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EXISTENCE OF SEPARABLE UNIFORMLY HOMEOMORPHIC NONISOMORPHIC BANACH SPACES

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ABSTRACT
An example is given proving that there exist two separable Banach spaces which are uniformly homeomorphic but not isomorphic.

Uniform homeomorphism and Lipschitz homeomorphism are natural equivalence relations among Banach spaces. Aharoni and Lindenstrauss [1] have given an example of two Banach spaces which are not isomorphic but which are Lipschitz homeomorphic, and thus also uniformly homeomorphic. The spaces of that example are non-separable and non-reflexive. Results have also been given in the opposite direction, i.e., showing how uniformly or Lipschitz homeomorphic Banach spaces must be related; see Enflo [3], Heinrich and Mankiewicz [4], and Ribe [7, 8]. Further references to previous work can also be found in those papers.

In this paper the following result is proved.

THEOREM 1. *Let $q > 1$, and let $(p_i)_{i=1}^{\infty}$ be a sequence of numbers such that $p_i > 1$ and $p_i \to 1$. Let $S = l_q(L_{p_i})$ denote the l_q-sum of the spaces $L_{p_i}(0,1)$. Then the spaces S and $S \oplus L_1$ are uniformly homeomorphic.*

REMARKS. So in particular reflexivity is not generally invariant under uniform homeomorphism, in contrast to superreflexivity [7]. The spaces S and $S \oplus L_1$ are not Lipschitz homeomorphic. For if L_1 were Lipschitz imbeddable into S, then it would also be isomorphically imbeddable into S, by a result of Mankiewicz [5]. Further, Theorem 2 of Ribe [7] gives some indications about what a uniform homeomorphism cannot look like when the two spaces are non-isomorphic and separable, and at least one of them is a dual space.

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The idea of the proof of Theorem 1 is that uniform homeomorphisms between larger and larger balls in L_p-spaces can be pasted together in a certain way. These homeomorphisms uniform on balls were defined by Mazur [6] and considered by Day [2].

LEMMA 1. *(Mazur [6].) For $1 \leqq p,\ q < \infty$, let $m_{p,q} : L_p \to L_q$ be the mapping defined by $m_{p,q}(x(\,.\,)) = x(\,.\,)\,|\,x(\,.\,)|^{p/q-1}$. Then the class of these mappings $m_{p,q}$ is equi-uniformly homeomorphic on the sets of x such that $\| x \|_p \leqq e^{1/|p-q|}$.*

By the last sentence is of course meant that for every $\varepsilon > 0$ there is a $\delta > 0$ such that whatever p and q, for x and y in L_p with length $\leqq e^{1/|p-q|}$ we have that

$$\| x - y \|_p < \delta \quad \text{implies} \quad \| m_{p,q}(x) - m_{p,q}(y) \|_q < \varepsilon,$$

and

$$\| m_{p,q}(x) - m_{p,q}(y) \|_q < \delta \quad \text{implies} \quad \| x - y \|_p < \varepsilon.$$

PROOF. Since $m_{p,q} = m_{q,p}^{-1}$, we only have to prove the first of the last-mentioned implications. We consider the case of real scalars. Let us write

$$x - y = (x - x_1) + (x_1 - z) + (z - y),$$

where $x_1(\,.\,) = |\,x(\,.\,)|\,\text{sign}\,(y(\,.\,))$ and $z(\,.\,) = \max\,(|\,x(\,.\,)|,\ |\,y(\,.\,)|)\,\text{sign}\,(y(\,.\,))$. Then we have

$$\| m_{p,q}(x) - m_{p,q}(x_1) \|_q \leqq 2 \| x - y \|_p,$$

$$\| x_1 - z \|_p \leqq \| x - y \|_p,$$

$$\| z - y \|_p \leqq \| x - y \|_p.$$

It now remains to show that $\| m_{p,q}(z) - m_{p,q}(y) \|_q$ becomes small for $\| z - y \|_p$ small; for the treatment of $\| m_{p,q}(x_1) - m_{p,q}(z) \|_q$ is similar. Write $y(\,.\,) = k(\,.\,)z(\,.\,)$, so that $0 \leqq k(\,.\,) \leqq 1$, and for a fixed number $d > 0$, let D be the set of those t in $[0,1]$ for which $1 - k(t) < d$. Now,

$$\| m_{p,q}(z) - m_{p,q}(y) \|_q^q = \int |\,z(t)|^p (1 - k(t)^{p/q})^q dt;$$

for $p \leqq q$ this is clearly dominated by $\| z - y \|_p^p$, so let $p > q$. Then, with $E = [0,1] \setminus D$, the last integral is dominated by

$$(p/q)^q \int_0^1 |\,z(t)|^p (1 - k(t))^q dt = (p/q)^q \left(\int_D + \int_E \right)$$

$$\leqq (p/q)^q (\| z \|_p^p d^q + \| z - y \|_p^p d^{q-p}).$$

By assumption $\| z \|_p \leqq 2e^{1/(p-q)}$, and taking $d = \| z - y \|^{p/(2(p-q))}$, we then obtain that the last expression becomes small whenever $\| z - y \|_p$ is small.

