ for some $1 \leqq p < m$, or

(β) $E = [S_p]$ for some $1 \leqq p \leqq m$,

then E is not universal in Q^+.

PROOF OF (α). If $a \in S_p$, $b \notin \Phi_{p+1}$, $1 \leqq p < m$, then by 4.2(5) b belongs to a set $A \in \mathcal{M}(Q/\Phi_{p+1})$. $\{a\} \cup A$ is included in a m.f. of Q^+ (of type (a) if p is odd and of type (b) if p is even). But $[a,b] \subset [a,A]$, hence $[a,b]$ is not universal in Q^+.

(β) follows immediately from Lemma 4.7. □

In the rest of this section we deal with the problem of embedding the complex $\mathcal{B}(Q,u)$ of u-universal faces of Q (see Definition 3.1) in the boundary complex of a neighborly $2(m-u)$-polytope.

This material will not be needed in subsequent sections.

If $|\text{vert}\, Q| = 2m+2$, then the structure of Q is well-known (see [6, pp. 98, 108]). Q has two vertex-disjoint missing m-faces Δ_1, Δ_2, and $\mathcal{B}(Q,u)$ is the free join of $\text{skel}_{m-u-1}\Delta_1$ and $\text{skel}_{m-u-1}\Delta_2$, i.e. if $S \subset \text{vert}\, Q$ then $[S] \in \mathcal{B}(Q,u)$ iff $|S \cap \Delta_1| \leqq m-u$ and $|S \cap \Delta_2| \leqq m-u$.

It follows that every u-universal $(2m-2u-2)$-face of Q is included in precisely $u+2$ u-universal $(2m-2u-1)$-faces of Q, consequently, $\mathcal{B}(Q,u)$ is not embeddable in the boundary complex of any $2(m-u)$-polytope, unless $u=0$.

From now on, assume that $|\text{vert}\, Q| = v \geqq 2m+3$.

For cyclic polytopes the situation is simple: If $Q = C(v,2m)$, $v \geqq 2m+3$, then $\mathcal{B}(Q,u) \cong \mathcal{B}(C(v,2(m-u)))$ for $u = 0,1,\cdots,m-1$, and the isomorphism is given by any mapping of vert Q onto vert $C(v,2(m-u))$ which preserves the natural cyclic order of the vertices. Moreover, if the vertices of Q lie on the moment curve $x(t)=(t,t^2,\cdots,t^{2m})$, or on the trigonometric moment curve $x(\theta)=(\cos\theta,\sin\theta,\cdots,\cos m\theta,\sin m\theta)$, then $C(v,2(m-u))$ can be taken as the image of Q under the orthogonal projection of R^{2m} onto the subspace $R^{2(m-u)}=\{(x_1,\cdots,x_{2(m-u)},0,\cdots,0): x_i\in R\}$.

In the case $u=m-1$, Theorem 3.10 solves the problem stated above: $\mathcal{B}(Q,m-1)$ is isomorphic to a subcomplex of the boundary complex of a convex v-gon.

Now we shall see that the embeddability of $\mathcal{B}(Q,u)$ is preserved by the sewing construction.

THEOREM 4.11. *Suppose Q_u is a $2(m-u)$-polytope $(0<u<m)$, $|\operatorname{vert} Q_u|=|\operatorname{vert} Q|$, and α is an isomorphism of $\mathcal{B}(Q,u)$ into $\mathcal{B}(Q_u)$. (This implies that Q_u is neighborly.) Assume also that $Q^+=[Q,x]$ is obtained from Q by sewing at x through a* u.t. $\mathcal{T}$.

Then there is a polytope $Q_u^+=[Q_u,x_u]$, obtained from Q_u by sewing at x_u through an appropriate u.t. *$\mathcal{T}_u$, and an isomorphism α^+ of $\mathcal{B}(Q^+,u)$ into $\mathcal{B}(Q_u^+)$. Moreover, $\alpha^+(q)=\alpha(q)$ for all $q\in\operatorname{vert} Q$.*

PROOF. Define $\mathcal{T}_u=\{\alpha\Phi_j\}_{j=1}^{m-u}$. It is easily checked that $\mathcal{T}_u$ is u.t. in Q_u. Choose x_u to be a point that lies exactly beyond $\mathcal{C}(Q_u,\mathcal{T}_u)$, and let $Q_u^+=[Q_u,x_u]$.

Define $\alpha^+:\operatorname{vert} Q^+\to\operatorname{vert} Q_u^+$ by $\alpha^+(x)=x_u$, $\alpha^+(q)$ for $q\in\operatorname{vert} Q$. In order to show that α^+ extends to an isomorphism of $\mathcal{B}(Q^+,u)$ into $\mathcal{B}(Q_u^+)$, we must prove that if $S\subset\operatorname{vert} Q^+$ and $[\alpha^+S]\notin\mathcal{B}(Q_u^+)$ then $[S]\notin\mathcal{B}(Q^+,u)$. It suffices to prove this for $S\subset\operatorname{vert} Q^+$ such that $[\alpha^+S]$ is a m.f. of Q_u^+.

