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ON AUTOMORPHISMS OF THE DEGREES THAT PRESERVE JUMPS†

BY

LINDA JEAN RICHTER

ABSTRACT

If F is an automorphism of the degrees of unsolvability that preserves the jump operator, then F leaves fixed all the degrees $\mathbf{c}$ that satisfy $\mathbf{c} \geqq \mathbf{0}^{(3)}$.

1. Let D be the set of all degrees of unsolvability, let $\leqq$ be the usual partial ordering on D, let $\oplus$ be the binary join operator on D (least upper bound), and finally let $'$ be the jump operator on D. No nontrivial automorphisms of the structure $\mathcal{D}_j = \langle D, \leqq, ' \rangle$ are known. Indeed no nontrivial automorphisms of $\mathcal{D} = \langle D, \leqq \rangle$ are known. It is conceivable that the identity is the only automorphism of $\mathcal{D}$, and thus that the automorphisms of $\mathcal{D}$ fix all the degrees. However, so far $\mathbf{0}$ is the only degree known to be fixed by all the automorphisms of $\mathcal{D}$.

More is known about the fixed points of the jump-preserving automorphisms of $\mathcal{D}$ (that is, automorphisms of $\mathcal{D}_j$). The main theorem of this paper, Theorem 3.1, states that for all degrees $\mathbf{c} \geqq \mathbf{0}^{(3)}$ and all automorphisms F of $\mathcal{D}_j$, $F(\mathbf{c}) = \mathbf{c}$. This theorem is proved independently in Epstein [1, p. 82] using distributive lattices rather than the nondistributive ones that will be used here.

This main theorem follows from Theorem 1.1, an immediate generalization of Jockusch and Solovay [2, theorem 2]. In Jockusch and Solovay [2] the theorem is stated and proved for $n = 4$, and it is used to prove that the degrees $\mathbf{c} \geqq \mathbf{0}^{(4)}$ are fixed under the automorphisms of $\mathcal{D}_j$.

THEOREM 1.1. *If for all degrees* $\mathbf{a}$ *and all automorphisms* F *of* $\mathcal{D}_j$, $\mathbf{a} \leqq (F(\mathbf{a}))^{(n)}$, *then for all degrees* $\mathbf{c} \geqq \mathbf{0}^{(n)}$ *and all automorphisms* F *of* $\mathcal{D}_j$, $F(\mathbf{c}) = \mathbf{c}$.

In Section 2 we will construct some nondistributive upper semilattices and embed them onto segments of the degrees of unsolvability. The embedding will

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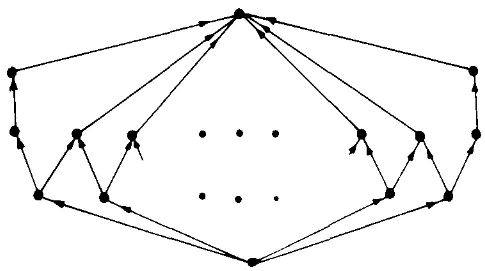
lead to Lemma 3.1: for all degrees $\boldsymbol{a}$ and automorphisms F of $\mathcal{D}_j$, $\boldsymbol{a}^{(2)} \leqq (F(\boldsymbol{a}))^{(3)}$. Theorem 3.1 will then be immediate from Theorem 1.1.

2. Consider for each natural number i the finite upper semilattice $\mathcal{L}_i = \langle L_i, \leqq_i, V_i \rangle$ pictured graphically in Fig. 1. For $a, b \in L_i$, define $a \leqq_i b$ if and only if there is a path in the graph which can be traced from a to b by moving only in the directions indicated by the arrows, and define $a V_i b$ in the usual way as the least upper bound of a and b relative to $\leqq_i$. Note that because of the arrangement of incomparable elements $\mathcal{L}_i$ is embeddable in $\mathcal{L}_j$ if and only if $i = j$.



Any countable collection of these finite upper semilattices can be combined into a countable upper semilattice. When $S \subseteq \omega$ and s_i is the ith element of S listed in numerical order, we can combine the collection $\{\mathcal{L}_i : i \in S\}$ into the countable upper semilattice $\mathcal{L}_S$ by identifying the greatest element of $\mathcal{L}_{s_i}$ with the least element of $\mathcal{L}_{s_i+1}$.

Because (1) two elements from different components of $\mathcal{L}_S$ will always be comparable, and (2) the embeddability of $\mathcal{L}_i$ in any lattice depends on the arrangement of incomparable elements, $\mathcal{L}_i$ is embeddable in $\mathcal{L}_S$ if and only if $i \in S$.