The next lemma is the main step in the mentioned pasting of the Mazur homeomorphisms $m_{p,q}$.

LEMMA 2. *For* $1 \leqq p, q, r \leqq 2$ *and* $0 \leqq t \leqq 1$ *there are bijective mappings*

$$f_{p,q} : L_p \oplus L_q \to L_q \qquad \textit{and}$$

$$F_{p,q,r,t} : L_p \oplus L_q \oplus L_r \to L_q \oplus L_r,$$

which for $B = e^{1/(C-1)}$, *where* $C = \max(p, q, r)$, *enjoy the properties*:

(i) $F_{p,q,r,0} = f_{p,q} \oplus \mathrm{id}$, *where* id *is the identity mapping on* L_r.

(ii) $F_{p,q,r,1} = \tau F_{p,r,q,0} \sigma$, *where* σ *and* τ *are the mappings interchanging* L_q *and* L_r *in* $L_p \oplus L_q \oplus L_r$ *and in* $L_q \oplus L_r$, *resp.*

(iii) *The class of mappings* $F_{p,q,r,t}$ *is equi-uniformly homeomorphic on the sets of points* x *with* $\| x \| \leqq B$.

(iv) *Denote by* pr_0, pr_1, *and* pr_2 *the natural projections from* $L_p \oplus L_q \oplus L_r$ *onto* L_p, L_q, *and* L_r, *resp. With*

$$\nu(\,.\,) = \| \mathrm{pr}_0(\,.\,) \|_p^p + \| \mathrm{pr}_1(\,.\,) \|_q^q + \| \mathrm{pr}_2(\,.\,) \|_r^r$$

defined on $L_p \oplus L_q \oplus L_r$ *we have*

$$\nu(F_{p,q,r,t}(x)) = \nu(x).$$

(v) *The class of mappings* $t \to F_{p,q,r,t/b}(x)$, *where* $b \leqq B$ *and* $\| x \| \leqq b$, *is equi-uniformly continuous on the sets* $[0, b]$.

PROOF. We shall consider L_q as $l_q(L_q)$, i.e., as the l_q-sum of countably many copies of itself. And L_r will be considered similarly. According to this view, a point x in $L_p \oplus L_q \oplus L_r$ is described as

$$x = (x_0(\,.\,)(x_{1,1}(\,.\,), x_{1,2}(\,.\,), \dots), (x_{2,1}(\,.\,), x_{2,2}(\,.\,), \dots)).$$

Here $x_0(\,.\,)$ is a function on $[0, 1]$ which is in L_p, the $x_{1,i}(\,.\,)$ are in the L_q-copies in $l_q(L_q)$, and the $x_{2,i}(\,.\,)$ are in the L_r-copies in $l_r(L_r)$. Likewise a point x in $L_q \oplus L_r$ is described as

$$x = ((x_{1,1}(\,.\,), x_{1,2}(\,.\,), \dots), (x_{2,1}(\,.\,), x_{2,2}(\,.\,), \dots)).$$

For x in $L_p \oplus L_q \oplus L_r$ and $0 \leqq u \leqq v \leqq 1$, write

$$I_0(x, u, v) = \int_u^v |\, x_0(s) \,|^p ds.$$

And for x in $L_p \oplus L_q \oplus L_r$ or in $L_q \oplus L_r$, put

$$I_i(x,u,v) = \sum_j \int_u^v |x_{i,j}(s)|^{q'} ds,$$

where $i = 1,2$, and where $q' = q$ for $i = 1$ and $q' = r$ for $i = 2$. Now write

$$I(x,u,v) = I_0 + I_1 + I_2,$$

$$J(x,u,v) = I_1 + I_2.$$

We shall now define the mapping $F_{p,q,r,t}$ by defining the image point $\bar{x} = F_{p,q,r,t}(x)$ of a given point x. Let $a(x)$ $(0 \leqq a(x) \leqq 1)$ be the least number such that $I(x,0,a(x)) = tI(x,0,1)$. Then, with $m_{p,q}$ as in Lemma 1, we define $\bar{x}$ by

