ON PERMANENTS OF (1, − 1)-MATRICES†

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ABSTRACT

A preliminary study on permanents of (1, − 1)-matrices is given. Some inequalities are derived and a few unsolved problems, believed to be new, are mentioned.

1. Introduction

It is well known that the evaluation of the permanent of an arbitrary matrix is a formidable problem. For the special case when the matrices are nonnegative, e.g., (0, 1)-matrices, the upper and lower bounds for the permanent have been studied extensively in the literature, mainly for their combinatorial interpretation and significance. On the other hand, the permanent of a (1, − 1)-matrix, though of combinatorial interest, has hardly been worked on.This problem is conceivably more difficult than the corresponding one of (0, 1)-matrices. For one thing, the permanent of a (0, 1)-matrix would not decrease if a zero is replaced by a one, yet it is totally independent of the number of − 1's in a (1, − 1)-matrix. In fact, it is easy to see that a (1, − 1)-matrix of even order can attain its maximum permanent value $n!$ even when all the entries are − 1 or exactly half of the entries are − 1. (cf. Example 1 in Section 2.)

The purpose of this paper is, therefore, to give a preliminary study on permanents of (1, − 1)-matrices. Denote by Ω_n the set of all $n \times n$ (1, − 1)-matrices and by J_n the one with all entries equal to 1. The permanent of $A \in \Omega_n$ will be denoted by per A. The $(n-1)\times(n-1)$ submatrix obtained from A by deleting the ith row and the jth column will be denoted by A_{ij}.

Two matrices A and B in Ω_n are said to be equivalent, denoted by $A \sim B$, if one is obtainable from the other by a sequence of operations of the following types:

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(I) interchange any 2 rows or any 2 columns.

(II) negate any row or any column.

This relation is obviously an equivalence relation and hence partitions Ω_n into equivalence classes. It is equally obvious that |per| is invariant in each class. The converse, however, is false in general; i.e., $|\text{per } A| = |\text{per } B|$ does not necessarily imply that $A \sim B$. For example, consider the following matrices in Ω_4:

$$A = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Then $\text{per } A = \text{per } H = 8$, yet A and H cannot be equivalent since H is a Hadamard matrix and it is easy to see that any matrix equivalent to a Hadamard matrix must itself be a Hadamard matrix.

2. Some results

For $A \in \Omega_n$, it is well known (e.g., [1, p. 102]) that $\det A \equiv 0 (\text{mod } 2^{n-1})$. We show that there is an analogy for permanent.

PROPOSITION 1. *Let $A \in \Omega_n$. Then*

$$\text{per } A \equiv 0 \begin{cases} (\text{mod } 2^{n/2}) & \textit{if} \quad n \textit{ is even}; \\ (\text{mod } 2^{(n-1)/2}) & \textit{if} \quad n \textit{ is odd}. \end{cases}$$

PROOF. Since $\text{per}(B) = 0$ or ± 2 for all $B \in \Omega_2$, the assertion follows from the Laplace expansion and induction on n.

As an interesting consequence of Proposition 1, we obtain the following:

COROLLARY 1. *For each positive integer k, there exist at most a finite number of $(1, -1)$-matrices A (of any order) such that $|\text{per } A| = k$.*

REMARK 1. Concerning Proposition 1, we do not know whether it is the best possible in general; i.e., it is not known whether for each n, there exists $A \in \Omega_n$

such that per A is divisible by $2^{n/2}$ (if n is even) or $2^{(n-1)/2}$ (if n is odd) but by no higher power of 2. For $n \leqq 5$, examples are available to show the answer is in the affirmative.

We next determine those $A \in \Omega_n$ the permanents of which attain the maximum absolute value.

PROPOSITION 2. *Let $A \in \Omega_n$. Then $|\text{per } A| = n!$ if and only if $A \sim J_n$.*

PROOF. The sufficiency is obvious. To show the necessity, suppose per $A = n!$ or per $A = -n!$ Then all the diagonal products of A have a common nonzero value. Hence by a result of Marcus and Minc [3, p. 577], A must have rank 1, and it is easily seen that any $n \times n$ $(1, -1)$-matrix of rank 1 is equivalent to J_n.

EXAMPLE 1. The $n \times n$ chessboard matrix $A \in \Omega_n$ is defined by $A_{ij} = (-1)^{i+j}$, $i, j = 1, 2, \cdots, n$. Since it is easy to see that $A \sim J_n$, we have $|\text{per } A| = n!$ by Proposition 1. In fact, per $A = n!$ since we can transform A to J_n by negating rows and columns an even number of times.