Lemma 2.1 below will specify the property of $\mathcal{L}_S$ which is important for the proof of Lemma 3.1, but some definitions are needed first. If A is a set, let $\bar{A}$ be the complement of A. If A and B are subsets of ω, let $A \oplus B = \{2i : i \in A\} \cup \{2i + 1 : i \in B\}$. If R is a relation on a set A, and $B \subseteq A$, let $R \restriction B$ be the restriction of R to B. Finally, if A is a set or relation, let $\deg(A)$ be the degree of unsolvability containing A.

Lemma 2.1. *Let B be a subset of ω with* $\deg(B) = \boldsymbol{b}$; *let $\mathcal{M} = \langle M, \leqq_{\mathcal{M}}, V_{\mathcal{M}} \rangle$ be a countable upper semilattice. If $S = B \oplus \bar{B}$ and $\mathcal{M} \simeq \mathcal{L}_S$, then* $\deg(\leqq_{\mathcal{M}}) \geqq \boldsymbol{b}$.

Proof. We see by the definitions of $\oplus$ and $\mathcal{L}_S$ that $i \in B$ if and only if $\mathcal{L}_{2i}$ is embeddable in $\mathscr{L}_S \simeq \mathscr{M}$, and $i \notin B$ if and only if $\mathscr{L}_{2i+1}$ is embeddable in $\mathscr{L}_S \simeq \mathscr{M}$. Also, the pattern of incomparable elements in $\mathscr{L}_S \simeq \mathscr{M}$ insures that $\mathscr{L}_j$ is embeddable in $\mathscr{M}$ if and only if the partial ordering $\langle L_j, \leqq_j \rangle$ which is the reduction of $\mathscr{L}_j$ is embeddable in $\langle M, \leqq_{\mathscr{M}} \rangle$, the reduction of $\mathscr{M}$. Thus we can effectively determine whether $i \in B$ or $i \notin B$ by searching all finite subsets F of M for one such that the partial ordering $\langle F, \leqq_{\mathscr{M}} \restriction F \rangle$ is isomorphic to $\langle L_{2i}, \leqq_{2i} \rangle$ or $\langle L_{2i+1}, \leqq_{2i+1} \rangle$. Hence $\boldsymbol{b} = \deg(B) \leqq \deg(M) \oplus \deg(\leqq_{\mathscr{M}})$. But if r is the least element of the upper semilattice $\mathscr{M}$, then $x \in M$ if and only if $r \leqq_{\mathscr{M}} x$. Thus $\deg(M) \leqq \deg(\leqq_{\mathscr{M}})$, and we can conclude that $\boldsymbol{b} \leqq \deg(\leqq_{\mathscr{M}})$. ∎

Define the degree of a presentation of a structure $\mathscr{L}$, written $\deg(\mathscr{L})$, as the join of the degrees of the universe, functions and relations. Define the degree of the isomorphism class of $\mathscr{L}$ to be the least degree among the degrees of presentations isomorphic to $\mathscr{L}$, if such a least degree exists. With this terminology Lemma 2.1 can be modified to the Corollary below.

Corollary 2.1. *For any degree $\boldsymbol{b}$, if B is a set such that* $\deg(B) = \boldsymbol{b}$ *and if $S = B \oplus \bar{B}$, then the degree of the isomorphism class of $\mathscr{L}_S$ exists and is $\boldsymbol{b}$.*

Proof. If $\mathscr{M} = \langle M, \leqq_{\mathscr{M}}, V_{\mathscr{M}} \rangle \simeq \mathscr{L}_S$ then in the proof of Lemma 2.1 we showed that $\boldsymbol{b} \leqq \deg(M) \oplus \deg(\leqq_{\mathscr{M}})$. Thus $\deg(\mathscr{M}) = \deg(M) \oplus \deg(\leqq_{\mathscr{M}}) \oplus \deg(V_{\mathscr{M}}) \geqq \boldsymbol{b}$. Since $\deg(\mathscr{L}_S) = \boldsymbol{b}$ we see that $\boldsymbol{b}$ is the least degree among the degrees of presentations isomorphic to $\mathscr{L}_S$. ∎

The following theorem is a strengthening of the relativized form of the main theorem from Lachlan and Lebeuf [3]. The theorem involves an embedding of an upper semilattice onto a segment of $\mathscr{D}$ with universe $\{\boldsymbol{c} \in D : \boldsymbol{a} \leqq \boldsymbol{c} \leqq \boldsymbol{b}\}$. This substructure will be denoted $\mathscr{D}(\boldsymbol{a}, \boldsymbol{b})$.

