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EQUIVALENT NORMS ON SPACES OF BOUNDED FUNCTIONS

BY
J. R. PARTINGTON

ABSTRACT
Let ω_1 denote the first uncountable ordinal, $m_\omega(\omega_1)$ the Banach space of all bounded real functions on ω_1 with countable support (with the supremum norm). It is shown that any space isomorphic to $m_\omega(\omega_1)$ contains a subspace isometric to $m_\omega(\omega_1)$. Several similar results concerning higher cardinals are obtained.

Let α be an infinite cardinal, and let S be a set of cardinality at least α. We write $m_\alpha(S)$ to denote the linear space of all bounded real functions on S with support of cardinality at most α. This is naturally a Banach space with the supremum norm. Clearly $m_\alpha(S)$ contains an isometric copy of $l_\infty(\alpha)$, the space of all bounded functions on a set of cardinality α.

In this article we will examine the subspace structure of any space obtained by giving $m_\alpha(S)$ an equivalent norm (which is the same as specifying an isomorphic image of $m_\alpha(S)$). In particular, we shall be considering the presence of subspaces isometric to spaces of the same form.

Greek letters will be used to designate ordinal numbers throughout and cardinal numbers will be identified with the corresponding initial ordinals.

Let $\omega_0 = \omega$, the first infinite ordinal; if α is an ordinal, let $\omega_{\alpha+1}$ denote the first ordinal of cardinality strictly greater than ω_α; if ε is a limit ordinal, let ω_ε denote $\bigcup_{\alpha<\varepsilon}\omega_\alpha$.

Day [1] showed that $m_\omega(\omega_1)$ cannot be given an equivalent strictly convex norm, although $l_\infty(\omega)$ can be. Moreover $c_0(S)$, the completion in $l_\infty(S)$ of the space of all finitely-supported functions, can always be given a locally uniformly convex norm.

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It was proved by James [2] (see also [4]) that, given an equivalent norm $|\cdot|$ on $c_0(\omega)$, it is possible to find a subspace Y whose Banach–Mazur distance from $c_0(\omega)$, $d((Y,|\cdot|),(c_0,\|\cdot\|))$, is arbitrarily close to one.

Schäffer [5] discussed the concept of a flat space, a notion related to super-reflexivity, and asked whether there is an isomorphism class consisting entirely of flat spaces. He conjectured that, if S has sufficiently large cardinality, then any isomorph of $l_\infty(S)$ contains an isometric copy of $l_\infty(\omega)$, and is thus flat.

Using constructions similar to those of Day, we shall show that all spaces isomorphic to $m_\omega(\omega_1)$ contain an isometric copy of $m_\omega(\omega_1)$, and hence that Schäffer's conjecture is true for all uncountable sets S. Our methods yield similar results about higher cardinals.

Assuming the continuum hypothesis, namely that $\omega_1 = c$, we will show that all spaces isomorphic to $l_\infty(\omega_1)$ contain an isometric copy of $l_\infty(\omega_1)$, rather than merely $m_\omega(\omega_1)$. Similar results can be proved for higher cardinals.

That this result does not extend to all uncountable cardinals will be shown by considering $l_\infty(\omega_\omega)$, on which an equivalent norm will be specified under which no subspace isometric to $l_\infty(\omega_\omega)$ exists, although copies of $m_{\omega_n}(\omega_\omega)$ exist for all finite n.

If α is an initial ordinal, define $\mathrm{cf}(\alpha)$ to be the smallest initial ordinal β such that $\alpha = \bigcup_{\gamma<\beta}\alpha_\gamma$ for a family $\{\alpha_\gamma : \gamma<\beta\}$ of ordinals strictly less than α. For example, $\mathrm{cf}(\omega_1)=\omega_1$ and $\mathrm{cf}(\omega_\omega)=\omega$. A cardinal α is regular if $\mathrm{cf}(\alpha)=\alpha$.

