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IF A TWO-POINT EXTENSION OF A BERNOULLI SHIFT HAS AN ERGODIC SQUARE, THEN IT IS BERNOULLI*

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ABSTRACT
A notion of "nesting" for very weakly Bernoulli distributions is developed and used to prove that any two-point extension of a Bernoulli shift, if it has ergodic square, must itself be Bernoulli.

A two-point extension of a Bernoulli shift is the following. Let T, acting on Ω, be a Bernoulli transformation, and $\{r, b\}$, a two point space. There are only two invertable maps of $\{r, b\}$ to itself, the identity I, and the interchange f. Let $g: \Omega \to \{I, f\}$ be a measurable map. Now define $T_g(\omega, *) = (T(\omega), g(\omega, *))$, a measure preserving invertable map of $\Omega \times \{r, b\}$ to itself. For any such g, T_g is a two-point extension of T.

It is not too difficult to see that if $\hat{T}$ is any transformation which has an invariant factor algebra with two-point fiber on which $\hat{T}$ is isomorphic to a Bernoulli shift T, then $\hat{T}$ is isomorphic to a two-point extension of T.

Our intention here is to characterize, up to isomorphism, all such extensions. There are two trivial extensions, given by $g \equiv I$ and $g \equiv f$. In both cases $(T_g)^2$ is nonergodic. A general result due to Parry [5] implies that if T_g is not K, then T_g is isomorphic to one of these. Our result will be a strengthening of this to say that if T_g is not Bernoulli, it must be isomorphic to one of these.

This fact was first conjectured by P. Shields. The author is also much indebted to D. Ornstein, J.-P. Thouvenot and B. Weiss for their insight ideas and interest.

To simplify matters, assume g is fixed and call T_g just T. We want to put a nice generating partition on $\bar{\Omega} = \Omega \times \{r, b\}$. Let $P = \{P_i\}$ be a finite partition which generates the invariant algebra of measurable sets in Ω, and for which the two

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sets where $g = I$ and $g = f$, are unions of P_i's. Now let $\bar{P} = \{(P_i \cap r), (P_i \cap b)\}$. Now any $\bar{\omega} \in \bar{\Omega}$ has a T, P-name, its "uncolored name", which it shares with one other point, and a $T, \bar{P}$-name, its "colored name", (r = red, b = black) which is determined by knowing the color of any point in the trajectory of $\bar{\omega}$, and the uncolored T, P-name of $\bar{\omega}$.

We know that T is a Bernoulli shift iff $(T, \bar{P})$ is v.w.b. (I am assuming throughout these arguments that the reader is quite familiar with Ornstein's work on the isomorphism theory of Bernoulli shifts [2, 7]). We know that (T, P), the uncolored process is f.d. and v.w.b. The heart of our argument here lies in two ideas. First, it is possible to show that in (T, P) we have the necessary structure to attempt a nesting type proof that $(T, \bar{P})$ is v.w.b. (see Ornstein [3] and Weiss [8, section 8] for discussions of the notion of nesting). Second, if there is an obstruction to this nesting argument, it must arise from a splitting of the distributions $\bigvee_{i=1}^{n} T^i(\bar{P})$ into sets each of which is, in a sense, $\bar{d}$ rigid, and this rigidity will force T^2 to be nonergodic.

Our first step is to develop, within (T, P), the basic machinery for nesting. We do this in a series of Lemmas. This first Lemma is originally due to J.-P. Thouvenot and D. Ornstein.

Lemma 1. *Let (T, P) be a f.d. process. Given any $\varepsilon > 0$, there is a δ and an N, for any $n \geqq N$, if $\{F_k\}_{k=1}^{m}$ is a partition of Ω so that for all but δ, in measure, of the F_k, $\mu(F_k) > 2^{-n\delta}$, then for some subset $K \subset \{1, \cdots, m\}$, with $\sum_{k \in K} \mu(F_k) > 1 - \varepsilon$,*

$$\bar{d}\left(\bigvee_{i=1}^{n} T^i(P), \bigvee_{i=1}^{n} T^i(P)/F_k\right) \leqq \varepsilon.$$

Proof. By the Shannon–Mcmillan Theorem, given any $\bar{\varepsilon}$, there is an $\bar{N}$ so that for $n > \bar{N}$, all but $\bar{\varepsilon}$ of the atoms $E \subset \bigvee_{i=1}^{n} T^i(P)$ have $\mu(E)$ within $2^{-n(h(T,P) \pm \bar{\varepsilon})}$.

