CARDINAL REPRESENTATIVES

By

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ABSTRACT

In the absence of the axiom of choice it is sometimes, but not always, possible to define the notion of cardinal number such that for any x, $x \approx |x|$.

I. Introduction and remarks in *ZF*

A central achievement of axiomatic set theory is that it lays a proper foundation for the theory of cardinal numbers. If x is a well-orderable set. $|x|$ is defined as the least (Von Neumann) ordinal α such that $x \approx \alpha$. ($x \approx y$ if and only if there is a 1:1 function from x onto y.) This definition is inadequate for non well orderable x, which may exist if the axiom of choice AC is false. Scott [18], however, using the axiom of foundation, extended the definition of $|x|$ to all sets x. (We discuss here only the usual axiom of foundation or regularity [1]. A somewhat weaker version suffices for Scott's definition.) If x is not well orderable, $|x|$ is defined as the *set* of all y of *least possible rank* satisfying $x \approx y$. Lévy [10] and, independently, Gauntt [2] proved that no definition of $|x|$ is possible in *ZF* set theory minus the axiom of foundation. (*ZF* denotes, at our convenience, Zermelo Fraenkel or Gödel Bernay's set theory with foundation and without *AC*. *ZFE* denotes *ZF* with Gödel's axiom of strong choice.)

A rather basic property of $|x|$ for well orderable x is that $x \approx |x|$. This means that $|x|$ is a representative for the $\approx$-equivalence class of x. For non well orderable x there is clearly no reason to expect Scott's definition to behave this way. The present paper deals with the possibility of finding another definition of $|x|$ which satisfies $x \approx |x|$ on all, or at least a larger class of x. An equivalent formulation of the problem is that of finding a class R of sets such that for each $\mathcal{M} \in \mathbf{C}$ (the class of cardinal numbers as previously defined) there is a unique $x \in R$ satisfying $|x| = \mathcal{M}$. Such a class R will be called a class of *cardinal representatives*. If $\mathbf{D} \subset \mathbf{C}$ can similarly speak of a class of cardinal representatives *for* $\mathbf{D}$.

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In view of the form that Scott's definition takes it would be very surprising if a class of cardinal representatives can be proved to exist. And indeed our first theorem is the following.

THEOREM 1.1. *It is consistent with ZF set theory that no class of cardinal respresentatives exists.* (We of course assume the consistency of *ZF* throughout.)

Rather more surprising, and considerably more difficult to prove is the following theorem.

THEOREM 1.2. *It is consistent with $ZF + \sim AC$ that a class of cardinal representatives exists.*

Following our announcement of the above results, Azriel Levy posed to us the following question (which we slightly rephrase).

QUESTION. What is the largest $\mathbf{D} \subset \mathbf{C}$ which has a definable class of cardinal representatives?

Levy remarked that such a **D** should contain the cardinals of the ranks as well as those of the well orderable sets.

We conjecture that the answer to the question is the class **DR** of cardinals $\mathcal{M}$ such that $\mathcal{M} = |x|$ for some ordinal definable x (see [13]). **DR** has a definable class of representatives (represent $\mathcal{M}$ by the x of least definition satisfying $|x| = \mathcal{M}$). We propose that **DR** is widest in the following sense.

CONJECTURE. If R is a definable class which can be proved in *ZF* to represent a class $\mathbf{D} \subset \mathbf{C}$ then *ZF* proves $\mathbf{D} \subset \mathbf{DR}$.

In the absence of a proof of the conjecture we content ourselves with Theorem 3 and its corollaries below. The fact that **DR** can be unequal to **E** in Corollary 4 means that **DR** is more comprehensive than a natural closure of the well ordered cardinals under +, x, and power set. Corollary 1.5 shows that the class **DC** of ordinal definable cardinal numbers is, in some sense, too big to answer the question. These results indicate, at least to us, that the answer should be **DR**. We hope that the remarks preceeding the statement of Theorem 3 have some independent interest.

In the following definitions we completely ignore the finite cardinals. The reader can modify the definitions to include them if he wishes. If α and β are ordinal numbers, $R_\beta(\alpha)$ is defined inductively via: $R_0(\alpha) = \alpha$, $R_{\gamma+1}(\alpha) = \mathcal{P}(R_\gamma(\alpha))$, and $R_\beta(\alpha) = \bigcup_{\gamma<\beta} R_\gamma(\alpha)$ for limit β ($\mathcal{P}(x)$ denotes the power set of x). We define the following class of **C**.

E is the smallest class of cardinals containing all the $|R_\beta(\alpha)|$ and closed under $+$, $\times$ and exponentiation

Idp is the class of cardinals $\mathcal{M}$ satisfying $\mathcal{M} \times \mathcal{M} = \mathcal{M}$.

Idm is the class of cardinals $\mathcal{M}$ satisfying $\mathcal{M} + \mathcal{M} = \mathcal{M}$.

Df is the class of cardinals $\mathcal{M}$ satisfying $\aleph_0 \not\leq \mathcal{M}$.

REMARK A. $\mathrm{Idp} \subset \mathrm{Idm} \subset \mathbf{C} - \mathrm{Df} \subset \mathbf{C}$ and no equalities are provable in ZF ([5] and [7]).

REMARK B. $|R_\beta(\alpha)| \in \mathrm{Idp}$.

PROOF. The following fact explicity proves the result for successor β in such a way that the inductively generated maps patch for limit β.

FACT. If X is transitive and $\Phi: X \times X \to X$ is $1:1$ and onto then there is a $1:1$ onto extension $\Phi^*, \Phi^*: \mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$ such that Φ^* is explicitly definable from Φ.

In proving the fact we let $y \approx^e z$ mean that there is a $\psi: y \to z$ which is 1:1, onto, and explicitly definable from Φ. The fact will be clear once it is shown that

$$\mathcal{P}(X) - X \approx^e (\mathcal{P}(X) \times \mathcal{P}(X)) - (X \times X).$$

We list the following easy consequences of the laws of exponents, the Cantor-Bernstein theorem, and the fact that $2 \subset X$ (X is transitive):

$$X \approx^e 2 \times X \approx^e 1 \times X$$
$$\mathcal{P}(X) \approx^e \mathcal{P}(2 \times X) \approx^e \mathcal{P}(X) \times \mathcal{P}(X) \approx^e X \times \mathcal{P}(X) \approx^e \mathcal{P}(X) \times X$$
$$\mathcal{P}(X) \approx^e \mathcal{P}(1 + X) \approx^e 2 \times \mathcal{P}(X) \approx^e 4 \times \mathcal{P}(X) \approx^e 3 \times \mathcal{P}(X).$$

The one slightly sticky point is to show

$$\mathcal{P}(X) \approx^e \mathcal{P}(X) - X.$$

To see this let ψ be a $1:1$, onto, Φ-definable function $\psi: \mathcal{P}(X) \to \mathcal{P}(X) \times \mathcal{P}(X)$. Let ρ be the y-axis projection ρ: $\mathcal{P}(X) \times \mathcal{P}(X) \to \mathcal{P}(X)$. $(\rho \circ \psi) \restriction X$ is a Φ-definable map from X to $\mathcal{P}(X)$. Cantor's diagonal argument gives a Φ-definable $B \in \mathcal{P}(X)$ - Range $(\rho \circ \psi) \restriction X$. The map $z: \mathcal{P}(X) \to \mathcal{P}(X) - X$ defined by

$$z(A) = \psi^{-1}(A, B)$$

shows via the Cantor-Bernstein theorem that $\mathcal{P}(X) \approx^e \mathcal{P}(X) - X$.

The equivalences now in hand give as desired:

$$\mathcal{P}(X) \times \mathcal{P}(X) - X \times X = (\mathcal{P}(X) - X) \times X \cup X \times (\mathcal{P}(X) - X) \cup (\mathcal{P}(X) - X) \times (\mathcal{P}(X) - X)$$

$$\approx^e 3\mathcal{P}(X) \approx^e \mathcal{P}(X) \approx^e \mathcal{P}(X) - X.$$

REMARK C. $E \subset \mathrm{Idm}$. (Immediate from Remark B.)

REMARK D. $E \subset \mathrm{Idp}$ is an equivalent to AC. (This formally proves Tarski's "$\mathbf{C} \subset \mathrm{Idp}'$ is an equivalent to AC" [21].)