Assume $[\alpha^+S]$ is a m.f. of Q_u^+.

Case I: $\alpha^+S=\alpha(S_1\cup S_3\cup\cdots\cup S_{2j-1})\cup\alpha A$ where $0\leqq j<\frac{1}{2}(m-u+1)$ and $\alpha A\in\mathcal{M}(Q_u/\alpha\Phi_{2j})$, or $j=\frac{1}{2}(m-u+1)$ and $A=\varnothing$ (see Lemma 4.7). It follows that $S=S_1\cup S_3\cup\cdots\cup S_{2j-1}\cup A$ and $[\alpha A,\alpha\Phi_{2j}]\notin\mathcal{B}(Q_u)$. Therefore $[A,\Phi_{2j}]\notin\mathcal{B}(Q,u)$, hence there is a set $B\subset\operatorname{vert} Q$, $|B|\leqq u$ and $[A,B,\Phi_{2j}]\notin\mathcal{B}(Q)$.

If F is a facet of Q and $F\supset S\cup B$, then $F\in(\mathcal{F}_1\setminus\mathcal{F}_2)\cup\cdots\cup(\mathcal{F}_{2j-3}\setminus\mathcal{F}_{2j-2})\cup\mathcal{F}_{2j-1}$ and $F\notin\mathcal{F}_{2j}$. Hence $[S,B]\notin\mathcal{B}(Q^+)$, and consequently $[S]\notin\mathcal{B}(Q^+,u)$.

Case II: $\alpha^+S=\alpha(S_2\cup S_4\cup\cdots\cup S_{2j})\cup\alpha A\cup\{x_u\}$ where $0\leqq j<\frac{1}{2}(m-u)$ and $\alpha A\in\mathcal{M}(Q_u/\alpha\Phi_{2j+1})$, or $j=\frac{1}{2}(m-u)$ and $A=\varnothing$.

As above, we find that $S=S_2\cup S_4\cup\cdots\cup A\cup\{x\}$ and $[A,\Phi_{2j+1}]\notin\mathcal{B}(Q,u)$. There is a set $B\subset\operatorname{vert} Q$ such that $|B|\leqq u$ and $[A,B,\Phi_{2j+1}]\notin\mathcal{B}(Q)$.

If F is a facet of Q and $F \supset B \cup (S \setminus \{x\})$, then $F \in (\mathcal{F}_0 \setminus \mathcal{F}_1) \cup \cdots \cup (\mathcal{F}_{2j} \setminus \mathcal{F}_{2j+1})$, hence $[S, B] \notin \mathcal{B}(Q^+)$, and therefore $[S] \notin \mathcal{B}(Q^+, u)$. □

Theorem 4.11 states that, in an appropriate sense, $(Q^+)_u = (Q_u)^+$.

It is probably not true that for every neighborly $2m$-polytope Q (with $|\operatorname{vert} Q| \geqq 2m+3$) and for all $1 \leqq u \leqq m-2$, the complex $\mathcal{B}(Q, u)$ can be embedded in the boundary complex of a neighborly $2(m-u)$-polytope Q_u, though we have no counterexample.

5. Reconstruction theorems

If Q, $\mathcal{T}$, $\mathcal{C}$, x and Q^+ are as in Convention 4.5, then we say that Q^+ *is obtained from Q at x by sewing through the tower $\mathcal{T}$.* We claim that the tower $\mathcal{T}$ is determined by Q^+ and x in the following sense:

If Q^+ is obtained from Q at x by sewing through any tower $\mathcal{T}'$, then $\mathcal{T}' = \mathcal{T}$.

In order to prove this it suffices to show that if $\mathcal{C}(Q, \mathcal{T}) = \mathcal{C}(Q, \mathcal{T}')$ then $\mathcal{T} = \mathcal{T}'$, because Q^+ and its vertex x determine $\mathcal{C}$ (see Theorem 2.10).

LEMMA 5.1. *Let Φ be a* u.f. *of Q with $2j$ vertices, Ψ a* u.f. *of Q with $2j+2$ vertices and $\Psi \supset \Phi$. Then $\Phi = \bigcap (\mathcal{F}_\Phi \setminus \mathcal{F}_\Psi)$ (see Definition* 4.3).

PROOF. Obviously $\Phi \subset \bigcap (\mathcal{F}_\Phi \setminus \mathcal{F}_\Psi) \subset Q$. Suppose $y \in \operatorname{vert} Q \setminus \Phi$. We have to show that Q has a facet F such that $F \supset \Phi$, $F \not\supset \Psi$ and $y \notin F$. If $y \in \Psi \setminus \Phi$ then every facet of Q which includes Φ and does not contain y is in $\mathcal{F}_\Phi \setminus \mathcal{F}_\Psi$.