With this fact, a mildly involved but not very difficult computation tells us that given $\hat{\varepsilon}$, if $\bar{\varepsilon}$ and δ are small enough, for all but $\hat{\varepsilon}$ of the F_k,

$$\text{(i)} \qquad \frac{1}{h} h\left(\bigvee_{i=1}^{n} T^i(P)/F_k\right) \text{ is within } 2^{n(h(T,P) \pm \hat{\varepsilon})}.$$

By the ergodic Theorem, given this $\hat{\varepsilon}$ and any m, there is an N so that for $n \geqq N$, for all but $\hat{\varepsilon}^2$ of the atoms $E \subset \bigvee_{i=1}^{n} T^i(P)$,

$$\left| \frac{1}{n-m} \sum_{l=1}^{n-m} \operatorname{dist}\left(\bigvee_{i=1}^{m+l-1} T^i(P)/E\right) - \operatorname{dist}\left(\bigvee_{i=1}^{m} T^i(P)\right) \right| < \hat{\varepsilon}^2$$

and hence for all but $\hat{\varepsilon}$ of the F_k, for all but $\hat{\varepsilon}$ of the atoms $\bar{E} \subset \bigvee_{i=1}^{n} T^i(P)/F_k$,

$$\text{(ii)} \qquad \left| \frac{1}{n-m} \sum_{l=1}^{n-m} \operatorname{dist}\left(\bigvee_{i=l}^{m+l-1} T^i(P)/\bar{E} \right) - \operatorname{dist}\left(\bigvee_{i=1}^{m} T^i(P) \right) \right| < \hat{\varepsilon}^2.$$

Let F_k be a set good in both (i) and (ii). Build a process (T_k, P_k) as the doubly infinite independent concatenation of $\bigvee_{i=1}^{n} T^i(P)/F_k$, labeling the beginning of each cycle of length n with a marker B. By (i) we have

$$\text{(i')} \qquad |h(T,P) - h(T_k, P_k)| < \hat{\varepsilon}$$

and by (ii) we have

$$\text{(ii')} \qquad \left| \operatorname{dist} \bigvee_{i=1}^{m} T^i(P) - \operatorname{dist} \bigvee_{i=1}^{m} T_k^i(P_k) \right| < \frac{4m}{n} + \hat{\varepsilon}.$$

As (T,P) is f.d., if m and n/m are large enough, and $\hat{\varepsilon}$ small enough, we get

$$\text{(iii')} \qquad \bar{d}(T,P;\, T_k, P_k) < \varepsilon.$$

This means we have a single process (T', P', P'_k) with

$$(T', P') \sim (T,P), \quad (T', P'_k) \sim (T, P_k) \quad \text{and} \quad d(P', P'_k) < \varepsilon.$$

Let F be the set of points in (T', P'_k) given markers B. Then $(T', \{F, F^c\})$ is an n-rotation, with o-entropy. As (T', P') is Bernoulli, $\bigvee_{-\infty}^{\infty} T'^i(P') \perp F$, and hence,

$$\bigvee_{i=1}^{n} T'^i(P')/F \quad \text{and} \quad \bigvee_{i=1}^{n} T'^i(P'_k)/F$$

are copies of $\bigvee_{i=1}^{n} T^i(P)$, and $\bigvee_{i=1}^{n} T^i(P)/F_k$ and this joining tells us

$$\bar{d}\left(\bigvee_{i=1}^{n} T^i(P),\ \bigvee_{i=1}^{n} T^i(P)/F_k \right) < \varepsilon. \qquad \blacksquare$$

It should be noted here that in fact Lemma 1 has a converse, and this notion of $\bigvee_{i=1}^{n} T^i(P)$ being "extremal" is in fact equivalent to being Bernoulli. We now want to strengthen Lemma 7 to hold close to (T,P) in $\bar{d}$.