For subsets S, T of an ordinal, we write $S > T$ if $\sigma\in S$, $\tau \in T$ implies that $\sigma > \tau$. Given a bounded set $\{u_\alpha : \alpha<\beta\}$ of elements of an l_∞ space which have pairwise disjoint supports, $\Sigma_{\alpha<\beta} u_\alpha$ will be used to denote the formal pointwise sum of the u_α.

THEOREM 1. *Let α be an infinite cardinal and β a regular cardinal, such that $\alpha<\beta$. Let $|\cdot|$ be a norm on $m_\alpha(\beta)$ equivalent to the usual norm $\|\cdot\|$. Then there exist pairwise disjoint elements $\{w_\gamma : \gamma<\beta\}$ such that the map from $(m_\alpha(\beta),\|\cdot\|)$ into $(m_\alpha(\beta),|\cdot|)$ taking $a=(a_\gamma)_{\gamma<\beta}$ to $\Sigma a_\gamma w_\gamma$ is an isometry.*

PROOF. Let U denote the sphere $\{x\in m_\alpha(\beta): \|x\|=1\}$. We assert first that there exist vectors $\{u_\gamma : \gamma<\beta\}$ on U, supported on disjoint sets S_γ of cardinality at most α, such that if $\gamma<\delta$ then $S_\gamma<S_\delta$, and such that $|u_\gamma| = |u_\gamma + y|$ whenever $\|y\|\leqq 1$ and $\mathrm{supp}(y) > S_\gamma$.

Adapting an argument given by Day [1], this follows by transfinite induction as follows.

Suppose we have constructed u_γ for all $\gamma<\gamma_0$; we restrict to the subspace $X = X(\gamma_0)$ of all functions f with $\mathrm{supp}(f) > \bigcup_{\gamma<\gamma_0} S_\gamma$.

For $x \in U \cap X$, let

$$M(x) = \sup\{|y| : y \in X \quad \text{and} \quad y(\delta) = x(\delta) \quad \text{whenever} \quad x(\delta) \neq 0\}$$

and

$$m(x) = \inf\{|y| : y \in X \quad \text{and} \quad y(\delta) = x(\delta) \quad \text{whenever} \quad x(\delta) \neq 0\}.$$

As in [1], $M(x) + m(x) \geqq 2|x|$, and we can choose u_{γ_0} to be the pointwise limit of a sequence (x_n) with $M(x_n) - |x_n| \to 0$ and $x_n(\delta) = x_{n-1}(\delta)$ whenever $x_{n-1}(\delta) \neq 0$. Then $M(u_{\gamma_0}) = m(u_{\gamma_0}) = |u_{\gamma_0}|$, and the induction proceeds.

We now show that there exists a positive number a, and an ordinal $\gamma_0 < \beta$ such that $|u_\delta| = a$ for all $\gamma_0 \leqq \delta < \beta$. For if $\gamma_1 < \gamma_2$, then $|u_{\gamma_1} \pm u_{\gamma_2}| = |u_{\gamma_1}|$ and so $|u_{\gamma_2}| \leqq |u_{\gamma_1}|$. Thus $|u_\gamma|$ decreases as γ increases, and, if it is not eventually constant, then the sequence (γ_n), defined by $\gamma_n = \inf\{\gamma : |u_\gamma| < \inf_{\delta<\beta} |u_\delta| + 1/n\}$ is cofinal with β, a contradiction.

We define w_γ to be $u_{(\gamma_0+\gamma)}/a$. To complete the proof, we require the following simple result.

Lemma. *If x and y are elements in a normed space such that $\|x\| = \|x \pm y\| = 1$, then $\|x + sy\| = 1$ for $-1 \leqq s \leqq 1$.*

Proof. We may assume that $s > 0$. Since $x + sy = s(x + y) + (1 - s)x$, we have $\|x + sy\| \leqq 1$. Also $x = t(x + sy) + (1 - t)(x - y)$, where $t = 1/(1 + s)$, and thus if $\|x + sy\| < 1$, then $\|x\| < 1$, a contradiction. This proves the lemma.