Suppose $y \notin \Psi$. By 4.2(5) there is a set $A \subset \operatorname{vert} Q \setminus (\Psi \cup \{y\})$ such that $[A, \Psi] \notin \mathcal{B}(Q)$ and $|A| = m - j$ (Proposition 4.2(7)). Since $y \notin [\Phi, A]$, it follows that Q has a facet F such that $\Phi \subset F$, $A \subset F$, $y \notin F$ and necessarily $\Psi \not\subset F$. □

THEOREM 5.2. *For $i = 1,2$, let $\mathcal{T}_i$ be a universal tower in Q. If $\mathcal{C}(Q, \mathcal{T}_1) = \mathcal{C}(Q, \mathcal{T}_2)$ then $\mathcal{T}_1 = \mathcal{T}_2$.*

PROOF. Let $\mathcal{T} = \{\Phi_i\}_{i=1}^m$ be a u.t. in Q. We shall prove that $\mathcal{C} = \mathcal{C}(Q, \mathcal{T})$ and $\mathcal{F}(Q)$ determine Φ_i $(i = 1, \cdots, m)$.

Denote by $\mathcal{F}_0$ the set of all facets of Q. We proceed to define by induction a sequence $\mathcal{C}_1, \mathcal{C}_2, \cdots$ of subsets of $\mathcal{F}_0$, as follows:

$$\mathcal{C}_1 = \mathcal{C}, \quad \text{and} \quad \mathcal{C}_j = \{F \in \mathcal{F}_0 \setminus \mathcal{C}_{j-1} : \bigcap \mathcal{C}_{j-1} \subset F\} \quad \text{for } j > 1.$$

We claim that $\Phi_j = \bigcap \mathcal{C}_j$ for $j = 1, 2, \cdots, m$. Assume $1 \leqq j \leqq m$ and $\Phi_i = \bigcap \mathcal{C}_i$ for all i, $1 \leqq i < j$. It is easy to prove, by induction on i, that $\mathcal{C}_i = \bigcup'_{\nu \geqq 0} (\mathcal{F}_{i+2\nu} \setminus \mathcal{F}_{i+2\nu+1})$ for $1 \leqq i \leqq j$. In particular $\mathcal{F}_j \setminus \mathcal{F}_{j+1} \subset \mathcal{C}_j \subset \mathcal{F}_j$. For $j < m$, Lemma 5.1 yields $\Phi_j = \bigcap (\mathcal{F}_j \setminus \mathcal{F}_{j+1}) \supset \bigcap \mathcal{C}_j \supset \bigcap \mathcal{F}_j = \Phi_j$, hence $\Phi_j = \bigcap \mathcal{C}_j$. If

$j = m$, then $\mathscr{C}_m = \mathscr{F}_m$, hence $\Phi_m = \cap \mathscr{C}_m$. For $j > m$, it can be easily checked that $\mathscr{C}_j = \emptyset$. □

THEOREM 5.3. *Let Q, Q^+ be neighborly $2m$-polytopes, $\operatorname{vert} Q^+ = \operatorname{vert} Q \cup \{x\}$, $x \notin Q$. Suppose $a, b \in \operatorname{vert} Q$, $a \neq b$. If $[a,x]$ and $[b,x]$ are universal edges of Q^+, then $[a,b]$ is a universal edge of Q.*

Theorem 5.3 is a special case of Corollary 3.9. It can also be deduced directly from the following theorem:

THEOREM 5.4. *Let Q, Q^+ and x be as in Theorem 5.3. If $a \in \operatorname{vert} Q$, then $[a,x]$ is a* u.e. *of Q^+ iff all the facets of Q that x covers contain a.*

PROOF. Suppose $[a,x]$ is a u.e. of Q^+. For $0 < \lambda < 1$ let $z = z(\lambda) = (1-\lambda)a + \lambda x$. It is easy to see that $[Q,z]$ is an m-neighborly polytope. If Q had a facet F such that $a \notin F$ and x lies beyond F, then λ could be chosen such that $z \in \operatorname{aff} F$, contradicting the simpliciality of $[Q,z]$.

Now suppose that all the facets of Q that x covers contain a. Assume $A \subset \operatorname{vert} Q$, $|A| = m-1$. We have to show that $[a,x,A] \in \mathscr{F}(Q^+)$. Since $[a,A] \in \mathscr{F}(Q^+)$, it suffices to prove that Q has a facet F such that x lies beyond F and $\{a\} \cup A \subset F$. But $[A,x] \in \mathscr{F}(Q^+)$, hence Q has a facet F that includes A and is covered by x. Necessarily, $a \in F$. □

PROOF OF THEOREM 5.3. Suppose $[a,x]$, $[b,x]$ are u.e.s of Q^+. If $A \subset \operatorname{vert} Q$, $|A| = m-1$, then $[A,x] \in \mathscr{F}(Q^+)$, hence Q has a facet F such that $F \supset A$ and x lies beyond F. By Theorem 5.4, $F \supset \{a,b\}$, and therefore $[a,b,A] \in \mathscr{F}(Q)$. □

The same technique leads to a proof of the next theorem.