LEMMA 2. *Let (T,P) be a finitely determined process. Given any ε, there is a δ, N so that if $n \geqq N$, and*

$$\bar{d}\left(\bigvee_{i=1}^{n} B_i,\ \bigvee_{i=1}^{n} T^i(P) \right) < \delta,$$

$\{B_i\}_{i=1}^{n}$ a collection of partitions of a space Y, and $\{F_k\}_{k=1}^{m}$ is a partition of Y with all but δ in measure of the F_k have $\mu(F_k) > 2^{-n\delta}$, then for a subset $K \subset \{1, \cdots, m\}$ with $\sum_{k \in K} \mu(F_k) > 1 - \varepsilon$,

$$\bar{d}\left(\bigvee_{i=1}^{n} T^i(P), \bigvee_{i=1}^{n} B_i / F_k\right) < \varepsilon.$$

PROOF. Choose $\delta < \varepsilon^2/4$, and N from Lemma 1 (with $\varepsilon^2/4$ as ε).
As

$$\bar{d}\left(\bigvee_{i=1}^{n} B_i, \bigvee_{i=1}^{n} T^i(P)\right) < \frac{\varepsilon^2}{4},$$

there are copies $\bigvee_{i=1}^{n} \bar{B}_i$, $\bigvee_{i=1}^{n} \bar{P}_i$ of $\bigvee_{i=1}^{n} B_i$ and $\bigvee_{i=1}^{n} T^i(P)$, on some space $\bar{Y}$, with $(1/n)\sum_{i=1}^{n} d(\bar{B}_i, \bar{P}_i) < \varepsilon^2/4$. Let $\{\bar{F}_k\}$ be a partition of Y obtained by breaking up $\bigvee_{i=1}^{n} \bar{B}_i$ as $\{F_k\}$ breaks up $\bigvee_{i=1}^{n} B_i$.

Now by Lemma 1, if δ is small enough, for all but $\varepsilon^2/4$ of the $\bar{F}_k$,

$$d\left(\bigvee_{i=1}^{n} T^i(P), \bigvee_{i=1}^{n} \bar{P}_i / \bar{F}_k\right) < \frac{\varepsilon}{2},$$

and we know, for all but $\varepsilon/2$ of the F_k, in measure,

$$\frac{1}{n}\sum_{i=1}^{n} d(\bar{P}_i / \bar{F}_k, \bar{B}_i / \bar{F}_k) < \frac{\varepsilon}{2}.$$

Hence $\bar{d}(\bigvee_{i=1}^{n} \bar{P}_i / \bar{F}_k, \bigvee_{i=1}^{n} B_i / F_k) < \varepsilon/2$ and we are done. ∎

Notice that in this argument it was necessary to work through "copies" $\bigvee_{i=1}^{n} \bar{B}_i$ of $\bigvee_{i=1}^{n} B_i$, on some other space Y. This situation will arise many times. Hence we make it precise. For any sequence of partitions on any Lebesque space, $\bigvee_{i=1}^{n} B_i$, a copy $\bigvee_{i=1}^{n} \bar{B}_i$ of it is a sequence of partitions of $[0,1]$ with $\operatorname{dist}(\bigvee_{i=1}^{n} B_i) = \operatorname{dist}(\bigvee_{i=1}^{n} \bar{B}_i)$. To simplify writing we use

$$d_n\left(\bigvee_{i=1}^{n} B_i, \bigvee_{i=1}^{n} P_i\right) = \frac{1}{n}\sum_{i=1}^{n} d(B_i, P_i).$$

Next we want to define a notion basic to the concept of nesting. In a nesting procedure we want to fix a joint distribution on some subset $S \subset \{1,\cdots,n\}$ and argue about the conditional distributions across the rest S^c. The sets S and S^c must behave nicely though. Here is the condition we want. A set $S \subset \{1\cdots n\}$ will be called an (α,δ,m)-subset if $\operatorname{card}(S) < \alpha n$, and there is a subset $B \subset \{1,\cdots,n\}$, $\operatorname{card}(B) < \delta n$, so that if $j \notin B$, then for some $0 \leqq l < m$, $\{j-l, j-l+1,\cdots,j,\cdots,j-l+m\} \subset S$ or $\subset S^c$. That is to say, outside a small set B, points in $\{1,\cdots,n\}$ come in blocks all in S or all in S^c, at least m long.

For a f.d. process (T,P) and (α,δ,m)-subset S of $\{1,\cdots,n\}$ we would like to know what $(\bigvee_{i\in S^c} T^i(P)/E)$ looks like for E an atom of $\bigvee_{i\in S} T^i(P)$. We proceed as follows.