$$\begin{aligned}
\bar{x}_{1,1}(s) &= m_{p,q}(x_0)(s) && \text{for } s > a(x)\\
\bar{x}_{1,i}(s) &= x_{1,i-1}(s) && \text{for } s > a(x), \quad i \geqq 2\\
&= x_{1,i}(s) && \text{for } s \leqq a(x)\\
\bar{x}_{2,i}(s) &= x_{2,i}(s) && \text{for } s > a(x)\\
\bar{x}_{2,1}(s) &= m_{p,r}(x_0)(s) && \text{for } s \leqq a(x)\\
\bar{x}_{2,i}(s) &= x_{2,i-1}(s) && \text{for } s \leqq a(x), \quad i \geqq 2.
\end{aligned}$$

From this definition of $F_{p,q,r,t}$ it is clear that also the mappings $f_{p,q}$ fulfilling (i) are well-defined; and likewise (ii) follows. For (iv) notice that $\nu(.) = I(.,0,1)$ resp. $J(.,0,1)$ on $L_p \oplus L_q \oplus L_r$ resp. $L_q \oplus L_r$. Further, (v) is a rather straightforward consequence of the definition made of $F_{p,q,r,t}$.

It remains to verify (iii). First notice that

$$I(x,u,v) = J(\bar{x},u,v)$$

always holds. It follows that the number $a(x)$ is determined by the point $\bar{x}$ as the least number such that $J(\bar{x},0,a(x)) = tJ(\bar{x},0,1)$. This shows that $F_{p,q,r,t}$ is a bijective mapping, since it clearly is so on every set of points x such that $a(x) = s$ for some s.

To prove the required equi-uniform continuity of mappings $F_{p,q,r,t}$, let x and y be two points of norm at most B in $L_p \oplus L_q \oplus L_r$. Let $\bar{x}$ and $\bar{y}$ denote their images under $F_{p,q,r,t}$, and assume that $a(x) \leqq a(y)$. We must show that the number

$$J(\bar{x}-\bar{y},0,1) = J(\bar{x}-\bar{y},0,a(x)) + J(\bar{x}-\bar{y},a(x),a(y)) + J(\bar{x}-\bar{y},a(y),1)$$

is small whenever x is close to y. By Lemma 1 and the definition of $F_{p,q,r,t}$ this is certainly true for the first and the third term. And the second term is at most

$$2^{C-1}(J(\bar{x}, a(x), a(y)) + J(\bar{y}, a(x), a(y)).$$

The two terms here are treated similarly. The first one can be rewritten and estimated as

$$\begin{aligned} I(x, a(x), a(y)) &= (1-t)I(x,0,1) - I(x, a(y), 1) \\ &= (1-t)(I(x,0,1) - I(y,0,1)) \\ &\quad + (I(y, a(y), 1) - I(x, a(y), 1)) \\ &\leqq 2\max(C, C\|x\|^{C-1}, C\|y\|^{C-1})\|x-y\| \\ &\leqq 4e\|x-y\|. \end{aligned}$$

By a corresponding argument the mappings $F^{-1}_{p,q,r,t}$ are equi-uniformly continuous, which completes the proof.

We shall also need the following fact.

Lemma 3. *Let $f : X \to Y$ be a uniformly continuous mapping from a normed space onto another. Suppose that f is such that for some convex functionals ϕ and ψ on X resp. Y and some set D of non-negative numbers, the following conditions are fulfilled:*

(i) $\phi(x), \psi(x) \geqq 0$ *for x in X resp. Y; with $\phi(x) = 0$, resp. $\psi(x) = 0$, if and only if $x = 0$.*

(ii) $\max(\phi(x_0 + x), \psi(y_0 + y)) \leqq C\min(\phi(x_0), \psi(y_0))$ *whenever x_0, x in X and y_0, y in Y are such that $\|x_0\| = \|y_0\| = \|x\| = \|y\|$, for some constant $C \geqq 1$.*

(iii) *The class of restrictions of f to sets $\phi^{-1}(d)$, $d \geqq 0$, is equi-uniformly homeomorphic.*

(iv) *The restriction of f to the set $\phi^{-1}(D)$ is a uniform homeomorphism onto $\psi^{-1}(D)$.*

(v) $\psi(f(x)) = \phi(x)$ *whenever $\phi(x)$ is not in D.*

Then f is a uniform homeomorphism of X onto Y.