PROOF. In the presence of regularity (without which the remark is false) Tarski's argument derives AC from

$$(\forall \alpha)(\forall \beta)[|R_\alpha(\phi)| + |\beta| \in \mathbf{Idp}]$$

(α, β, γ, are understood to range over ordinals).

THEOREM 3. *It is consistent with ZF that there is an $\mathcal{M} \in \mathbf{DR} \cap Df$ such that $\{\mathcal{M} - n\}_{n \in \omega}$ has no class of cardinal representatives.*

Theorem 3 clearly implies Theorem 1 as well as the following two corollaries.

COROLLARY 4. $\mathbf{E} \subset \mathbf{DR} \subset \mathbf{DC} \subset \mathbf{C}$ *and no equalities are provable in ZF.*

PROOF. $\mathbf{E} = \mathbf{DR}$ is not provable since $\mathbf{E} \subset \mathrm{Idm}$ while $\mathbf{DR} \cap \mathrm{Df} \neq \varnothing$ in the model of Theorem 3. $\mathbf{DR} = \mathbf{DC}$ is not provable since in the model of Theorem 1.3$\{\mathcal{M} - n\}_{n \in \omega} \subset \mathbf{DC}$ but $\{\mathcal{M} - n\}_{n \in \omega} \not\subset \mathbf{DR}$. $\mathbf{DR} = \mathbf{DC}$ is not provable since there are models [1] in which $\mathrm{Df} \neq \varnothing$ and 2^ω cannot be will ordered. A theorem of Tarski [21] shows that Df, hence $\mathbf{C}$, has a subset of cardinal 2^ω. Therefore in such a model $\mathbf{C}$ cannot be well ordered while $\mathbf{DC}$ can.

COROLLARY 5. $\mathbf{DC}$ *cannot be proved to have a class of representatives.*

PROOF. In the model of Theorem 3 $\{\mathcal{M} - n\}_{n \in \omega} \subset \mathbf{DC}$.

A final remark concerning cardinal representatives is that without AC one cannot have a canonical similarity from x to $|x|$. In fact, we can say the following.

REMARK E. AC is equivalent to "For every family of pairs of similar sets there is a function associating each pair with a similarity between its terms".

PROOF. An element of X is explicitly definable from any similarity between $X \times \omega$ and $(X \times \omega) \cup \{\varnothing\}$.

The remainder of the paper is organized as follows. Theorem 3, hence also Theorem 1, is proved in Section II. Theorem 2 is proved in Section III which also obtains some bonus results which are discussed below.

In [9] Levy attempted unsucessfully to give a Fraenkel-Mostowski consistency proof for the following statement Λ: "The axiom of choice for families of finite sets holds while the axiom of choice for families of countable sets fails." Howard [4] (also see [6]) found a mistake in Levy's work. He in fact proved that a Fraenkel-Mostowski consistency proof for Λ is *impossible*. From (3.17) and (3.19) follows Theorem 6.

THEOREM 6. *Λ is consistent with ZF.*

Λ is thus an example of an abstract (that is, not concerned with sets of $\cdots$ of sets of ordinals) *ZF* independence which is not a Fraenkel-Mostowski independence. This complements a number of examples of Fraenkel-Mostowski independences which are not *ZF* independences ([6] and [15]).

Sageev [19] recently settled a long standing question by proving *AC* independent of $\mathbf{C} = \mathrm{Idm}$. (This result has been independently achieved in a Fraenkel-Mostowski model by Halpern and Howard.) He actually proved *AC* independent of $\mathrm{Order} + Z(\aleph_0) + \mathbf{C} = \mathrm{Idm}$ where Order is the linear ordering principle and $Z(\aleph_0)$ is: "There is a function which selects a (finite or) countable subset from every set." He raises two questions, and indicates that the second is probably false.

$$Z(\aleph_0) \rightarrow \mathrm{Order}?$$

$$\mathrm{Order} + Z(\aleph_0) \rightarrow \mathbf{C} = \mathrm{Idm}?$$

We have proven both of these false. The independence of Order from $Z(\aleph_0)$ will be included in [17]. The other is proved in (3.23) and is stated as follows

THEOREM 7. $\mathbf{C} = \mathrm{Idm}$ *is independent, in ZF, of* $\mathrm{Order} + Z(\aleph_0)$.

We close this introductory section with the following problem.

PROBLEM. Prove the independence in *ZF* of *AC* from "$\mathbf{C} = \mathbf{E}$".

In view of the above discussion a solution to the problem would immediately give both Sageev's theorem and our Theorem 1.2. (Segeev's model [19] can also be used for Theorem 6.)

II. Proof of Theorem 3

The proof of Theorem 3 is partly based on our original proof of Theorem 1, which appears in [6]. We sketch below a somewhat simpler proof of Theorem 1 which we recently noticed.

It suffices by [15, II B 2] to give a Fraenkel-Mostowski model for the nonexistence of a class of cardinal representatives. The arguments of this section will show that the ordered model of [12] suffices. It is even easier to consider the unordered model of [12] (also see [7]). In this model a very simple proof is available for the analogue of our Lemma 2.12.

In view of the above paragraph it is natural to wonder why we base our present model on Mostowski's ordered model rather than his unordered model. The problem is that in order to create a bad *OD* (ordinal definable) set one apparently has to introduce pseudoindividuals in $\mathcal{P}\mathcal{P}\mathcal{P}(\omega)$ rather than in $\mathcal{P}\mathcal{P}(\omega)$ (as is more usual ([5], [6], and [14]). Things work particularly well if each pseudoindividual resembles the set of individuals in Mostowski's unordered while the set of pseudoindividuals resembles the set of individuals in Mostowski's ordered model. If, however, the pseudoindividuals resemble the individuals of Mostowski's unordered model, Lemma 2.10 is false and considerable complexity is introduced.

It should not be inferred that Theorem 3 could not be proved if Mostowski's unordered model were used. The obstacles can be circumvented and doing so has the following interesting by-product.

THEOREM 2.1. *Let* $\Phi(x)$ *be a boundable formula* (see [5]). *If* $(\exists x \in OD)\Phi(x)$ *has a Fraenkel-Mostowski model then it is consistent with ZF.*

This detour is not taken since in order to prove Theorem 1.3 one would have to generalize the above theorem to surjectivity boundable formulae (see [15]). We do not know how to do this.

2.2. *Constants.* We follow here the set-up of Cohen [1] with a few refinements from Shoenfield [20]. The model is built from three basic kinds of constant symbols.

(1) Ground model constants. These are constants which denote the members of a fixed ground model M of *ZFE.*

(2) Undefined (new) constants.

(3) Abstraction constants.

Constants of types 2 and 3 are discussed in [1]. Those of type (3) are discussed in [20]. The constants are arranged in a hierarchy which is completely determined once one is told where the new constants fit. In the resulting model each element is named by several constants. The element and its constant are usually identified. In ambiguous situations a constant for a is denoted a. The new constants here are as follows.

(a) There is a single new constant $<$ at level $\omega + 6$.

(c) There is a set of new constants $\{Z_q\}_{q \in Q}$ at level $\omega + 3$. The index set Q is the set of rational numbers.

(d) For each Z_q there is a countable set $\{a_q, b_q, \cdots\}$ of new constants at level

$\omega+2$. The sets are disjoint for distinct q.

(e) For each new constant a at level $\omega+2$ there is a countable set $\{r_a, s_a, \cdots\}$ of new constants at level $\omega+1$. The sets are disjoint for distinct a.

2.3. *Conditions.* An *interval* in $\mathcal{P}(\omega)$ is an $E \subset \mathcal{P}(\omega)$ which has description of the form:

$$E = \{r \in \mathcal{P}(\omega): n_1 \in r \wedge \cdots \wedge n_p \in r \wedge m_1 \notin r \wedge \cdots \wedge m_k \notin r\}.$$

The intervals form the basis for the usual topology on $\mathcal{P}(\omega)$ (identified with 2^ω). The statement "$r \in E$" has an equivalent formulation involving r and finitely many parameters from ω.