In view of Proposition 1, a positive integer $k \leqq n!$ equals $|\text{per } A|$ for some $A \in \Omega_n$ only if k is divisible by $2^{n/2}$ or $2^{(n-1)/2}$ depending on whether n is even or odd. This condition, however, is not sufficient as shown in the next proposition.

PROPOSITION 3.† *Let $A \in \Omega_n$, $n \geqq 2$. If $|\text{per } A| \neq n!$, then $|\text{per } A| \leqq (n-2)(n-1)!$. For $n > 3$, the equality holds if and only if $A \sim J_n^*$ where J_n^* has -1 in the $(1,1)$ position and 1 elsewhere. For $n = 3$, equality holds if and only if $A \sim J_3^*$ or $A \sim J_3^{**}$ where J_3^{**} has -1 in the $(1,2)$ and $(2,1)$ positions and 1 elsewhere.*

PROOF. Let p and q denote the number of diagonals of A with diagonal product 1 and -1 respectively. Then $p + q = n!$ and $|p - q| = |\text{per } A|$. Since $|\text{per } A| \neq n!$, A can not have rank 1 and hence by the result of Marcus and Minc [3, p. 577] we obtain $p \leqq (n-1)(n-1)!$ and $q \leqq (n-1)(n-1)!$ Therefore, $p - q = 2p - n! \leqq 2(n-1)(n-1)! - n! = (n-2)(n-1)!$ and $p - q = n! - 2q \geqq n! - 2(n-1)(n-1)! = -(n-2)(n-1)!$ which imply that $|\text{per } A| \leqq (n-2)(n-1)!$. The "if" part of the assertion for equality is trivial. To prove the "only if" part, we first notice that for $n = 2$, it is clear. Suppose then that $A \in \Omega_n$ such that $|\text{per } A| = (n-2)(n-1)!$, $n \geqq 3$. Consider the submatrices $A_{1i} \in \Omega_{n-1}$, $i = 1, 2, \cdots, n$. If for all i, $|\text{per } A_{1i}| \neq (n-1)!$, then $|\text{per } A_{1i}| \leqq (n-3)(n-2)!$ implies that $|\text{per } A| \leqq n(n-3)(n-2)! < (n^2 - 3n + 2)(n-2)! =$

† I am grateful to Dr. P. Gibson for pointing out a flaw in the original statement of this proposition.

$(n-2)(n-1)!$, a contradiction. Hence $|\text{per } A_{1i}| = (n-1)!$ for some i, and by Operation I, we can assume that $i = 1$; i.e., $|\text{per } A_{11}| = (n-1)!$. Therefore $A_{11} \sim J_{n-1}$ by Proposition 2 and we have:

$$A \sim A' = \left[\begin{array}{c|ccc} a_{11} & a_{12} & \cdots & a_{1n} \\ \hline a_{21} & & & \\ \cdot & & & \\ \cdot & & J_{n-1} & \\ \cdot & & & \\ a_{n1} & & & \end{array}\right]$$

By Operation II, we can also assume that $a_{11} = 1$. Let h be the number of ones among a_{1i} and k be the number of ones among a_{i1}, $i = 2, 3, \cdots, n$, $0 \leqq h, k \leqq n-1$. Then it is easily seen that $\text{per } A' = (n-1)! + (n-1-2h)(n-1-2k)(n-2)!$ Since $|\text{per } A'| = |\text{per } A| = (n-2)(n-1)!$, we obtain $|(n-1)+(n-1-2h)(n-1-2k)| = (n-2)(n-1)$, and thus there are two cases:

Case (i). $(n-1)+(n-1-2h)(n-1-2k) = (n-2)(n-1)$ or $(n-1-2h)(n-1-2k) = (n-3)(n-1)$. If $h = 0$ or $k = 0$, we obtain the solutions $h = 0$, $k = 1$ and $h = 1$, $k = 0$. If $h \neq 0$ and $k \neq 0$, then we can assume that $1 \leqq k \leqq n-2$ or $1 \leqq h \leqq n-2$ since $h = k = n-1$ is obviously impossible. If $1 \leqq k \leqq n-2$, we rewrite $(n-1-2h)(n-1-2k) = (n-3)(n-1)$ as $h(n-k-2)+(n-1-h)(k-1) = 0$ from which $k = n-2$ follows. If $k \neq 1$, then we obtain the solution $h = n-1$, $k = n-2$. If, however, $k = 1$, then $n = 3$, and we obtain 2 solutions: $h = k = 1$ and $h = 2$, $k = 1$. Similarly, if $1 \leqq h \leqq n-2$, then we obtain the solution $h = n-2$, $k = n-1$ and for the case $n - 3$, 2 solutions: $h = k = 1$ and $h = 1$, $k = 2$.