Proof. First notice that if y_1 and y_2 are any two points in Y we have the estimate

$$\text{(E)} \qquad \|y_1 - y_2\| \geqq (1/4C^2)\operatorname{dist}(\phi^{-1}(\psi(y_1)), \phi^{-1}(\psi(y_2)).$$

To see this, first consider the case when y_1 and y_2 are such that

$$2\|y_1 - y_2\| \leqq \|y_1\| \quad \text{and}$$
$$2\,\mathrm{dist}\,(\phi^{-1}(\psi(y_1)),\ \phi^{-1}(\psi(y_2)) \leqq \mathrm{dist}\,(\phi^{-1}(\psi(y_1)), 0).$$

Then let x_1 be a point in $\phi^{-1}(\psi(y_1))$, and let $\bar{x}$ and $\bar{y}$ be any two points within distance $\|x_1\|/2$ resp. $\|y_1\|/2$ from x_1 resp. y_1. Let u and v be elements in the subgradient of ϕ resp. ψ at $\bar{x}$ resp. $\bar{y}$ (i.e., u is a real linear functional of X such that $u(x-\bar{x}) \leqq \phi(x) - \phi(\bar{x})$ for x in X). Then condition (ii) implies that

$$\phi(x_1)/2C \leqq \|u\|\,\|x_1\| \leqq 2C\phi(x_1) \quad \text{and}$$
$$\phi(x_1)/2C \leqq \|v\|\,\|x_1\| \leqq 2C\phi(x_1).$$

The estimate (E) follows by the mean value theorem. And we can generalise (E) to an arbitrary pair of points y_1, y_2 in Y, by successively applying it to suitable pairs of points on the line segment between y_1 and y_2.

Next, let δ_1 be a given positive number. By condition (iii) there is a number $\delta_2 > 0$ such that for any two points in X which lie at least distance $\delta_1/2$ apart and which have the same ϕ-value, their image points under f lie at least distance δ_2 apart. Then by the uniform continuity of f there is a number $\delta_3 > 0$ such that for any two points in X of distance apart less than δ_3, their image points under f are of distance apart less than $\delta_2/2$.

To prove the assertion of the lemma, we must show that for points x_1, x_2 in X such that $\|x_1 - x_2\| \geqq \delta_1$ for a given number $\delta_1 > 0$, the numbers $\|f(x_1) - f(x_2)\|$ have a uniform lower bound. In view of condition (iv) it will do if we assume that x_1 and x_2 are both outside $\phi^{-1}(D)$. Consider two cases: First if $\mathrm{dist}\,(\phi^{-1}(\phi(x_1)), x_2) < \delta_3$, then the choices of δ_2 and δ_3 imply that $\|f(x_1) - f(x_2)\| \geqq \delta_2/2$. Second, in the opposite case, condition (v) and the estimate (E) yield a lower bound for $\|f(x_1) - f(x_2)\|$. So in both cases the required uniform lower bound is obtained.

Proof of Theorem 1. We shall construct a uniform homeomorphism $h : L_1 \oplus S \to S$, by making use of the mappings $F_{p,q,r,s}$ of Lemma 2. The number sequence $(p_i)_i$ defining the space S can be replaced with a subsequence without loss of generality, so we can assume that $p_i \leqq 1 + 1/20i$. For simplicity in notation we also assume that $q \leqq 2$. The proof can be carried over to the case of general $q > 1$ by changes of constants.

Let $n \geqq 1$ be an integer. We shall now define $h(x)$ for $10^{7(n-1)} \leqq \|x\| \leqq 10^{7n}$. First we need some auxiliary notions.

Let

$$\mathrm{pr}^0 : L_1 \oplus S \to L_1$$
$$\mathrm{pr}_i : L_1 \oplus S \to L_{p_i} \qquad (i \geqq 1)$$

be the natural projections. The norm on $L_1 \oplus S$ is assumed to be

$$\| x \| = (\| \mathrm{pr}^0(x) \|_1^q + \| x - \mathrm{pr}^0(x) \|_S^q)^{1/q}.$$

We introduce the convex functional

$$U_n(x) = \left((\| \mathrm{pr}^0(x) \|_1 + \| \mathrm{pr}_n(x) \|_{p_n}^{p_n} + \| \mathrm{pr}_{n+1}(x) \|_{p_{n+1}}^{p_{n+1}})^q + \sum_{i \neq n, n+1} \| \mathrm{pr}_i(x) \|_{p_i}^q \right)^{1/q},$$

defined on $L_1 \oplus S$. In view of the conditions on p_i and q it can be seen that

$$(*) \qquad \| x \| / 9 \leqq U_n(x) \leqq 9 \| x \| \qquad \text{for } 1 \leqq \| x \| \leqq 10^{7^n}.$$