Now to prove that $|\Sigma_{\gamma<\beta} a_\gamma w_\gamma| = \sup |a_\gamma|$, when this element lies in $m_\alpha(\beta)$, we may suppose that the supremum is attained, as such elements form a dense set. Let γ_0 be the least index at which the supremum is attained. If γ_0 is the first non-vanishing coefficient (in particular if $\gamma_0 = 0$) the result follows from the construction. In general, we apply transfinite induction, assuming the result to hold for all $\gamma_0 < \delta$. We may assume that $a_\delta = 1$, and that $s = \sup_{\gamma<\delta} |a_\gamma|$ is attained at $\gamma_1 < \delta$ and $s \neq 0$. Now let $v = \Sigma_{\gamma<\delta} a_\gamma w_\gamma$ and $z = \Sigma_{\delta<\gamma<\beta} a_\gamma w_\gamma$. Then $|w_\delta + z| = 1$ and also $|\pm s^{-1}v + w_\delta + z| = 1$ by the inductive hypothesis. Applying the lemma with $x = w_\delta + z$ and $y = s^{-1}v$, the result follows. This completes the proof.

Corollary. *Every space isomorphic to $l_\infty(\omega_1)$ contains an isometric copy of $m_\omega(\omega_1)$, and hence of every separable space.*

Our next theorem gives an improvement on the above corollary, assuming the continuum hypothesis, namely that $\omega_1 = c$. We will prove a result only for $l_\infty(\omega_1)$: at the end of the proof we will indicate what can be obtained for higher cardinals.

THEOREM 2. *Assume that $\omega_1 = c$. Let $|\cdot|$ be a new norm on $l_\infty(\omega_1)$ equivalent to the usual norm $\|\cdot\|$. Then there exist pairwise disjoint elements $\{w_\gamma : \gamma < \omega_1\}$ such that the new norm of a vector $\Sigma_{\gamma<\omega_1} a_\gamma w_\gamma$ lying in $l_\infty(\omega_1)$ is $\sup|a_\gamma|$.*

PROOF. Let T be the set of all functions $f: \omega_1 \to \omega_1$ for which there exists an $\alpha < \omega_1$ such that $f(\gamma) = 0$ for all $\gamma \geqq \alpha$.

Assuming that $\omega_1 = c$, the cardinal of T is $\Sigma_{\alpha<\omega_1} |\omega_1|^{|\alpha|}$ which is ω_1 (see [3], for example).

We order T by writing $f \leqq g$ if $f = g$ or $f(\gamma) < g(\gamma)$ when γ is the least ordinal such that $f(\gamma) \neq g(\gamma)$.

Regarding $l_\infty(\omega_1)$ as $l_\infty(T)$, let X be the subspace of all functions with bounded support (that is, there is an $f \in T$ such that $x_g = 0$ for all $g \geqq f$).

We will now use transfinite induction to select $\{u_\alpha : \alpha < \omega_1\}$ with $\mathrm{supp}(u_\alpha) < \mathrm{supp}(u_\beta)$ if $\alpha < \beta$, such that $\|u_\alpha\| = 1$ and $|u_\alpha| = |u_\alpha + y|$ whenever $y \in X$, $\|y\| \leqq 1$, and y is a formal sum $\Sigma_{\alpha<\gamma<\omega_1} a_\gamma u_\gamma$.

Using Day's process, as in the proof of Theorem 1, we can select u_0 with these properties, since any increasing sequence of bounded subsets of T has a bounded union.

If we have chosen $\{u_\alpha : \alpha < \beta\}$, and $0 < \beta < \omega_1$, with $\mathrm{supp}(u_\alpha)$ bounded by $f_\alpha \in T$, consider $T_\beta \subseteq T$, the subset of all f with $f(\alpha) = f_\alpha(\alpha) + 1$ for $0 \leqq \alpha < \beta$. Restricting to the subspace of all funetions in X with support contained in T_β, we may choose u_β satisfying the above conditions.