THEOREM 5.5. *Let $\mathscr{C}$ and $\mathscr{D}$ be sets of facets of Q. Let x, y be points in $R^{2m} \setminus Q$, such that x lies exactly beyond $\mathscr{C}$ and y lies exactly beyond $\mathscr{D}$. Assume both $[Q,x]$ and $[Q,y]$ are neighborly $2m$-polytopes with $|\operatorname{vert} Q| + 1$ vertices. If $\mathscr{C} \subset \mathscr{D}$ then $\mathscr{C} = \mathscr{D}$.*

PROOF. For $0 < \lambda < 1$, define $z = z(\lambda) = (1-\lambda)x + y$. It is easy to see that $[Q,z]$ is a neighborly polytope (z covers enough, but not too many facets of Q) and the simpliciality of $[Q,z]$ for every choice of λ, $0 < \lambda \leqq 1$, implies that $\mathscr{D} \setminus \mathscr{C} = \emptyset$. □

As a matter of fact, $|\mathscr{C}|$ is a function of $v = |\operatorname{vert} Q|$ and m only:

$$|\mathscr{C}| = \binom{v-m-1}{m-1}.$$

The next theorem is a converse to Theorem 4.6.

THEOREM 5.6. *Let Q, Q^+ be neighborly $2m$-polytopes. Assume* $\operatorname{vert} Q^+ = \operatorname{vert} Q \cup \{x\}$, $x \notin Q$ *and let* $\mathcal{T} = \{\Phi_j\}_{j=1}^m$ *be a tower in* Q *with* $|\operatorname{vert} \Phi_j| = 2j$ *for* $j = 1, 2, \cdots, m$. *If*

(1) Φ_j *is a* u.f. *of* Q^+ *for every even* j, $1 \leqq j \leqq m$, *and*

(2) $[\Phi_j, p, x]$ *is a* u.f. *of* Q^+ *for every even* j, $0 \leqq j < m$ *and* $p \in \operatorname{vert} \Phi_{j+1} \setminus \Phi_j$ *(here* $\Phi_0 = \emptyset$*),*

then $\mathcal{T}$ *is a universal tower in* Q, *and* Q^+ *is obtained from* Q *at* x *by sewing through* $\mathcal{T}$.

REMARK. If P is a subpolytope of P^+, and Φ is a face of both P and P^+, then there is a flat H such that the corespondence $\Psi \to \Psi \cap H$ is an isomorphism of the upper segment $[\Phi, P^+]$ of $\mathcal{F}(P^+)$ onto $\mathcal{F}(P^+ \cap H)$, and the same correspondence is an isomorphism of the upper segment $[\Phi, P]$ of $\mathcal{F}(P)$ onto $\mathcal{F}(P \cap H)$ (see [8, pp. 72–73]).

PROOF. Denote by $\mathcal{D}$ the set of facets of Q covered by x. We have to show that $\mathcal{T}$ is a u.t. in Q and that $\mathcal{D} = \mathcal{C}(Q, \mathcal{T})$ (see Definition 4.3).

Assume j is even, $0 \leqq j < m$. Consider the polytopes Q/Φ_j and Q^+/Φ_j. They are neighborly $2(m-j)$-polytopes, and $\operatorname{vert} Q^+/\Phi_j = \operatorname{vert} Q/\Phi_j \cup \{[\Phi_j, x]/\Phi_j\}$ by the remark above. By Corollary 5.3, Φ_{j+1}/Φ_j is a u.f. of Q/Φ_j. Therefore $[\Phi_{j+1}, A]/\Phi_j$ is a face of Q/Φ_j for every $A \subset \operatorname{vert} Q \setminus \Phi_j$, $|A| \leqq m - j - 1$. Hence Φ_{j+1} is a u.f. of Q. It follows that $\mathcal{T}$ is a u.t. in Q.

Denote by x' the vertex $[\Phi_j, x]/\Phi_j$ of Q^+/Φ_j. From condition (2) and Theorem 5.4 it follows that if x' lies beyond the facet $F' = F/\Phi_j$ of Q/Φ_j then $F \supset \Phi_{j+1}$. (Note that x' lies beyond F' iff x lies beyond F.) It follows that $\mathcal{D} \cap \mathcal{F}_j \subset \mathcal{F}_{j+1}$ for even j, $0 \leqq j < m$. The same conclusion follows from condition (1) for $j = m$, if m is even ($\mathcal{F}_{m+1} = \emptyset$). Therefore $\mathcal{D} \cap (\mathcal{F}_j \setminus \mathcal{F}_{j+1}) = \emptyset$ for even j, $0 \leqq j \leqq m$, hence $\mathcal{D} \cap \bigcup \{(\mathcal{F}_j \setminus \mathcal{F}_{j+1}) : j \text{ even}\} = \emptyset$, i.e., $\mathcal{D} \subset \mathcal{C}(Q, \mathcal{T})$. By Theorem 4.6 and Theorem 5.5 we conclude that $\mathcal{D} = \mathcal{C}(Q, \mathcal{T})$. □

Theorem 5.6 says that we can tell from partial information about the structure of $\mathcal{F}(Q)$ whether Q is obtained by sewing at a given vertex x through a given tower $\mathcal{T}$. The next theorem shows how this implies "commutativity" of our sewing procedure.

DEFINITION 5.7. If $\mathcal{T}$ is a u.t. in Q, let $Q(\mathcal{T})$ denote the class of all polytopes that are obtained from Q by sewing through $\mathcal{T}$.