LEMMA 3. *If (T,P) is an ergodic process, given any ε and α, there is a δ, m, N so that for $n > N$, if S is an (α,δ,m)-subset of $\{1,\cdots,n\}$, then for all but ε of the atoms E in $\bigvee_{i\in S} T^i(P)$,*

$$\frac{1}{\operatorname{card}(S^c)} h\Big(\bigvee_{i\in S^c} (T^i(P)/E)\Big) \text{ is within } h(T,P) \pm \varepsilon.$$

PROOF. Fix $\bar{\varepsilon}$, whose value we set later. Let P have k sets. Make $k^\delta < 2^{\bar{\varepsilon}^2/100}$. Find an $\bar{m}$ so that $(1/\bar{m})h(\bigvee_{i=1}^{\bar{m}} T^i(P))$ is within $h(T,P)\pm\bar{\varepsilon}/10$. Use the ergodic Theorem to pick M', so that for all but δ^3 of the points $\omega\in\Omega$, its T,P,m'-name gives density to sets in $\bigvee_{i=1}^{\bar{m}} T^i(P)$ within $\bar{\varepsilon}/10$ of their mass, for all $m' \geqq M'$, and now choose $m > 100M'/\delta^2$. Now choose N large enough that $\{1,\cdots,n\}$ for $n \geqq N$ actually has an (α,δ,m)-subset S.

Break $\{1,\cdots,n\}\backslash B$ into a sequence of blocks

$$\{j_1, j_1+1,\cdots, j_2-1\}\{j_2, j_2+1,\cdots, j_3-1\},\cdots,\{j_l, j_l+1,\cdots, j_{l+1}-1\}$$

of consecutive integers, $j_{i+1}-j_i > m$ and each block is entirely in S or S^c. By our choice for m, for all but a set C of blocks occupying at most δ of $\{1,\cdots,n\}$,

$$\left|\frac{1}{j_{i+1}-j_i-\bar{m}} \sum_{l=0}^{j_{i+1}-j_i-\bar{m}-1} \operatorname{dist}\Big(\bigvee_{k=j_i+l}^{j_i+l-\bar{m}} T^k(P)\Big) - \operatorname{dist}\Big(\bigvee_{i=1}^{\bar{m}} T^i(P)\Big)\right| < \delta.$$

If δ is small enough, this implies that

$$\frac{1}{j_{i+1}-j_i} h\Big(\bigvee_{k=j_i}^{j_{i+1}-1} T^k(P)\Big) \leqq \frac{1}{m} h\Big(\bigvee_{i=1}^{\bar{m}} T^i(P)\Big) + \frac{\bar{\varepsilon}^2}{10} \leqq h(T,P) + \frac{\bar{\varepsilon}^2}{5},$$

a one sided bound.

But

$$h\Big(\bigvee_{i\in B^c} T^i(P)\Big) = \sum_{p=1}^{l} h\Big(\bigvee_{j_p}^{j_{p+1}-1} T^i(P) \Big| \bigvee_{\substack{k<j_p\\ k\in B^c}} T^k(P)\Big)$$

and so

$$\frac{1}{\operatorname{card} B^c} h\Big(\bigvee_{i\in B^c} T^i(P)\Big) = \sum_{p=1}^{l} \Big(\frac{j_{p+1}-j_p}{\operatorname{card} B^c}\Big)\Big(\frac{1}{j_{p+1}-j_p} h\Big(\bigvee_{j_p}^{j_{p+1}-1} T^i(P) \Big| \bigvee_{\substack{k<j_p\\ k\in B^c}} T^k(P)\Big)\Big).$$

But $(1/\operatorname{card} B^c) h(\bigvee_{i\in B^c} T^i(P))$ is within $h(T,P)\pm\bar{\varepsilon}^2/10$, and so we must have for all but a set of blocks $\{j_p,\cdots,j_{p+1}-1\}$ of density $\bar{\varepsilon}$ in $\{1,\cdots,n\}$,

$$\frac{1}{j_{p+1}-j_p} h\Big(\bigvee_{j_p}^{j_{p+1}} T^i(P) \Big| \bigvee_{\substack{k<j_p\\ k\in B^c}} T^k(P)\Big) \text{ is within } \bar{\varepsilon} \text{ of } h(T,P).$$

It follows that

$$\frac{1}{\operatorname{card}(S^c \cap B^c)} h\Big(\bigvee_{i \in S^c \cap B^c} T^i(P) \Big/ \bigvee_{i \in S \cap B^c} T^i(P) \Big) \text{ is within } \bar{\varepsilon} \text{ of } h(T,P),$$

and as

$$\frac{1}{\operatorname{card}(S^c \cap B^c)} h\Big(\bigvee_{i \in S^c \cap B^c} T^i(P) \Big) \geqq h(T,P) + \frac{\bar{\varepsilon}^2}{5},$$

for all but $\sqrt{2\bar{\varepsilon}}$ of the atoms $E \subset \bigvee_{i \in S \cap B^c} T^i(P)$, we must have

$$\frac{1}{\operatorname{card}(S^c \cap B^c)} h\Big(\bigvee_{i \in S^c \cap B^c} T^i(P)/E \Big) \text{ is within } \sqrt{2\bar{\varepsilon}} \text{ of } h(T,P).$$