Consider $L_1 \oplus S$ as the direct sum of the two spaces $L_1 \oplus L_{p_n} \oplus L_{p_{n+1}}$ and $\bigoplus_{i \neq n, n+1} L_{p_i}$, and let P_n and Q_n be the projections from $L_1 \oplus S$ onto the first resp. second of those spaces, annihilating the other space. Then, considering S similarly as the direct sum of $L_{p_n} \oplus L_{p_{n+1}}$ and $\bigoplus_{i \neq n, n+1} L_{p_i}$, we define

$$h(x) = F_{1, p_n, p_{n+1}, t(x)}(P_n(x)) \oplus Q_n(x)$$

for $10^{7^{(n-1)}} \leqq \| x \| \leqq 10^{7^n}$, where

$$t(x) = \begin{cases} 0 & \text{if } U_n(x) \leqq 2 \cdot 10^{7^n - 3}, \\ (U_n(x) - 2 \cdot 10^{7^n - 3})/(8 \cdot 10^{7^n - 3}) & \\ \quad \text{if } 2 \cdot 10^{7^n - 3} \leqq U_n(x) \leqq 10^{7^n - 2}, & \\ 1 & \text{if } U_n(x) \geqq 10^{7^n - 2}. \end{cases}$$

The integer $n \geqq 1$ being arbitrary, $h(x)$ has thus been defined for all x with $\| x \| \geqq 1$. And for $\| x \| \leqq 1$, take

$$h(x) = F_{1, p_1, p_2, 0}(P_1(x)) \oplus Q_1(x).$$

For those x with $\| x \| = 10^{7^n}$ for some $n \geqq 0$, the value $h(x)$ has actually been defined twice, but the definition is indeed consistent, as is guaranteed by (i) and (ii) of Lemma 2. For in view of $(*)$ we always have $t(x) = 0$ resp. 1 for those x.

By its definition and Lemma 2, the mapping h is clearly uniformly continuous. We shall verify that Lemma 3 can be applied to prove that h is a uniform homeomorphism. First we shall construct the convex functionals ϕ and ψ, defined on $L_1 \oplus S$ resp. S. Write

$$V_n(x) = \begin{cases} \max(0, N_n(x), 100 U_n^-(x), 10000 N_{n+1}(x)) & \text{for } n \geqq 1, \\ 10000 N_1(x) & \text{for } n = 0, \end{cases}$$

where

$$N_n(x) = \|x\| - 9 \cdot 10^{7n-8},$$

$$U_n^-(x) = U_n(x) - 9 \cdot 10^{7n-6}.$$

Then in view of inequality (*) the following relations hold for $n \geqq 1$:

$$(\dagger) \qquad V_n(x) = \begin{cases} N_n(x) & \text{for } \|x\| = 10^{7(n-1)}, \\ 100U_n^-(x) & \text{for } 10^{7n-4} \leqq \|x\| \leqq 10^{7n-1}, \\ 10000N_{n+1}(x) & \text{for } \|x\| \geqq 10^{7n}. \end{cases}$$

Now we define the desired convex functional

$$\phi(x) = \sup_{n \geqq 0} (10^{4n} V_n(x)).$$

And for ψ we take the restriction of ϕ to S, as a subspace of $L_1 \oplus S$.

Finally, for the number set D in Lemma 3 we take the union of all the intervals $[0, 10^8]$ and $[10^{11i-2}, 10^{11i+8}]$ $(i \geqq 1)$.

If x is a point in X outside $\phi^{-1}(D)$, then (*) and (†) imply that

$$10^{7i-4} \leqq \|x\| \leqq 10^{7i-1}, \qquad \text{whence} \qquad \phi(x) = 10^{4i+2} U_i^-(x),$$

for some $i \geqq 1$. In view of (iv) of Lemma 2 it then follows that $\psi(h(x)) = \phi(x)$, so that condition (v) of Lemma 3 is fulfilled. For condition (iv) of Lemma 3, notice that the functionals $t(.)$ used in the definition of h only assume the values 0 and 1 on the set $\phi^{-1}(D)$. And clearly h is onto. Lastly, conditions (i)–(iii) of Lemma 3 are clearly fulfilled.

REMARK. Maybe a little unexpectedly, the proof would apparently fail if one tried to replace L_1 in Theorem 1 with an L_p-space with $p > 1$, and the condition $p_i \to 1$ with $p_i \to p$. For in the estimation at the end of the proof of Lemma 2, the factor $C\|x\|^{C-1}$ can remain uniformly bounded only if C is taken close to 1 when $\|x\|$ is large.

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