A *condition* is a finite set of statements having the form:

$$P = \{r_1 \in E_1, \cdots, r_k \in E_k\}$$

where the r_i are distinct new constants at level $\omega+1$ and the E_i are intervals. If

$$Q = \{s_1 \in F_1, \cdots, S_h \in F_h\}$$

is another condition then $P \leq Q$ iff $\{r_i\}_{i=1}^k \subset \{s_j\}_{j=1}^h$ and $r_i = s_j \rightarrow F_j \subset E_i$.

2.4. *Forcing.* As in [1] it is only required to state when a constant is strongly forced (the strong forcing relation is denoted $\vdash^*$) to be a member of a new constant.

(a) $P \vdash^*$ "$c \in <$" iff $P \vdash^*$ "$c = (Z_p, Z_q)$" and $p < q$ in Q.

(b) $P \vdash^*$ "$c \in I$" iff $P \vdash^*$ "$c = Z_p$" for some $p \in Q$.

(c) $P \vdash^*$ "$c \in Z_p$" iff $P \vdash^*$ "$c = a$" for some a in the set of new constants at level $\omega+2$ which is associated with p.

(d) $P \vdash^*$ "$c \in a$" iff $P \vdash^*$ "$c = r$" for an r in the set of constants at level $\omega+1$ which is associated with a.

(e) $P \vdash^*$ "$c \in r$" iff $P \vdash^*$ "$c = n$" for some $n \in \omega$ and there is some interval E with $n \in \cap E$ and "$r \in E$" $\in P$.

2.5. *The model.* As in [1] the above set-up produces a model V of ZF containing M as a transitive proper subclass. The following facts are easily verified and can be stated in V.

(a) I and $\bigcup I$ are disjoint sets of infinite sets. $\bigcup \bigcup I \subset \mathcal{P}(\omega) <$ linearly orders I densely without endpoints.

If $x \in V$ and $G \subset I \cup \bigcup I \cup \bigcup \bigcup I$ is finite we say $x \in \nabla G$ if x is definable (in V) from parameters in $M \cup \{I, <\} \cup G$. If H well orders G and $\alpha \in \mathrm{On}$ (the class of ordinals) $T(H,\alpha)$ denotes the αth element in the "least definition" well ordering of ∇G induced by H and a fixed well ordering of

$M \cup \{I, <\}$. We list the following standard properties of ∇ and T.

(b) The replacement schema extends to formulae involving ∇ and T.

(c) $(\forall x \in V)\ (\exists G)\ [x \in \nabla G]$.

(d) ∇G contains all ordinals and well orderings of G.

(e) If x is definable (using $\in$ I, ∇, and T) from parameters in ∇G, then $x \in \nabla G$.

(f) If H well orders G then $\Phi(\alpha) = T(H, \alpha)$ is a $1:1$ onto function from On to ∇G.

We say that G is *full* if for any $g \in I \cup \bigcup I$, $g \cap G \neq \emptyset \rightarrow g \in G$. The disjointness of $I \cup \bigcup I$ guarantees that every G is contained in a unique smallest (finite) full G^*. A well ordering H of G defines canonically a well ordering H^* of G^* so $\nabla G = \nabla G^*$.

2.6. *Density and continuity.* The following statements describe in V the relation between truth, in V, and forcing. Such a description was first given in [3]. Proofs are omitted here. Some detail is worked out in a similar but more difficult situation in (3.9) - (3.13),

(a) *Density.* Every $a \in \bigcup I$ is dense in $\mathcal{P}(\omega)$, that is, $a \cap E \neq \emptyset$ for any interval E.

(b) *Continuity* (Schema). Let $\Phi(x_1, \cdots, x_n, y_1, \cdots, y_m, z_1, \cdots, z_k)$ have parameters in $M \cup \{I, <\}$ (and exactly the x's, y's, and z's free). Suppose Φ is true at distinct $r_1, \cdots, r_n \subset \bigcup \bigcup I$, $a_1, \cdots, a_m \in \bigcup I$, and $Z_1, \cdots, Z_k \in I$. There are disjoint intervals $E_1, \cdots, E_n$ with $r_i \in E_i$ which satisfy in addition the following. For any full G, G' and $\varphi : G \rightarrow G'$ such that:

(1) $r_1, \cdots, r_n$, $a_1, \cdots, a_m$, $Z_1, \cdots, Z_k \subset G$;

(2) φ is an $\in$-isomorphism;

(3) $\varphi \restriction I$ is a $<$-isomorphism;

(4) $\varphi(r_i) \in E_i$;

$\Phi(\varphi(r_1), \cdots, \varphi(r_n), \varphi(a_1), \cdots, \varphi(a_m), \varphi(Z_1), \cdots, \varphi(Z_k))$ holds.

SUPPORT LEMMA 2.7. (a) *If G_1 and G_2 are full then* $\nabla G_1 \cap \nabla G_2 = \nabla G_1 \cap G_2$.

(b) *If $x \in V$ there is a unique smallest full G (denoted G_x) with $x \in G$.*

Part (b) follows from (a) and the finiteness of the G's. Part (a) is proved by a now standard process of combining the arguments of Mostowski [11](for both the ordered and unordered models) with those of Halpern and Levy [3]. We offer as references also [15, Prop. 3.5] and [6] (where this process is called the support lemma). Somewhat more detail is given for an analogus lemma in Section III (Lemma 3.16).

We say $X \approx^* Y$ if there are cofinite $X' \subset X$, $Y' \subset Y$ such that $X' \approx Y'$. $\approx^*$ is an equivalence relation. The set A is said to be *amorphous* if A is infinite but is not the disjoint union of two infinite subsets.

LEMMA 2.8. (a) *Every subset of I is a finite union of $<$-open intervals (possibly using $\pm\infty$ as endpoints) and singletons.*

(b) *Every $Z \in I$ is amorphous.*

(c) *If Z_1, $Z_2 \in I$ then either $Z_1 = Z_2$ or $Z_1 \not\approx^* Z_2$.*

PROOF. The three assertions of the lemma mirror standard facts about Mostowski's ordered and unordered models. Since they are somewhat less difficult to prove than Lemma 2.7 a we include their proofs as an illustration of how the continuity schema can be used to mimic permutation arguments.

If the lemma is false the offending subset A or function f is in ∇G for some fixed full G. The negations of the desired conclusions imply:

(a) there is a minimal $<$-interval $K \subset I$ with endpoints in $(G \cap I) \cup \{\pm\infty\}$ and there is a $Z_1 \in K \cap A$ and a $Z_2 \in K - A$;

(b) some $a \in Z - G$ is in A;

(c) $f(a) = b$ for some $a \in Z_1 - G$, $b \in Z_2 - G$.

One now finds a $G_1 \supset G$ such that the Z_1 and a which exist above are in G'. Then one finds a $G_2 \supset G$ and $\varphi : G_1 \to G_2$ which satisfies the criteria of continuity such that φ takes:

(a) Z_1 to Z_2;

(b) a to some $b \in Z - (A \cup G)$;

(c) b to some b' also in $Z_2 - G$, a to a.

By continuity, one concludes the contradictions: (a) $Z_2 \in (K \cap A) \cap (K - A)$;

(b) $b \in A \cap (Z - A)$;

(c) $b = f(\mathrm{a}) = b'$.

LEMMA 2.9. *If G is full and $G' \in \nabla G$ (G' is a finite subset of $I \cup \bigcup I \cup \bigcup \bigcup I$), then $G' \subset G$.*

PROOF. The argument breaks into three cases.

Case 1. There is an $r \in (G' \cap \bigcup \bigcup I) - G$.

Let $r \in a \in \bigcup I$. By continuity, using a statement listing r, a, and the members of G, G' contains $a \cap E$ some interval E. This, and the finiteness of G' contradicts density (2.6a).

Case 2. $G' - G \cap \bigcup \bigcup I = \varnothing$ but there exists $a \in (G' - G) \cap \bigcup I$.

Let $a \in Z \in I$. As in (2.8), continuity shows that $Z - G \subset G'$, contradicting the finiteness of G'.

Case 3. $(G' - G) \subset I$.

Suppose $Z \in G' - G$. Let K be the minimal $< -$ interval containing Z and with endpoints from $G \cup \{\pm\infty\}$. It follows by continuity that $K \subset G'$, again contradicting the finiteness of G'.