THEOREM 5.8. *For* $1 \leqq i \leqq p$, *let* $\mathcal{T}_i$ *be a* u.t. *in* Q. *Assume the sets* $\bigcup \mathcal{T}_i$, $1 \leqq i \leqq p$, *are pairwise disjoint (i.e.,* $A \in \mathcal{T}_i$, $B \in \mathcal{T}_i$ *and* $i \neq j$ *imply*

$A \cap B = \emptyset$). Then there are points x_i, $1 \leqq i \leqq p$, such that $[Q,\{x_i : i \in \Gamma\}, x_j] \in [Q,\{x_i : i \in \Gamma\}](\mathcal{T}_j)$ for every subset Γ of $\{1,\cdots,p\}$ and for every j in $\{1,\cdots,p\} \setminus \Gamma$.

PROOF. By Lemma 4.4 and Theorem 4.8 it follows easily, by induction on k, $1 \leqq k \leqq p$, that there are points $x_1,\cdots,x_p$ such that the sequence $(x_1,\mathcal{T}_1),\cdots,(x_p,\mathcal{T}_p)$ satisfies

$$(*) \qquad [Q,x_1,\cdots,x_k] \in [Q,x_1,\cdots,x_{k-1}](\mathcal{T}_k) \quad \text{for } 1 \leqq k \leqq p.$$

Theorem 5.8 follows from the next lemma. □

LEMMA 5.9. *Under the assumptions of Theorem 5.8, if the sequence $\{(x_j,\mathcal{T}_j)\}_{j=1}^p$ satisfies $(*)$, then the sequence $\{(x_{\pi(j)},\mathcal{T}_{\pi(j)})\}_{j=1}^p$ satisfies $(*)$ for any permutation π of $\{1,\cdots,p\}$.*

PROOF. For $p = 1$ there is nothing to prove.

Assume $p = 2$. Consider the faces of $[Q,x_1,x_2]$ whose universality is asserted in Theorem 4.6, parts (3) and (5), with $x = x_2$ and $\mathcal{T} = \mathcal{T}_2$. Obviously, each such face is a u.f. of $[Q,x_2]$, and by Theorem 5.6, $[Q,x_2] \in Q(\mathcal{T}_2)$. Now consider the faces of $[Q,x_1]$ whose universality is asserted in Theorem 4.6, parts (3) and (5), with $x = x_1$ and $\mathcal{T} = \mathcal{T}_1$. By Theorem 4.8, these faces are u.f.s also in $[Q,x_1,x_2]$, and by Theorem 5.6, $[Q,x_2,x_1] \in [Q,x_2](\mathcal{T}_1)$. Therefore the sequence $(x_2,\mathcal{T}_2)$, $(x_1,\mathcal{T}_1)$ satisfies $(*)$.

Now assume $p > 2$. We say that π is an admissible permutation of $\{1,\cdots,p\}$ if $\{x_{\pi(j)},\mathcal{T}_{\pi(j)})\}_{j=1}^p$ satisfies $(*)$ whenever $\{(x_j,\mathcal{T}_j)\}_{j=1}^p$ satisfies $(*)$. Since the set of admissible permutations is closed under multiplication, it suffices to show that the transpositions $(i, i+1)$, $1 \leqq i < p$, are admissible.

Let $x_1,\cdots,x_p$ be points such that $(*)$ holds for $(x_1,\mathcal{T}_1),\cdots,(x_p,\mathcal{T}_p)$. Assume $1 \leqq i < p$. We shall show that the sequence $(x_1,\mathcal{T}_1),\cdots,(x_{i+1},\mathcal{T}_{i+1})$, $(x_i,\mathcal{T}_i),\cdots,(x_p,\mathcal{T}_p)$ also satisfies $(*)$. The only parts of $(*)$ that are not self-evident for this sequence are $[Q',x_{i+1}] \in Q'(\mathcal{T}_{i+1})$ and $[Q',x_{i+1},x_i] \in [Q',x_{i+1}](\mathcal{T}_i)$, where $Q' = [Q,x_1,\cdots,x_{i-1}]$. But these assertions follow from the case $p = 2$, applied to Q' and to the sequence $(x_i,\mathcal{T}_i)$, $(x_{i+1},\mathcal{T}_{i+1})$. □

Note that the proof of Theorem 5.8 given here holds even if the assumption that the sets $\bigcup \mathcal{T}_i$ are pairwise disjoint is replaced by the following slightly weaker condition: Assume $\mathcal{T}_i = \{\Phi_{i,1},\cdots,\Phi_{i,m}\}$ for $1 \leqq i \leqq p$, then $\Phi_{i,m-1} \cap \Phi_{j,m} = \emptyset$ and $|\Phi_{i,m} \cap \Phi_{j,m}| \leqq 1$ for $i \neq j$, $1 \leqq i,j \leqq p$.

It would be interesting to know what is the freedom in choosing the points $x_1,\cdots,x_p$, in case we do assume that the towers $\mathcal{T}_1,\cdots,\mathcal{T}_p$ have pairwise disjoint unions.