Case (ii). $(n-1)+(n-1-2h)(n-1-2k) = -(n-1)(n-2)$ or $(n-1-h)(n-1-k)+hk = 0$ which yields the solutions $h = 0$, $k = n-1$ and $h = n-1$, $k = 0$.

In summary, if $n > 3$, the ordered pair (h, k) must be $(0, 1)$, $(1, 0)$, $(n-1, n-2)$, $(n-2, n-1)$, $(0, n-1)$ or $(n-1, 0)$. In all these cases, it is easy to see that $A' \sim J_n^*$ and thus $A \sim J_n^*$. When $n = 3$, one more solution is possible, i.e., $(h, k) = (1, 1)$. In this case, it is readily seen that $A \sim J_3^{**}$.

Remark 2. Since $\det A$ is always divisible by 2^{n-1} for $A \in \Omega_n$, if follows from Propositions 1 and 3 that if $A \in \Omega_n$ is nonsingular, $n = 1, 2, 3, 4$, then $|\text{per } A| \leqq |\det A|$. This is, however, false in general; e.g., let

$$A = \begin{bmatrix} 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{bmatrix} \in \Omega_5.$$

Then we have $\det A = 16$, $\operatorname{per} A = 24$.

Since the maximum value $n!$ is always attainable for $|\operatorname{per} A|$, $A \in \Omega_n$, it is natural to ask whether the minimum value 0 is always attainable. This is clearly impossible for $n = 1$. In general, we have the following partial answer to this question.

PROPOSITION 4. *If* $n \geqq 2$ *is even or* $n \equiv 1 \pmod 4$, *then there exist* $A \in \Omega_n$ *such that* $\operatorname{per} A = 0$.

PROOF. When n is even, this is trivial. For $n \equiv 1 \pmod 4$, i.e., $n = 4k + 1$ for some $k \geqq 1$, consider the following matrix:

$$A = \begin{array}{l} \\ r\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \\ s\left\{\begin{array}{l} \\ \\ \\ \\ \end{array}\right. \end{array} \left[\begin{array}{c|c} 1 & \overbrace{1 \cdots\cdots 1}^{p}\quad \overbrace{-1 \cdots\cdots -1}^{q} \\ \hline \begin{array}{c} 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \\ -1 \\ \cdot \\ \cdot \\ \cdot \\ -1 \end{array} & J_{n-1} \end{array}\right] \in \Omega_n,$$

where $p+q=r+s=n-1=4k$.

Since $\operatorname{per} A=(n-1)!+(p-q)(n-2)!$, we obtain:

$$\operatorname{per} A=0 \text{ if and only if } n-1=(p-q)(s-r) \text{ or } (2k-q)(2k-r)=k.$$

The last equation clearly holds if we let $q=2k-1$ and $r=k$.

REMARK 3. For $n \equiv 3 \pmod 4$, it is not known in general whether there exist $A \in \Omega_n$ such that $\operatorname{per} A=0$. In the following, we give some results pertaining to this problem.

PROPOSITION 5. *There exist no* 3×3 $(1,-1)$*-matrix* A *such that* $\operatorname{per} A=0$.

PROOF. Let $A \in \Omega_3$. Since $|\operatorname{per} A|$ is invariant under Operation II, $|\operatorname{per} A|=|\operatorname{per} A'|$ where $A \sim A'$ and A' has all entries 1 in the first row and the first column. A simple case by case check then shows that $|\operatorname{per} A'|=2$ or 6.

PROPOSITION 6. *If* $n \equiv 3 \pmod 4$ *and* $A \in \Omega_n$ *such that* $\operatorname{per} A=0$, *then*

(i) *A cannot contain a submatrix B such that $B \sim J_{n-1}$.*

(ii) *A cannot be equivalent to some $B \in \Omega_n$ which has $n-3$ rows (or columns) with all entries* 1.

PROOF. (i) Assume the contrary. Then A is equivalent to some matrix of the form given in the proof of Proposition 4. Since $\operatorname{per} A=0$ if and only if $(p-q)(s-r)=n-1=4k+2$, where $p+q=r+s=n-1=4k+2$, we obtain $(4k+2-2q)(4k+2-2s)=4k+2$, a contradiction.