LEMMA 2.10. *If X is amorphous there is a unique $Z \in I$ such that $X \approx *Z$.*

PROOF. This argument is a slight refinement of the arguments in (2.9). The uniqueness is clear by (2.8c). Then given X is in ∇G for some fixed full G. $X \not\subset \nabla G$ since X would otherwise be well orderable and not amorphous. Therefore there is a $w \in X$ such that $G_w - G \neq \emptyset$.

Case 1. There is an $r \in G_w - G \cap \bigcup \bigcup I$.

We prove this case to be contradictory. Fix a well ordering H of $(G \cup G_w) - \{r\}$. Denote concatenation of orderings by $^\wedge$. For suitable α and β:

$$X = T(H \restriction G, \alpha)$$

$$w = T(H^\wedge\{r\}, \beta)$$

and the sentence

$$(*) \qquad T(H^\wedge\{r\}, \beta) \in T(H \restriction G, \alpha) \wedge r \in G_{T(H^\wedge\{r\},\beta)}$$

holds. By continuity this sentence continues to hold with r replaced by an arbitrary $r' \in a \cap E$ where $r \in a \in \bigcup I$ and E is a suitable interval avoiding $(G \cup G_w) - \{r\}$. If $r' \neq r''$ and

$$T(H^\wedge\{r'\}, \beta) = T(H^\wedge\{r''\}, \beta)$$

the support lemma shows

$$G_{T(H^\wedge\{r'\},\beta)} \subset (G \cup G_w) - \{r\} \text{ and } r' \notin G_{T(H^\wedge\{r'\},\beta)}.$$

This contradicts the fact that (*) holds at r'. Therefore by density if E' and E'' are disjoint subintervals of E then

$$X'' = \{T(H^\wedge\{r'\}, \beta) : r' \in a \cap E'\}$$

$$X'' = \{T(H^\wedge\{r''\}, \beta) : r'' \in a \cap E''\}$$

are infinite disjoint subintervals of X, and I as not amorphous.

Case 2. $(G_w - G) \cap \bigcup \bigcup I = \emptyset$ but there is an $a \in (G_w - G) \cap \bigcup I$.

Follow the argument for Case 1 replacing r by a. Produce the analogous sentence (*) which holds whenever a is replaced by $a' \in Z - (G_w \cup G)$, where $a \in Z \in I$. Also argue as in Case 1 that if $a' \neq a''$ then $T(H^\wedge\{a'\},\beta) \neq T(H^\wedge\{a''\},\beta)$. The map $\Phi: Z - (G_w \cup G)$ defined by

$$\Phi(a') = T(H^\wedge\{a'\},\beta)$$

establishes that $Z \approx^* Y$ for an infinite $Y \subset X$. Since X is amorphous $Y \approx^* X$ and $Z \approx^* X$ as desired.

Case 3. $G_w - G \subset I$.

This case will be proved contradictory. Again follow the $<$ argument of Case 1 this time replacing $r \in \bigcup \bigcup I$ by $Z \in I$. One obtains, by the arguments of Cases 1 and 2, a subset $Y \subset X$ which is similar to an open$<$ − interval of I. Since $<$ densely orders I, Y can be split into infinite disjoint subsets and X is not amorphous.

LEMMA 2.11. $|I| \in \mathbf{DR} \cap \mathrm{Df}$.

PROOF. It is obvious from (2.8a) that $|I| \in Df$ (see [7]). To show $|I| \in \mathbf{DR}$ we must find an OD set similar to I. Such a set is the set of $\approx^*$-equivalence classes of amorphous subsets of $\mathcal{P}\mathcal{P}(\omega)$. The results of (2.8b) and 2.10) prove that the map $\Phi(Z) = [Z]_{\approx^*}$ is 1 : 1 and onto this set. Our definition of the set has no parameters whatever (since ω can be defined).

The key fact in proving that $\{|I| - n\}_{n \in \omega}$ has no set of cardinal representatives is the following.

LEMMA 2.12. *If* $|X| = |I| - n$ *then* $|G_X \cap I| \geqq n$.

PROOF. Fix a well ordering H of G_X. We will use H, and other parameters in G_X, to define a finite subset of I which contains at least n elements. This proves the lemma by (2.9).

Let

$$U = \{y \in V: \quad G_y - G_X \subset I\}.$$

SUBLEMMA 2.13. $X \subset U$.

PROOF. Assume there is a $w \in X$ with $G_w - G_X \neq \emptyset$. The argument breaks down into the same three cases as the proof of (2.10). Without going into detail we state what the arguments conclude in each case.

Case 1. $(G_w - G_X) \cap \bigcup \bigcup I \neq \emptyset$.

There is a mapping of X onto an infinite disjoint collection of subintervals of some interval E. This contradicts (2.8a).

Case 2. $(G_w - G_X) \cap \bigcup \bigcup I = \emptyset$ but $(G_w - G_X) \cap \bigcup I \neq \emptyset$.

In this case there is an amorphous subset of X, contradicting the fact that X can be linearly ordered (since I can). This is argued in [7].

Case 3. $G_w - G_X \subset I$.

This is as desired.

We resume the proof of Lemma (2.12). For $y \in U$, let H_y be the ordering of G_y defined (with the parameter H) by:

$$H_y \restriction G_y \cap (\bigcup I \cup \bigcup \bigcup I) = H \restriction G_y \cap (\bigcup I \cup \bigcup \bigcup I),$$

$$H_y \restriction G_y \cap I = \ < \restriction G_y \cap I.$$

Let α_y be the solution of $T(H_y, \alpha) = y$. Using only the parameter H we extend $<$ to all of U. In the following expression all parentheses are assumed to be on the right.

$$\begin{aligned} y < z \text{ iff } \quad & \alpha_y \leqq \alpha_z \wedge \alpha_y = \alpha_z \rightarrow \\ & |G_y - I| \leqq |G_z - I| \wedge |G_y - I| = |G_z - I| \rightarrow \\ & G_y - I \leqq_{H^*} G_z - I \wedge G_y - I = G_z - I \rightarrow \\ & |G_y \cap I| \leqq |G_z \cap I| \wedge |G_y \cap I| = |G_z \cap I| \rightarrow \\ & G_y \cap I <^* G_z \cap I. \end{aligned}$$

Above $\leqq_{H^*}$ and $<^*$ denote the first difference orderings induced by H and $<$ respectively. We remark that $<$ is rigged so that all intervals are sets. The subsequent use of *open* or *connected* refers to the order $<$ on $Y \subset U$.

SUBLEMMA 2.14. *If Φ is a 1:1 function from I onto $Y \subset U$ there are disjoint $<$-open intervals $J_1, \cdots, J_k$ of I such that:*

(a) *$I - (J_1 \cdots \cup J_k)$ is finite;*

(b) *$\Phi \restriction J_i$ is q $<$-preserving map whose range is an open connected subset of Y.*

PROOF. The intervals $J_1, \cdots, J_k$ are the minimal intervals with endpoints in $(G_\Phi \cap I) \cup \{\pm\infty\}$. If $Z \in J_i$ then $\Phi(Z) \in \nabla(G_\Phi \cup \{Z\})$ so $G_{\Phi(Z)} \subset G_\Phi \cup \{Z\}$. It is a typical application of continuity to show that for any $Z' \in J_i$, $\alpha_{Z'} = \alpha_Z$ and $G_{\Phi(Z')} = (G_{\Phi(Z)} - \{Z\}) \cup \{Z'\}$. (Note: the alternative is that Φ maps J_i into the well orderable $\nabla G_Z - \{Z\}$.) A study of the definition of $<$ on Y will show that $\Phi \restriction J_i$ is order preserving with open range J_i is clearly connected, in fact order complete, from (2.8a). Since the range of ΦJ_i is open and relatively connected it is connected.

SUBLEMMA 2.15. (a) *If $J \subset I$ is an open interval and Φ: $J \to I$ is order preserving then Φ is the identity.*

(b) *If $A \subset I$ and Φ: $A \to I$ is* 1 : 1 *then it is the identity except on finitely many points.*

PROOF. A continuity argument modeled on the proof of (2.8c) proves that Φ is the identity except on $G_\Phi \cup \Phi^{-1}[G_\Phi]$. This gives (b) immediately and (a) follows shortly thereafter.