The following might be true:

If $[Q, x_i] \in Q(\mathcal{T}_i)$ for $1 \leqq i \leqq p$, then $[Q, \{x_i : i \in \Gamma\}, x_j] \in [Q, \{x_i : i \in \Gamma\}](\mathcal{T}_j)$ for every $\Gamma \subset \{1, 2, \cdots, p\}$ and $j \in \{1, 2, \cdots, p\} \setminus \Gamma$.

6. Lower bounds

Repeated application of the sewing construction described in section 4 yields lower bounds for the number $g(v, 2m)$ of combinatorial types of neighborly $2m$-polytopes with v vertices.

We start with a cyclic $2m$-polytope $C(v, 2m)$ with v vertices, $v \geqq 2m + 3$, and "sew" it repeatedly.

The main results are as follows:

THEOREM 6.1. $g(2m+4, 2m) > \dfrac{(2m+2)!}{3 \cdot 2^{m+3}(m+2)!}$.

(The right-hand side is asymptotic to $(\sqrt{2}/6)(2m/e)^m$ as $m \to \infty$.)

THEOREM 6.2. *$g((2m+1)p + 2, 2m) \geqq \frac{1}{2}(pm - p)!$ for $p = 2, 3, \cdots$.*

PROOF OF THEOREM 6.1. Let K be $C(2m+3, 2m)$. Assume that $\operatorname{vert} K = \{a_1, a_2, \cdots, a_{2m+3}\}$, and that $a_1, a_2, \cdots, a_{2m+3}, a_1$ is the circuit of universal edges of K. $\operatorname{Aut} K$, the group of combinatorial automorphisms of K, is precisely the dihedral group of order $2(2m+3)$ consisting of rotations and reflections of the circuit $a_1, a_2, \cdots, a_{2m+3}, a_1$.

Consider pairs (Q, x) where Q is a neighborly $2m$-polytope with $2m+4$ vertices and x is a distinguished vertex of Q. Two such pairs (Q, x), (Q', x') are considered isomorphic if there is a combinatorial equivalence $\varphi : \operatorname{vert} Q \to \operatorname{vert} Q'$ with $\varphi(x) = x'$. The number of isomorphism types of such pairs is clearly at most $(2m+4)g(2m+4, 2m)$.

Now let us count the number of isomorphism types of pairs (K^+, x), where K^+ is obtained from K at x by sewing through a u.t. $\mathcal{T}$. Consider two such pairs (K_i^+, x_i), with $K_i^+ \in K(\mathcal{T}_i)$, $i = 1,2$. If these pairs are isomorphic by a mapping $\varphi : \operatorname{vert} K_1^+ \to \operatorname{vert} K_2^+$ such that $\varphi(x_1) = x_2$, then $\psi = \varphi \mid K$ is an automorphism of K, and ψ maps $\mathcal{C}(k, \mathcal{T}_1)$ onto $\mathcal{C}(K, \mathcal{T}_2)$. It follows from Theorem 5.2 that $\psi(\mathcal{T}_1) = \mathcal{T}_2$.

The converse is obvious: If $\mathcal{T}_1$ is mapped onto $\mathcal{T}_2$ by an automorphism of K, then the pairs (K_1^+, x_1), (K_2^+, x_2) are isomorphic.

Thus, the number of isomorphism types of pairs (K^+, x) considered above is precisely the number of equivalence classes of u.t.s of K under $\operatorname{Aut} K$.

Every u.t. in K can be transformed by a suitable rotation to a tower with $\Phi_1 = [a_1, a_2]$. With Φ_1 fixed, Φ_2 can be chosen in $2m+1$ ways, then Φ_3 in $2m-1$ ways and so on. (Note that K/Φ_i is a cyclic polytope of type $C(2(m-i)+3, 2(m-i))$, for $0 \leqq i < m$.) Therefore the number of u.t.s in K with $\Phi_1 = [a_1, a_2]$ is $(2m+1)(2m-1)\cdots 7\cdot 5$. There is only one automorphism ψ of K, except the identity, that maps the edge $\Phi_1 = [a_1, a_2]$ onto itself ($\psi(a_i) = a_{2m+6-i}$ for $3 \leqq i \leqq 2m+3$).

Only one u.t. is fixed by ψ. It follows that the number of equivalence classes of u.t.s in K under $\operatorname{Aut} K$ is precisely $\frac{1}{2}(1+(2m+1)(2m-1)\cdots 5)$.

We conclude that $(2m+4)g(2m+4, 2m) > \frac{1}{6}\prod_{i=1}^{m}(2i+1)$. □

If we repeat the reasoning in the proof of Theorem 6.1 with $K = C(2m+3, 2m)$ replaced by $K = C(v-1, 2m)$, we conclude that $g(v, 2m) \to \infty$ as $v \to \infty$. But the following construction yields a better lower bound.