Lemma 2.2. *Given $\mathscr{L} = \langle L, \leqq, V \rangle$ a countable upper semilattice with least and greatest elements such that* $\deg(\mathscr{L}) \leqq \boldsymbol{a}^{(2)}$, *then there is a degree $\boldsymbol{b}$ such that $\boldsymbol{b}^{(2)} \leqq \boldsymbol{a}^{(2)}$ and $\mathscr{D}(\boldsymbol{a}, \boldsymbol{b}) \simeq \mathscr{L}$.*

Proof. The main theorem of Lachlan and Lebeuf [3] constructs embeddings of countable upper semilattices with least and greatest elements as initial segments of $\mathscr{D}$. Let $\mathscr{L}$ be embedded onto $\mathscr{D}(a, b)$ by the relativized form of the Lachlan–Lebeuf method. Let B be a set of degree $\boldsymbol{b}$, and fix some indexing of the functions recursive in B. At a recursively determined stage of the construction of the embedding it is decided whether the eth function recursive in B is total or not total. (See Lemma 3.3 of Lachlan and Lebeuf [3].) Thus $B^{(2)}$ is recursive in the construction since $\boldsymbol{b}^{(2)} = \deg(B^{(2)})$ is the degree of $\{e : \text{the } e\text{th function recursive in } B \text{ is total}\}$. (For a proof, see Rogers [4, p. 264].) An examination of the construction shows that its relativization can be completed recursively in $a^{(2)} \oplus \deg(\mathscr{L})$. Thus $b^{(2)} \leqq a^{(2)}$ as desired. ∎

3. The last preliminary lemma is a modification of Jockusch and Solovay [2, corollary 2].

LEMMA 3.1. *If F is a jump-preserving automorphism of the degrees of unsolvability and* $F(a) = b$, *then* $a^{(2)} \leqq b^{(3)}$.

PROOF. Let A be any set of degree a, let $S = A^{(2)} \oplus \overline{A^{(2)}}$, and let $\mathscr{L}_S = \mathscr{L} = \langle L, \leqq, V \rangle$. Corollary 2.1 showed that $\deg(\mathscr{L}) \leqq a^{(2)}$. Thus Lemma 2.2 applies and a degree a_0 can be found such that $\mathscr{L}$ is isomorphic to $\mathscr{D}(a, a_0)$ and $a_0^{(2)} = a^{(2)}$. Let $b_0 = F(a_0)$, and note that since F preserves jumps, $b_0^{(2)} = b^{(2)}$. The image of $\mathscr{D}(a, a_0)$ under F must be $\mathscr{D}(b, b_0)$. Since $\mathscr{D}(b, b_0)$ is isomorphic to $\mathscr{L}$, by Lemma 2.1 $a^{(2)} \leqq \deg(\leqq_{\mathscr{D}(b, b_0)})$. Also $\leqq_{\mathscr{D}(b, b_0)}$ is relative recursiveness restricted to the degrees between b and b_0. Since relative recursiveness is determined by three quantifiers, $\deg(\leqq_{\mathscr{D}(b, b_0)}) \leqq b_0^{(3)}$. Combining these last two facts yields the result that $a^{(2)} \leqq b_0^{(3)} = b^{(3)}$. ∎

Now the main theorem follows easily.

THEOREM 3.1. *If F is a jump-preserving automorphism of the degrees of unsolvability, then* $F(c) = c$ *for all* $c \geqq 0^{(3)}$.

PROOF. The theorem follows immediately from Theorem 1.1 and Lemma 3.1. ∎

The following Corollary strengthens Lemma 3.1.

COROLLARY 3.1. *If F is a jump-preserving automorphism of the degrees of unsolvability, and* $F(a) = b$, *then* $a^{(3)} = b^{(3)}$.

PROOF. If a is any degree of unsolvability, then $a^{(3)} \geqq 0^{(3)}$. So by Theorem 3.1, $F(a^{(3)}) = a^{(3)}$. But since F preserves jumps, $F(a^{(3)}) = F(a)^{(3)} = b^{(3)}$. Thus $a^{(3)} = b^{(3)}$. ∎

For fixed $n < 3$ a further strengthening of Lemma 3.1 to show that $a \leqq F(a)^{(n)}$ could be used with Theorem 1.1 to establish a strengthening of Theorem 3.1 to degrees $c \geqq 0^{(n)}$. However the method of this paper, the embedding into $\mathscr{D}$ of partially ordered sets whose isomorphic presentations have well-behaved degrees, will not serve to strengthen the lemma to n less than 3 since the relative recursiveness of degrees less than b, which is the partial ordering relation in $\mathscr{D}$, is a relation of degree $b^{(3)}$.

References

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Mathematics Department
Wabash College
Crawfordsville, Indiana 47933 USA