SUBLEMMA 2.16. *Suppose $Y \subset U$ and $|Y| = |I|$. Y is* uniquely *expressed as the disjoint union of the least possible number of singletons and open connected subsets of Y which are isomorphic to open intervals in I.*

PROOF. Sublemma 2.14 guarantees that Y is a disjoint union of a finite number of such singletons and open sets. Let Γ_1 and Γ_2 be two such expressions. Let J be the leftmost interval of I which is isomorphic, under Φ, to an $R \in (\Gamma_1 \cup \Gamma_2) - (\Gamma_1 \cap \Gamma_2)$. Without loss of generality, say $R \in \Gamma_1$. We will show that Γ_1 does not involve the least possible number of open intervals.

By the choice of J there is an interval K with the same left endpoint as J and isomorphic under ψ to an $S \in \Gamma_2$. We claim that ψ and Φ agree on their common domain. Φ and ψ cannot differ on only finitely many points since then at a bad $Z \in J \cap K$ the Y–open interval $(\psi(Z), \Phi(Z))$ would be empty and Range ψ would be disconnected. On the other hand ψ and Φ can't differ on infinitely many points. The property of (2.15b) applies to Y because $Y \approx I$. Applying this to $\psi\Phi^{-1}$ or $\Phi\psi^{-1}$ gives that both are the identity except on finitely many points.

Since Φ and ψ agree on their common domain and J is the leftmost interval involved, it follows that J is a proper subinterval of K and, in particular, the right endpoint Z of J is in K.

Let L be the interval with left endpoint Z which is isomorphic, under ρ, to a $Q \in \Gamma_1$. L exists since distinct open sets of Γ_1 are isomorphic to disjoint subintervals of I. (If the subintervals intersect they do so in an interval. The argument of the last paragraph shows that the corresponding components of Γ_1 intersect.) Hence only finitely many points of I do ot occur in one such interval.

Arguing again from (2.15b), ρ and ψ agree on their common domain. Therefore $\rho \cup \psi \cup \Phi$ is a 1:1 onto $<$–preserving function $K \cup J \cup L$ to $Q \cup R \cup S$. $Q \cup R \cup S$ is open and connected from topological arguments hence so is $Q \cup \{\psi(Z)\} \cup R$. This is $<$–isomorphic to $J \cup \{Z\} \cup L$ so $Q \cup$

$\{\psi(Z)\} \cup R$ can replace both Q and R in Γ_1, thus reducing the number of intervals in Γ_1 by one.

We now return for the last time to the proof of Lemma 2.12. For the given X with $|X| = |I| - n$ let $Y = X \cup \{k_1, \cdots, k_n\}$ where the k_i are the first n integers not in $X(|X| \in \mathrm{Df}$ so $X \cap \omega$ is finite.). Let Γ be the decomposition of Y given by (2.16). From its definition, $\Gamma \in \nabla G_X$. Let ψ be the union of the isomorphisms defined from the open sets of Γ to intervals of I. Sublemma 2.15a quickly shows that ψ is uniquely defined and $1:1$. $\psi \in \nabla G_X$ hence so are $\psi[\{k_1, \cdots, k_n\} \cap \text{Domain } \psi]$, I–Range ψ, and

$$G = \psi[\{k_1, \cdots, k_n\} \cap \text{Domain } \psi] \cup (I\text{–Range } \psi).$$

Since $|I| \in \mathrm{Df}$ finite subtraction is well defined so $|I - \text{Range } \psi| = |Y - \text{Domain } \psi|$. Therefore

$$|G| = |\{k_1, \cdots, k_n\} \cap \text{Domain } \psi| + |Y\text{-Domain } \psi|$$

$$= |\{k_1, \cdots, k_n\} \cup (Y\text{-Domain } \psi)| \geqq |\{k_1, \cdots, k_n\}| = n.$$

G is the promised H–definable finite subset of I with $|G| \geqq n$. By (2.9), $|G_X| \geqq n$ and the lemma is proved.

2.17. Proof of Theorem 3. From Lemma 2.11, $|I| \in \boldsymbol{DR} \cap Df$. It remains to show that $\{|I| - n\}_{n \in \omega}$ has no set of representatives. Assume that it did, say R. Each $X \in R$ is in ∇G_R since it can be defined from R as "that member of R with cardinal $|I| - m$" for some m. Therefore $|G_X| \leqq |G_R|$. On the other hand, for $n > |G_R|$, if X has cardinal $|I| - n$ then, by Lemma 2.12, $|G_X| \geqq n > |G_R|$. This is a contradiction.

III. Proofs of Theorems 2, 6, and 7

In this section we employ a model of a new type. As the discussion of Theorem 6 indicates this model is quite unlike a Fraenkel-Mostowski model. It is a special case of the models we described in [16]. Sageev informs us that he has independently considered similar models. (The conjecture, in [19], of Theorem 7 was based on such considerations.) We describe the model here in what we believe to be sufficient detail for the interested reader. Some exhaustive checking, particularly in Lemmas 3.7 and 3.10 is omitted. A more general discussion of these models [17] will contain some of this checking as well as a proof of the Boolean prime ideal theorem in this and similar models. Here we content ourselves with a proof of the ordering theorem, as needed for Theorem 7. (It is a theorem of Mathias (see [11]) that the prime ideal theorem implies the ordering principle and not conversely.)

Theorem 2 does have an easy proof in the weak set theory ZF with atoms.

Levy [8] gives a model which resembles Mostowski's unordered model but has finite and countable supports. One can show that if G is a countable support then ∇G contains a set of every cardinal number. Since ∇G can be well ordered there is a class of cardinal representatives. We are aware, however, of no model of ZF in which a similar argument can be carried out. Indeed we are unable to prove Theorem 2 using any of the more classical models of ZF.

The present model is best intuitively described as an ω-iteration of the Cohen-Halpern-Levy model ([1, p. 136] and [3]). It is constructed from a set

$$I = \bigcup_{n \in \omega} I_n$$

where $I_{-1} = 2$ and I_{n+1} is an independent set of generic onto functions from ω to I_n. In this context the Cohen-Halpern-Levy model is the submodel based on I_0 and each I_n is a set of independent ways to well order the entire model based on $\bigcup_{i<n} I_i$. Models of this iterated sort were first introduced in [22] and it was shown there how the model could be completely described in the ground model. In the present case such a description is so simple that we can forget about the apparatus of iteration and describe the model in the style of Cohen [1].

3.1. *Constants.* As in (2.2) we need only specify the new constants. These are:

(a) countably many function symbols of order 0 at level $\omega + 3$ in the hierarchy;

(b) countably many function symbols of order $n + 1 \in \omega$ at the first level beyond that containing all (the usual canonical) symbols of the form (m, f) where $m \in \omega$ and f is a function symbol of order n;

(c) the symbol I at the first level beyond that containing all new function symbols.

In the sequel the variables $f, g, \cdots$ will range over new function symbols.

3.2. *Forcing terms.* In the model the function symbols of order n will become the functions of I_n. For this one would want forcing conditions to be finite sets of statements of the form $fp = g$ where f has order $n + 1$, $p \in \omega$, and g has order n (2 is the set of function symbols of order -1). To carry out standard arguments one needs, in addition, that if the condition P forces a statement involving $f_1, \cdots, f_n$ and I then so does the subcondition of P involving only the $f_1, \cdots, f_n$. A moment's consideration of the condition

$$P = \{fp = g,\ fq = h\}$$

and the statement "$f(p) \neq f(q)$" reveals that this is not true of the conditions described above.

The arguments can be rescued by demanding that a condition be closed under the induced equalities and inequalities. In the example considered above the inequality $fp \neq fq$ is induced (if $g \neq h$) so the condition will include this inequality and the restricted condition will force $fp \neq fq$. The problem now is to codify what equalities and inequalities are induced and prove that none have been overlooked. This is the reason for the following definitions of forcing term, subterm, and substitution.

(a) A *forcing term* is a sequence of the form $t = fp_1 \cdots p_m$, $m \geqq 0$, where f is a new function symbol of order $n \geqq m - 1$. The *order* of t is $n - m$. 0 and 1 are included as terms (but not function symbols) of order -1.

(b) t_1 is a *subterm* of $t_2(t_1 \leqq t_2)$ if t_1 is a nonvoid initial subsequence of t_2.