Proof of Theorem 6.2. Let K be a cyclic polytope $C(v, 2m)$ with $v = 2mp+2$ vertices, $p \geqq 2$. Assume that $\operatorname{vert} K = \{a_1, a_2, \cdots, a_v\}$ and $a_1, a_2, \cdots, a_v, a_1$ is the circuit of u.e.s of K. We say that a_i and a_j are successive vertices if $j = i+1$. A family $\sigma = \{\sigma(i,j),\ 1 \leqq i \leqq p,\ 1 \leqq j \leqq m\}$ is called a partition of $\operatorname{vert} K$ if:

(1) $\sigma(i,j)$ is a set of two successive vertices of K, $1 \leqq i \leqq p$, $1 \leqq j \leqq m$.

(2) $\sigma(i,1) = \{a_{2i}, a_{2i+1}\}$, $1 \leqq i \leqq p$.

(3) $\sigma(i,j) \subset \{a_{2p+3}, a_{2p+4}, \cdots, a_v\}$, $1 \leqq i \leqq p$, $1 < j \leqq m$.

(4) Every vertex of K, except a_1 and a_{2p+2}, is contained in some $\sigma(i,j)$.

From (1)–(4) it follows that the sets $\sigma(i,j)$ are pairwise disjoint.

For every partition σ define: $\mathcal{T}_i = \mathcal{T}_i(\sigma) = \{[\sigma(i,1), \cdots, \sigma(i,l)]\}_{l=1}^{m}$, $1 \leqq i \leqq p$. For $1 \leqq i \leqq p$, $\mathcal{T}_i$ is a u.t. in K, and the sets $\bigcup \mathcal{T}_i$, for $1 \leqq i \leqq p$, are pairwise disjoint. By Theorem 5.8 there are points $x_i = x_i(\sigma)$, $1 \leqq i \leqq p$, such that $[K, \{x_j : j \in \Gamma\}, x_i] \in [K, \{x_j : j \in \Gamma\}](\mathcal{T}_i)$ for $\Gamma \subset \{1, \cdots, p\} \setminus \{i\}$.

Define $K^\sigma = [K, x_1, \cdots, x_p]$. K^σ is a neighborly $2m$-polytope with $v+p = (2m+1)p+2$ vertices. We shall show that the number of distinct combinatorial types of polytopes K^σ is at least one half the number $(pm-p)!$ of partitions.

Define an equivalence relation on the set of partitions as follows: $\sigma \sim \tau$ if $K^\sigma \cong K^\tau$.

In order to complete the proof, it suffices to show that an equivalence class of a partition contains at most two elements.

Assume $\sigma \sim \tau$, and let $f : K^\sigma \to K^\tau$ be a combinatorial equivalence.

If E is a u.e. of K^σ then either E is a u.e. of K or $x_i = x_i(\sigma) \in \operatorname{vert} E$ for some $1 \leqq i \leqq p$. K^σ is sewed at x_i through $\mathcal{T}_i$ (Theorem 5.8), hence $[x_i, a_{2i}]$ and

$[x_i, a_{2i+1}]$ are u.e.s of K^σ for $1 \leqq i \leqq p$ (Theorem 4.6(5)). For $1 \leqq i \leqq p$, the edges $[a_{2i-1}, a_{2i}]$, $[a_{2i+1}, a_{2i+2}]$, $[\sigma(i,j)]$ $(1 \leqq j \leqq m)$ are not u.e.s of K^σ, since they are not u.e.s of $[K, x_i] \in K(\mathcal{T}_i)$, by Theorem 4.10.

It follows that each $x_i(\sigma)$, $1 \leqq i \leqq p$, belongs to exactly two u.e.s of K^σ, and no other vertex of K^σ has this property. The same holds for K^τ. It follows that f maps $\{x_1(\sigma), \cdots, x_p(\sigma)\}$ onto $\{x_1(\tau), \cdots, x_p(\tau)\}$ and the restriction of f to vert K is an automorphism of K, call it $\bar{f}$. Moreover, if $f(x_\alpha(\sigma)) = x_\beta(\tau)$, then $\bar{f}(\mathcal{T}_\alpha(\sigma)) = \mathcal{T}_\beta(\tau)$, and therefore $\tau(\beta, j) = \bar{f}(\sigma(\alpha, j))$, $1 \leqq j \leqq m$. In particular, $\bar{f}$ maps the set $\{a_\nu : 2 \leqq \nu \leqq 2p+1\}$ onto itself, hence $\bar{f}$ is either the identity (in which case $\tau = \sigma$) or a unique reflection (in which case $f(x_i(\sigma)) = x_{p+1-i}(\tau)$). $\square$

7. Remarks and open questions

(1) From the proof of Theorem 6.2 it easily follows that the estimation $g(v, 2m) \geqq \frac{1}{2}((m-1)p)!$ holds for all $v \geqq (2m+1)p+2$. This yields the asymptotic estimate $g(v, 2m) > A(m)e^{B(m)v \log v}$, where $B(m) \to \frac{1}{2}$ as $m \to \infty$. Our attempts to refine the construction in the proof of Theorem 6.2 did not yield anything significant but only a slight improvement in the coefficients $A(m)$, $B(m)$.

As regards upper bounds, it is easy to prove that $g(v, 2m) \leqq v^{2mv^m/m!}$. Using the fact that each vertex of a neighborly $2m$-polytope with v vertices covers exactly $\binom{v-m-2}{m-1}$ facets of the subpolytope determined by the remaining vertices (for $v \geqq 2m+4$) one can show that $g(v, 2m) \leqq C(m)e^{D(m)v^m}$, where $C(m)$, $D(m) > 0$ and $D(m) \to 0$ as $m \to \infty$.