Having constructed $\{u_\alpha : \alpha < \omega_1\}$, we conclude the proof as in the proof of Theorem 1: there is a $\beta < \omega_1$ such that $|u_\alpha| = |u_\beta|$ for all $\beta \leqq \alpha < \omega_1$, and so the disjoint elements $w_\gamma = u_{\beta+\gamma}/|u_\beta|$ $(\gamma < \omega_1)$ have the required property. This proves the theorem.

In general, the above proof can be adapted for any cardinal α of cofinality greater than ω, assuming that $|\alpha|^{|\beta|}$ is at most α, for any $\beta < \alpha$. In this case one considers the set of functions $f: \alpha \to \omega_1$ with bounded support to be the index set T.

It follows from Theorem 1 that any isomorph of $l_\infty(\omega_\omega)$ contains isometric copies of $l_\infty(\omega_n)$ for every $n \in \omega$, since ω_{n+1} is a regular cardinal. The following example shows that in this case the result is best possible.

EXAMPLE. Let $S_n = \{\alpha : \omega_{n-1} \leqq \alpha < \omega_n\}$, $n = 1, 2, 3, \cdots$ and $S_0 = \{\alpha : \alpha < \omega\}$. Let X be the space $l_\infty(\omega_\omega)$ renormed by putting $|x| = \|x\|_\infty + \Sigma_{n=0}^\infty 2^{-n} \|x|_{S_n}\|_\infty$.

Then $\|x\| \leqq |x| \leqq 3\|x\|$ and X contains no isometric copy of $c_0(\omega_\omega)$.

PROOF. We show that if $\{u_\beta : \beta < \alpha\}$ satisfy $|\Sigma_{i=1}^N a_i u_{\beta_i}| = \max|a_i|$ for all

$a_1, \cdots, a_N \in \mathbf{R}$ and $(\beta_i) < \alpha$, then $\alpha < \omega_\omega$.

Since $|\Sigma_{i=1}^N \pm u_{\beta_i}| = 1$ for all sign choices and for all choices of β_i, we see that, for each $\gamma < \omega_\omega$, $u_\beta(\gamma)$ can only be nonzero at most countably many times.

For otherwise $\Sigma_{i=1}^N \pm u_{\beta_i}(\gamma)$ could be made arbitrarily large by suitable choices of signs and of indices (β_i), contradicting the equivalence of the two norms.

Suppose then that $\alpha = \omega_\omega$. Let $x = u_0$; then for some $k \in \omega$, we have $\Sigma_{n=0}^k 2^{-n} \| x|_{S_n} \| = s > 0$.

Since for each γ all but countably many $u_\beta(\gamma)$ are zero, there is a $\beta < \alpha$ such that $y = u_\beta$ satisfies $y|_{S_n} = 0$ $(n = 0, \cdots, k)$ and $\|y\|_\infty > 1 - s/2$, for the second property is satisfied automatically if $y|_{S_n} = 0$ for all n such that $2^{-n} \geqq s/4$.

There exists a coordinate β for which $|y_\beta| > 1 - s/2$; changing the signs of x and y if necessary we may assume that $y_\beta > 1 - s/2$ and $x_\beta \geqq 0$.

Then

$$
\begin{aligned}
|x + y| &\geqq \|x + y\|_\infty + \sum_{n=0}^{k} 2^{-n} \|(x+y)|_{S_n}\|_\infty \\
&\geqq (1 - s/2) + s \\
&> 1.
\end{aligned}
$$

This is a contradiction and hence $\alpha < \omega_\omega$, as asserted.

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Added in proof. The above methods may be adapted to extend the result of James on c_0 to l_∞. The details will appear elsewhere.

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Trinity College
Cambridge CB2 1TQ
England