(c) If t_1 and t_2 are terms of order n and t is a term, the *substitution*

$$\mathrm{sub}(t_1, t_2, t) = \begin{cases} t \text{ unless } t_1 \leqq t \\ \overline{\qquad\qquad\qquad} \\ t_2 p_r \cdots p_m \text{ if } t = t_1 p_r \cdots p_m. \end{cases}$$

3.3. *Conditions.* A *condition* is a finite set of equalities and inequalities between terms which has the consistency and closure properties listed below. For convenience we use $\ominus$ and $\not\ominus$ within a condition so as not to confuse this situation with the more normal use of $=$ or $\neq$ elsewhere, $\odot$ is a variable which can stand for $\ominus$ or $\not\ominus$. A condition P satisfies:

(a) If $t_1 \odot t_2 \in P$ then t_1 and t_2 have the same order. t_1 *occurs* in P if some $t_1 \odot t_2 \in P$.

(b) $t_1 \ominus t_2 \in P \rightarrow t_1 \not\ominus t_2 \notin P$.

(c) $t_1 \odot t_2 \in P \rightarrow t_2 \odot t_1 \in P$.

(d) If t occurs in P and $t_1 \leqq t$ then $t_1 \ominus t_1 \in P$.

(e) If t_1 and t_2 are distinct new function symbols of the same order and both occur in P then $t_1 \not\ominus t_2 \in P$. $0 \not\ominus 1 \in P$.

(f) If $t_1 \ominus t_2 \in P$ and $t \ominus t' \in P$ then $\mathrm{sub}(t_1, t_2, t) \ominus t' \in P$.

(g) If t_1 and t_2 occur in P and have the same order then some $t_1 \odot t_2 \in P$. If t has order -1 then $t \ominus 0 \in P$ or $t \ominus 1 \in P$.

REMARK 3.4. For fixed P the relation $t_1 \approx_P t_2 \leftrightarrow t_1 \ominus t_2 \in P$ is an equivalence relation on the terms which occur in P. The equivalence class of t, denoted $[t]_P$ consists of terms with the same order as t.

A P-class is *basic* if it contains a (unique, by (2.4b and e)) function symbol, 0, or 1. P is basic if every class of P is basic. For an arbitrary P, $B(P)$ denotes the set of basic terms which occur in P. If B is a set of basic terms terms $P \restriction B$ is the subset of P consisting of those equalities and inequalities among terms whose function symbols are from B, together with the terms 0 and 1. $P \restriction B$ is verified to be a condition.

REMARK 3.5. Every condition can be extended to a basic condition (conditions are ordered under $\subset$).

REMARK 3.6. Let f and g be function symbols of order n and $n-1$ respectively. If P contains neither $fp \not\circleddash g$ nor $fp \circleddash h$ for a function symbol $h \neq g$ then some extension of P contains $fp \circleddash g$.

LEMMA 3.7. *P and Q have a common extension (are compatible) in the set of conditions if and only if both:*

(a) *$P \restriction B(P) \cap B(Q)$ and $Q \restriction B(P) \cap B(Q)$ are compatible; and*

(b) *if $[t]_P \cap [t']_Q \neq \emptyset$ and both are basic classes then both contain the same function symbol.*

PROOF. Parts (a) and (b) are clearly necessary. To show that they are sufficient one considers two special cases.

(1) $B(Q) \subset B(P)$ and $Q \supset P \restriction B(Q)$. In this case the common extension R is to satisfy $R \restriction B(Q) = Q$.

(2) $Q \restriction B(P) \cap B(Q) = P \restriction B(P) \cap B(Q)$.

If P and Q are arbitrary but satisfy (a) and (b) one can first use the special Case (1) to obtain a P' and Q' which satisfy the hypotheses of the special Case (2). Case (2) then finishes the job.

Case (1) is proved by a downward induction on the orders of the terms involved in P and Q. One proves for each such n that P has an extension R such that R and Q satisfy the hypotheses of (1) and R and Q have the same terms of orders $\leqq n$. Part (b) is the case $n = 0$.

Case (2) is proved by taking the transitive closure of the relations $\approx_P$ and $\approx_Q$ on the terms of $P \cup Q$, closing under substitution requirement (3.3 g), and introducing $\not\circleddash$ between different classes at the same level.

Both of the above arguments require considerable checking, the details of which we defer to [17].

3.8. *Forcing.* As in (2.4) it suffices to state what is strongly forced ($\vdash^*$) to be a member of a new constant.

(a) $P \vdash^* "c \in f"$ iff $P \vdash^* "c = (p, g)"$ for some $p \in \omega$ and $fp \circleddash g \in P$.

(b) $P \vdash^* "c \in I"$ iff $P \vdash^* "c = f"$ for some f.

The usual (weak) forcing relation discussed in [13] and [20] is denoted by $\vdash$.

3.9. *Forcing automorphisms.* Let π be an order-preserving permutation of the new function symbols. π is extended to terms via $\pi(fp_1 \cdots p_k) = (\pi f)p_1 \cdots p_k$ and to constants by letting $\pi(I) = I$ and otherwise following the recipe of [1]. It is standard that

$$P \vdash \Phi(c_1, \cdots, c_m) \leftrightarrow \pi P \vdash \Phi(\pi c_1, \cdots, \pi c_m)$$

where $\Phi(x_1, \cdots, x_n)$ is parameter-free. π is called a *forcing automorphism.*

LEMMA 3.10. *Let $P \restriction B(P) \cap B(Q)$ and $Q \restriction B(P) \cap B(Q)$ be compatible. There is a forcing automorphism π such that π fixes $B(P) \cap B(Q)$ and P is compatible with πQ.*

PROOF. From (3.7) one can assume $Q \supset P \restriction B(P) \cap B(Q)$, and every term of $B(P) - B(Q)$ has a P class which intersects a basic Q class. Inductively one can assume $|B(P) - B(Q)| = 1$ and map Q so that the function symbol in that class coincides with the one in the P class. It turns out that $\pi Q \supset P$. Again some detail is left to [17].

LEMMA 3.11. *Let Φ have parameters in $M \cup I \cup B$ where B is a set of new function symbols. Then*

$$P \vdash \Phi \leftrightarrow P \restriction B \vdash \Phi.$$

PROOF. One can now carry out an argument of [1, p. 139]. If for some $Q \supset P \restriction B$, $Q \vdash^* \sim \Phi$ then by (3.9) and (3.10) there is a forcing automorphism π such that $\pi Q \restriction B = Q \restriction B$ and P and πQ are compatible. If R is a common extension of πQ and P then by standard properties of $\vdash$ [1], $R \vdash \Phi \wedge \sim \Phi$, a contradiction.

3.12. *The model.* As in Cohen [1] the above set-up leads to a model V of ZF containing M as a transitive proper subclass. From the parameter I, one can define the sequence I_n via $I_0 = I \cap 2^\omega$ and $I_{n+1} = I \cap I_n^\omega$. From (3.5), (3.6) and (3.8) it follows in a standard way [1] that each $f \in I_{n+1}$ takes every value in I_n infinitely many times.

If $x \in V$ and $G \subset I$ is finite we say, as in (2.5), that $x \in \nabla G$ if x is definable in V from parameters in $M \cup \{I\} \cup G$. If H well orders G, $T(H, \alpha)$ denotes the αth element in the well ordering of ∇G which H induces. The standard properties of ∇ and T listed in (2.5 b-f) hold.

3.13. *Density and continuity.* An *interval* in I_n $(n \geqq 0)$ is a subset of the form

$$E = \{f \in I_n : fp_1 = d_1 \wedge \cdots \wedge fp_n = d_n\}$$

where $d_1, \cdots, d_n \in I_{n-1}$. The following two facts illustrate clearly the sense in which V is an ω-iteration of the Halpern-Levy model [3]. They should be compared with the corresponding statements of [3], and also those of (2.6).

(a) *Density.* Every interval of I_n is nonempty (hence infinite).

PROOF. This is a density statement since it shows that I_n is dense in the natural topology on I^{ω}_{n-1}. Its truth is established by standard arguments [1] from the following clear consequence of (3.5) and (3.6).

FACT. For every condition P and every new function symbol f not occuring in P there is an extension of P which contains an arbitrary finite set of equations of the form $fp = t$ where no $p \in \omega$ occurs twice and each t occurs in P.