There is still a huge gap between the lower bound and the upper bound, even for $m = 2$.

(2) Denote by $g(v, d)$ the number of combinatorial types of simplicial neighborly d-polytopes with v vertices. The numbers $g(2m+4, 2m+1)$ were determined explicitly by Altshuler and McMullen, see [2].

The inequality $g(v+1, 2m+1) \geqq g(v, 2m)$ can be proved as follows: W.l.o.g., assume $v \geqq 2m+3$. If $Q \subset R^{2m} \subset R^{2m+1}$ is a neighborly $2m$-polytope with v vertices, choose a vertex x of Q and a vector $u \in R^{2m+1} \setminus R^{2m}$, and define $Q^+ = [Q, x+u, x-u]$. Q^+ is a simplicial neighborly $(2m+1)$-polytope with $v+1$ vertices, $x+u$ and $x-u$ are the only universal vertices of Q^+, and Q is the vertex figure of Q^+ at $x+u$ and at $x-u$.

(3) Table 1 shows the present state of knowledge as to the exact number of combinatorial types of neighborly $2m$-polytopes with $2m+\beta$ vertices. In this table, "333 sewed, 287 totally sewed" means: Exactly 333 neighborly 4-polytopes

TABLE 1

m	β	No. of types	Sources	Comments
arbitrary	$\leqq 3$	1	[6, ch. 6, sec. 7.2]	cyclic
1	arbitrary	1		
2	4	3	[7]	All sewed
2	5	23	[4], [9]	18 sewed
2	6	333–432	[1], and recent computations by the author	333 sewed, 287 totally sewed
3	4	37	Recent computations by the author and by J. Bokowski	15 sewed

with 10 vertices are obtained by sewing from neighborly 4-polytopes with 9 vertices, but only 287 types are obtained by triple sewing from $C(7,4)$.

All this information seems to indicate that sewing alone yields only a vanishing minority of the neighborly $2m$-polytopes with $2m+\beta$ vertices, both for $\beta \geqq 4$ fixed and $m \to \infty$, as well as for $m \geqq 2$ fixed and $\beta \to \infty$.

(4) As mentioned in Section 4, Barnette's facet-splitting operation [5] is dual to our sewing construction. Barnette uses a tower $v \in F^2 \subset F^3 \subset \cdots \subset F^{d-1}$ of faces in a *-neighborly polytope P^* (i.e., the dual of a neighborly d-polytope P).

He applies a slight rotation to F^{d-1}, and intersects P^* with a half-space bounded by the affine hull of the rotated facet. This is dual to adding a new vertex near the vertex $\hat{F}^{d-1}$ of P. We consider only the case where $d = 2m$ is even. In this case the dual of Barnette's tower consists of universal faces of dimensions $0, 1, 2, \cdots, 2m-4, 2m-3, 2m-1$, whereas our universal tower contains odd-dimensional faces only.

As a matter of fact, we can prove the following:

Let $\mathcal{T}$ be a universal tower in a neighborly $2m$-polytope Q, and let $\mathcal{T}'$ be a tower in Q that includes $\mathcal{T}$. If x' lies exactly beyond $\mathcal{C}(Q,\mathcal{T}')$, then $[Q,x']$ is neighborly. Moreover, we can describe $\mathcal{F}([Q,x'])$ in terms of $\mathcal{F}(Q)$ and $\mathcal{T}'$.

Barnette applies the facet-splitting operation repeatedly, using a lemma (lemma 8 in [5], which is analogous to parts (3) and (5) of our Theorem 4.6) that guarantees the existence of a new tower near the old one, and thus he obtains a lower bound for $g(v,d)$ which is close to 4^v. Our superexponential bound (see Theorem 6.2 and Remark 1) utilizes Lemma 4.7, which gives a full description of $\mathcal{F}([Q,x])$ in terms of $\mathcal{F}(Q)$ and $\mathcal{T}$.

(5) Starting with the cyclic polytope $C(2m+3,2m)$ and applying successively (in any order) operations of sewing and omitting a vertex (see sections 4 and 2), we obtain a large class SO of neighborly $2m$-polytopes. For $m=2$, this class

contains only the "totally sewed" neighborly 4-polytopes, but for $m \geqq 3$ it is strictly larger. (For example, 13 types of neighborly 6-polytopes with 10 vertices are not obtained by sewing, but are obtained from $C(9,6)$ by double sewing and omitting a vertex.)

Every polytope in SO has two adjacent universal edges. This still holds if the notion of sewing is extended as in remark (4) above. It follows that some neighborly 4-polytopes with 9 vertices and some neighborly 6-polytopes with 10 vertices are not in SO. We conjecture that if m or β is large, then most neighborly $2m$-polytopes with $2m+\beta$ vertices are not in SO.

(6) Is there a neighborly $2m$-polytope which is not a proper subpolytope of any neighborly polytope of the same dimension?

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