(ii) Assume the contrary. Then $A \sim B$ where $B \in \Omega_n$ has $n-3$ rows with all entries 1. We evaluate $\operatorname{per} B$ by using the Laplace expansion. Let α, β, γ and δ denote the number of 3×3 submatrices formed from the other three rows with permanent $2, -2, 6$ and -6 respectively (cf. proof of Proposition 5). Then we obtain $2\alpha-2\beta+6\gamma-6\delta=0$ and $\alpha+\beta+\gamma+\delta=\binom{n}{3}$. Since $n=4k+3$,

$$\binom{n}{3}=\frac{(4k+3)(4k+2)(4k+1)}{3\cdot 2\cdot 1}$$

is clearly an odd integer m. Adding $\alpha-\beta+3\gamma-3\delta=0$ to $\alpha+\beta+\gamma+\delta=m$ then yields $2\alpha+4\gamma-2\delta=m$, a contradiction.

Since for $n \geqq 4$ all previously known examples of A for which $\operatorname{per} A=0$ are singular, it is natural to ask whether there exist nonsingular $A \in \Omega_n$ with $\operatorname{per} A=0$. The next example originally due to L. Beasley shows that this is indeed possible. (The proof given here is different from that of L. Beasley.)

EXAMPLE 2. Let $A = (a_{ij}) \in \Omega_n$ be as follows:

$$A = \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 1 & 1 & -1 & \cdots & -1 \\ 1 & 1 & 1 & \cdots & -1 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & 1 & 1 & \cdots & -1 \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

where $a_{ij} = 1$ for all $i \geqq j$ and $a_{ij} = -1$ otherwise, where n is even. Then it is easy to see that $\det A = 2^{n-1}$. We claim that per $A = 0$. By negating the first column and all the rows of A, we obtain per $A = -$per B, where

$$B = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & -1 & 1 & \cdots & 1 \\ 1 & -1 & -1 & \cdots & 1 \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ 1 & -1 & -1 & \cdots & 1 \\ 1 & -1 & -1 & \cdots & -1 \end{bmatrix}$$

Since B can clearly be transformed to A by applying Operation I only, we have per $B =$ per A. Therefore per $A = -$per A whence per $A = 0$.

Our next result shows that if $A \in \Omega_n$ satisfies some prescribed properties, then a better bound for per A might be possible.

PROPOSITION 7. *If $A \in \Omega_n$ is normal such that $|\lambda_i| \leqq n^{1/2}$ for all eigenvalues λ_i of A then $|\operatorname{per} A| \leqq n^{n/2}$.*

PROOF. This is an immediate consequence of a result of Marcus and Minc [4, Th. 1] which states that for any normal matrix N, $|\operatorname{per} N| \leqq (1/n)\sum_{i=1}^{n}|\lambda_i|^n$.

COROLLARY 2. *If $H \in \Omega_n$ is a Hadamard matrix, then $|\operatorname{per} H| \leqq |\det H|$.*

PROOF. Since $HH' = nI$, where I denotes the $n \times n$ identity matrix, it is clear that H is normal and $|\lambda_i| = n^{1/2}$ for all eigenvalues λ_i of H. The result follows from Proposition 7 since it is well known that $|\det H| = n^{n/2}$.

REMARK 4. For Hadamard matrices, the bound given in Corollary 2 is, in general, very rough. It gives the values 1, 2, 16 and 4096 for $n = 1, 2, 4$ and 8 respectively. However, it is known (cf. [6, p. 409]) that for $n = 1, 2, 4, 8$, (and 12), there is only one equivalence class of Hadamard matrices and the values of $|\operatorname{per} H|$ can easily seem to be 1, 0 and 8 for $n = 1, 2$ and 4 respectively. For $n = 8$, we find, with the aid of computer, that $|\operatorname{per} H| = 384$.

3. Some problems

In addition to those problems mentioned in Remarks 1, 3 and 4, there are many others. We mention here just a few of them.

PROBLEM 1. For each $n \geqq 4$, can one always find nonsingular $A \in \Omega_n$ such that $|\operatorname{per} A| = |\det A|$?

PROBLEM 2. Is there a decent upper bound for $|\operatorname{per} A|$ when $A \in \Omega_n$ is nonsingular?

PROBLEM 3. Can per $H = 0$ for a Hadamard matrix H other than the one of order 2?

PROBLEM 4. Does $|\operatorname{per} H|$ distinguish among non-equivalent Hadamard matrices? (In view of the fact mentioned in Remark 4, the first step would be to evaluate the values of $|\operatorname{per} H|$ when $n = 16$ in which case there are 5 non-equivalent classes [6, p. 409].

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