(b) *Continuity.* Let $\Phi(x_1, \cdots, x_k)$ have parameters in $M \cup \{I\} \cup \bigcup_{m<n} I_m$. Assume $\Phi(f_1, \cdots, f_k)$ is true in V for distinct $f_1, \cdots, f_k \in I_n$. There are disjoint intervals $E_1, \cdots, E_k \in I_n$ such that $f_i \in E_i$, all i, and $\Phi(f'_1, \cdots, f'_k)$ is true for any $f'_1, \cdots, f'_k$ with $f'_i \in E_i$, all i.

PROOF. Let $\Psi(x_1, \cdots, x_k)$ be the result of replacing each Φ-parameter in $\bigcup_{m<n} I_m$ by a term $x_i p_1 \cdots p_h$ where the given parameter is $f_i p_1 \cdots p_h$. This is possible since every $g \in I_m$ is onto I_{m-1}. If we include in E_i all the equations $fp_1 = g$, which arise in this way then for any $f' \in E_1$, $f' p_1 \cdots p_h = f_1 p_1 \cdots p_h$. Thus solving the problem for Ψ automatically solves the problem for Φ. Therefore we need only allow Φ to have parameters in $M \cup \{I\}$.

Since Φ is true there are now constants $\dot{f}_1, \cdots, \dot{f}_k$ and a condition P with $B(P) = \{\dot{f}_1, \cdots, \dot{f}_k\}$ such that

$$P \vdash \Phi(\dot{f}_1, \cdots, \dot{f}_k).$$

Take

$$E_i = \{f \in I_n : (\forall p \in \omega)[\dot{f}_i p \text{ occurs in } P \rightarrow fp = f_i p]\};$$

E_i is clearly an interval and $f_i \in E_i$. Assume $f'_1, \cdots, f'_k$ are distinct members of I_n with $f'_i \in E_i$, all i. P is true of the f'_i since terms at level $m < n$ have the same values for f'_i as for f_i. At level n we have guaranteed $f'_i = f'_j \leftrightarrow f_i \circleddash f_j \in P$ and $f'_i \neq f'_j \leftrightarrow f_i \not\circleddash f_j \in P$ by choosing the f'_is distinct.

Refine the E'_is as needed to make them disjoint and continue to satisfy $f_i \in E_i$. This is possible since the f_i are distinct. The new E_i satisfy the conclusion of continuity.

REMARK 3.14. $G \subset I$ is *level* if for some n, $G \subset I_n$. Every $x \in V$ is in ∇G for some level G. In fact if $x \in \nabla G$ then $x \in \nabla(G \cap I_n)$ for the highest n where $G \cup I_n \neq \emptyset$.

PROOF. Each $f \in I_n$ explicitly defines an enumeration of $\bigcup_{m<n} I_m$. The elements $g_m = f0 \cdots 0$ ($n - m - 1$ times) each map ω onto I_m. These maps can be made into enumerations of I_m by least preimage and these enumerations can be meshed into an enumeration of $\bigcup_{m<n} I_m$.

REMARK 3.15. If $G \subset I$ is finite and $G \in \nabla G'$ for $G' \subset I_n$ then $G \subset G' \cup \bigcup_{m<n} I_m$.

PROOF. I has a natural linear ordering defined inductively by ordering I_0 as a subset of 2^ω, I_{n+1} as a subset of I_n^ω, and I by letting I_{n+1} dominate I_n. Thus each $f \in G$ is also in $\nabla G'$ (defined as the kth element of G under $<$) so it suffices to assume that $G = \{f\}$.

First assume $f \in I_n - G$. Let $G = \{g_1, \cdots, g_k\}$. If $f \in \nabla G$ then for some α,

$$f = T(<g_1, \cdots, g_k>, \alpha).$$

An application of continuity shows $f' = T(<g_1, \cdots, g_k>, \alpha)$ for any f' in a suitable infinite interval E, a contradiction.

Next assume $f \in I_m$ for some $m > n$. Let f' be an arbitrary member of $I_n - G$. $f' = fp_1 \cdots p_j$ for some $p_1, \cdots, p_j \in \omega$. Thus if $f \in \nabla G$, $f' \in \nabla G$ contradicting the above paragraph.

SUPPORT LEMMA 3.16. (Compare with (2.7). *If $G_1 \cup G_2 \subset I_n$ and $x \in \nabla G_1 \cap \nabla G_2$ then either*

(a) *$G_1 \cap G_2 \neq \emptyset$ and $x \in \nabla G_1 \cap G_2$,*

(b) *$G_1 \cap G_2 = \emptyset$ and for some $G \subset I_{n-1}$, $x \in \nabla G$.*

PROOF. The situation is sufficiently like that in [3, Lem. 24] that we can virtually copy down the proof. Let $G_1 \cap G_2 = \{f_1, \cdots, f_r\}$, $G_1 - G_2 = \{g_1, \cdots, g_s\}$, $G_2 - G_1 = \{h_1, \cdots, h_t\}$. For suitable α and β,

$$x = T(<f_1, \cdots, f_r, g_1, \cdots, g_s>, \alpha) = T(<f_1, \cdots, f_r, h_1, \cdots, h_t>, \beta).$$

By continuity there are $E_1, \cdots, E_s$ not containing $f_1, \cdots, f_r$ or $h_1, \cdots, h_t$ such that whenever $g'_i \in E_i$, $i = 1, \cdots, s$;

$$T(\langle f_1, \cdots, f_r, g'_1, \cdots, g'_s \rangle, \alpha) = T(\langle f_1, \cdots, f_r, h_1, \cdots, h_t \rangle, \beta) = x.$$

Therefore

$$x = \{T(\langle f_1, \cdots, f_r, g'_1, \cdots, g'_s \rangle, \alpha) : g'_i \in E_i \text{ all } i = 1, \cdots, s\}.$$

The parameters of the above definition are $\{f_1, \cdots, f_r\}$ and the elements of I_{n-1} which occur in the definitions of the E_i. If $G_1 \cap G_2 \neq \emptyset$ these other parameters are all in $\nabla G_1 \cap G_2$ and Case (a) holds. If $G_1 \cap G_2 = \emptyset$ there are no fs and only parameters in I_{n-1} come in. This gives Case (b).

The following three corollaries prove Theorem 6.

COROLLARY 3.17. *AC fails for a countable family of countable sets.*

PROOF. The family $\{I_n\}_{n \in \omega}$ has no choice function. If ϕ were such a choice function and $\phi \in \nabla G$ for some $G \subset I_n$ then $\phi(I_{n+1}) \in \nabla G$ contradicting (3.15).

COROLLARY 3.18. *For every $x \in V$ there is a unique G_x such that*

(1) $x \in G_x$;

(2) *$G_x \subset I_n$ and n is least such that some $G \subset I_n$ satisfies* (1);

(3) *G_x has least cardinality among all $G \subset I_n$ which satisfy* (1) *and* (2).

PROOF. Clear from (3.16).

COROLLARY 3.19. *There is a linear ordering of V defined in V.*

PROOF. Let $H_x = < \restriction G_x$ where $<$ is the ordering of I. Let α_x be the solution of $x = T(H_x, \alpha)$. The embedding $x \to (G_x, \alpha_x)$ maps V into the clearly orderable class $(\bigcup_{n \in \omega} \mathcal{P}_{<\omega}(I_n)) \times \mathrm{On}$. ($\mathcal{P}_{<\omega}(x)$ denotes the set of finite subsets of x.) This defines an ordering on V.

$A \subset (\bigcup_{n \in \omega} \mathcal{P}_{<\omega}(I_n)) \times \mathrm{On}$ is said to be *solid* if for each level G,

$$\rho(A \cap (\{G\} \times \mathrm{On}))$$

is a well ordered cardinal (ρ denotes the y-axis projection).

LEMMA 3.20. *Every x is canonically similar to a solid set.*

PROOF. We begin with the observation that the well ordered V cardinals are exactly the M cardinals. This follows by standard arguments [20] once it is shown that the collection of conditions satisfies the countable chain condition. The collection of conditions is in fact countable.

Let ζ_α be the designated (by axiom E in M) 1:1 M-functions from α onto $|\alpha|$. If $B \subset \mathrm{On}$ let ψ_B denote the canonical order isomorphism $B \to$ Ordinal B. Map x to a solid set by the following steps.

Step 1. Use the mapping $\Phi \restriction x$ where Φ is, as in the proof of (3.19), an injection of V into $(\bigcup_{n \in \omega} \mathcal{P}_{<\omega}(I_n)) \times \mathrm{On}$.

Step 2. For each level G let $B(G)$ denote $\rho(\Phi(x) \cap (\{B\} \times \mathrm{On}))$. Define ψ on

$\Phi[x]$ by letting $\psi(G,\alpha)=\psi_{B(G)}(\alpha)$. $\psi\circ\Phi$ maps x to a set y such that $\rho(y\cap)\{G\}\times \mathrm{On}))$ is always an ordinal $\alpha(G)$.

Step 3. Define ζ on $\psi\circ\phi[x]$ via $(G,\beta)=\ \zeta_{\alpha(G)}(\beta)$. $\zeta\circ\psi\circ\Phi[x]$ is solid and $\zeta\circ\psi\circ\Phi$ is 1:1.

LEMMA 3.21. *Suppose $x\in\nabla G_1$ and $y\in\nabla G_2$ are solid and $x\approx y$. There is a solid $z\in\nabla G_1\cap\nabla G_2$ such that $x\ \approx\ y\ \approx\ z$.*

PROOF. For $x\in V$ and $n\in\omega$ let

$$[x]_n=\{(G,\alpha)\in x: G\in\bigcup_{n<m}\mathscr{P}_{<\omega}(I_m)\}.$$

Evidently $x\in\nabla G\rightarrow[x]_n\in\nabla G$. We claim that if x and y are solid and $x\ \approx\ y$ then for some n, $[x]_n=[y]_n$. The claim implies the lemma as follows. $x-[x]_n\subset\nabla G$ for any nonempty $G\subset I_n$. Therefore $x-[x]_n\ \approx\ \{\varnothing\}\times\alpha$ for some $\alpha\in\mathrm{On}$ since ∇G is well orderable. Let $z=[x]_n\cup(\{\varnothing\}\times\alpha)$. Evidently $z\approx x$, z is solid, and $z\in\nabla G_1$. $z\in\nabla G_2$ since also $Z=[y]_n\cup(\{\varnothing\}\times\alpha)$ so $z\in\nabla G_1\cap\nabla G_2$.

The claim is established as follows. Let Φ be a 1:1 onto map from x to y. Let n be such that $\Phi\in\nabla G$ for $G\subset I_n$. For $(G^*,\alpha)\in[x]_n$ the first coordinate G^{**} of $\Phi(G^*,\alpha)$ is in ∇G^*. By (3.13b), $G^{**}\subset G^*\cup\bigcup_{m<n}I_m$. Similarly using Φ^{-1}, $G^*\subset G^{**}\cup\bigcup_{m<n}I_m$ Since both G^* and G^{**} are level, one quickly concludes $G^*=G^{**}$ from (3.13b). Φ therefore restricts to a 1:1 onto function from $x\cap(\{G^*\}\times\mathrm{On})$ to $y\ \cap(\{G^*\}\times\mathrm{On})$. Hence $\rho(x\cap(\{G^*\}\times\mathrm{On}))=\rho(y\cap(\{G^*\}\times\mathrm{On}))$ because both are similar well ordered cardinals. Thus $x\cap(\{G^*\}\times\mathrm{On})=y\cap(\{G^*\}\times\mathrm{On})$ for any $G^*\in\bigcup_{n<m}\mathscr{P}_{<\omega}(I_m)$. This means $[x]_n=[y]_n$.

3.22. PROOF OF THEOREM 1.2. We prove that a class of cardinal representatives exists in V. For each $\mathscr{M}\in\mathbf{C}$ let $A_{\mathscr{M}}$ be the unique set A satisfying:

(1) A is solid and $|A|=\mathscr{M}$.

(2) $G_A\subset I_n$ and n is least possible with this property for any A satisfying (1).

(3) $|G_A|$ is minimal among those of all A satisfying (1) and (2).

(4) $A=T(G_A,\alpha)$ for α least among those A satisfying (1), (2) and (3).

The above definition is justified by Lemma 3.20, 3.21, and the Support Lemma 3.16. $\{A_{\mathscr{M}}:\mathscr{M}\in\mathbf{C}\}$ is a class of cardinal representatives.

3.23. PROOF OF THEOREM 1.7. We have already proved Order in V. We must now prove $Z(\aleph_0)$ and $\mathbf{C}\neq\mathrm{Idm}$ in V. To prove $Z(\aleph_0)$ we choose in a canonical similarities of Lemma 3.20 it suffices to do this for a solid x. If $x\neq\varnothing$

then $x - [x]_n \neq \emptyset$ for some unique least n. $\mathcal{P}_{<\omega}(I_n) \times \{0\}$ is then a canonically chosen countable set which intersects x.

To prove $\mathbf{C} \neq \mathrm{Idm}$ we use the set I. Applying the results of Lemma 3.21 to $I \times \{0\}$ and then back to I it follows that if $A \subset I$ and $A \approx I$ then $I - A \subset \bigcup_{n<m} I_m$ for some m. Thus $I - A$ is finite or countable. Since I is not countable (by (3.17) it follows that no two disjoint subsets of I are similar to I and $|I| \notin \mathrm{Idm}$.

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References

1. P. J. Cohen, *Set Theory and the Continuum Hypothesis*, Benjamin, 1966.
2. R. J. Gauntt, *Undefinability of cardinality*, Lecture notes, UCLA symposium on set theory, summer 1967.
3. J. D. Halpern and A. Levy, *The Boolean prime ideal theorem does not imply the axiom of choice*, Proc. Symp. Pure Math. XIII Part 1, pp. 83-134.
4. P. Howard, Doctoral Thesis, University of Michigan, 1970.
5. T. Jech and A. Sochor, *Applications of the θ model*. Bull. Acad. Polon. Sci. **14** (1966), 351–355.
6. T. Jech, *The Axiom of Choice*, North-Holland, 1973.
7. A. Levy, *Independence of various definitions of finiteness*, Fund. Math. **46** (1958), 1-13.
8. A. Levy, *Interdependence of various consequences of the axiom of choice*, Fund. Math. **54** (1964), 135-157.
9. A. Levy, *The Fraenkel-Mostowski method for independence proofs*, Symposium on theory of models, pp. 221-228, Berkeley, 1963, North-Holland, 1965.
10. A. Levy, *The definability of cardinal numbers*, Foundations of Mathematics, pp. 15–38, Gogel-Festschrift, Springer-Velag, 1969.
11. A. R. D. Mathias, *The order extension principle*, Lecture notes, UCLA symposium on set theory, summer 1967.
12. A. Mostowski, *Uber die unabhangigkeit des auswahlaxioms von ordnungsprinzip*, Fund. Math. **32** (1939), 201-252.
13. J. Myhill and D. Scott, *Ordinal definability*, Proc. Symp. Pure Math. XIII, Part 1, pp. 271-278.
14. D. Pincus, *Support structures for the axiom of choice*, J. Symbolic Logic **36** (1971), 28-38.
15. D. Pincus, *Zermelo Fraenkel consistency results by Fraenkel-Mostowski methods*, J. Symbolic Logic **37** (1972), 721-743.
16. D. Pincus, *Rigid models via iterated forcing*, Notices. Amer. Math. Soc. October 1970.
17. D. Pincus, *Adding dependent choice*, to appear.
18. D. Scott, *Definitions by abstraction in axiomatic set theory*, Bull. Amer. Math. Soc. **6** (1955), 442.

19. G. Sageev, *An independence result concerning the axiom of choice*, Doctoral Thesis, Hebrew University, Jerusalem, Israel, 1973.

20. J. R. Shoenfield, *Unramified forcing*, Proc. Symp. Pure Math. XIII Part 1, pp. 357-382.

21. W. Sierpinski, *Cardinal and Ordinal Numbers*, Warsaw, 1958.

22. R. M. Solovay and S. Tannenbaum, *Iterated Cohen extensions and Souslin's problem*, Ann. of Math. **94** (1971).

23. A. Tarski, *On the existence of large sets of Dedekind cardinals*, Notices Amer. Math. Soc., October, 1965.

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