As B has density δ, S^c has density at least $1-\alpha$, and P has k sets, adding those in modifies the bound by at most $k\delta/(1-\alpha-\delta) < \bar{\varepsilon}$, hence the result if $\varepsilon = 2\sqrt{2\bar{\varepsilon}}$. ∎

LEMMA 4. *If (T,P) is an ergodic process, given $\varepsilon, \alpha < 1$ and m', there is a δ, m, N so that for all $n \geqq N$, if S is an (α,δ,m)-subset of $\{1,\cdots,n\}$, and $S^c = \{j_1, j_2, \cdots, j_l\}$ in increasing order then for all but ε of the atoms $E \in \bigvee_{i \in S} T^i(P)$,*

$$\left| \frac{1}{l-m'} \sum_{k=1}^{l-m'+1} \operatorname{dist}\Big(\bigvee_{i=k}^{k+m'-1} T^{j_i}(P)/E \Big) - \operatorname{dist}\Big(\bigvee_{i=1}^{m'} T^i(P) \Big) \right| < \bar{\varepsilon}.$$

PROOF. We proved this, using the ergodic Theorem, in the process of demonstrating the previous Lemma. ∎

Combining Lemmas 3 and 4, we get the following result on f.d. processes.

LEMMA 5. *Let (T,P) be a f.d. process. Given any ε and $\alpha < 1$, there is a δ, m, N so that for any partitions $\bigvee_{i=1}^n B_i$ with $n \geqq N$ and*

$$\bar{d}_n\Big(\bigvee_{i=1}^n T^i(P), \bigvee_{i=1}^n B_i \Big) < \delta$$

and any (α,δ,m)-subset S of $\{1,\cdots,n\}$, for all but ε of the atoms

$$E \in \bigvee_{i \in S} B_i, \quad \bar{d}\Big(\bigvee_{i \in S^c} B_i/E, \bigvee_{i=1}^{l} T^i(P) \Big) < \varepsilon, \quad l = \operatorname{card}(S^c).$$

PROOF. Fix $\hat{\delta}$ whose value we set later. As (T,P) is f.d., there is an $\bar{\varepsilon}, m, N'$ so that for any distribution $\bigvee_{i=1}^{n'} \bar{B}_i$, $n' > N'$, with

$$\text{(i)} \qquad \left| h(T,P) - \frac{1}{n'} h\Big(\bigvee_{i=1}^{n'} \bar{B}_i \Big) \right| < \bar{\varepsilon}$$

and

(ii) $$\left| \frac{1}{n'-m'} \sum_{k=1}^{n'-m'+1} \operatorname{dist}\left(\bigvee_{i=k}^{k+m'-1} \bar{B}_i \right) - \operatorname{dist}\left(\bigvee_{i=1}^{m'} T^i(P) \right) \right| < \bar{\varepsilon},$$

then

(iii) $$\bar{d}_{n'}\left(\bigvee_{i=1}^{n'} T^i(P), \bigvee_{i=1}^{n'} \bar{B}_i \right) < \hat{\delta}.$$

For this $\bar{\varepsilon}, \alpha$ and m', use Lemmas 3 and 4 to find δ, m, N, so that if S is an (α, δ, m) subset of $\{1, \cdots, n\}$, $n \geqq N > N'/(1-\alpha)$, then for all but $\bar{\varepsilon}$ of the $\bar{E} \in \bigvee_{i \in S} T^i(P)$,

(i′) $$\left| h(T, P) - \frac{1}{l} h\left(\bigvee_{i \in S^c} T^i(P)/E \right) \right| < \bar{\varepsilon}$$

and

(ii′) $$\left| \frac{1}{l-m'} \bigvee_{k=1}^{l-m'+1} \operatorname{dist}\left(\bigvee_{i=k}^{k+m'-1} T^{j_i}(P)/\bar{E} \right) - \operatorname{dist}\left(\bigvee_{i=1}^{m'} T^i(P) \right) \right| < \bar{\varepsilon},$$

where $S^c = \{j_1, \cdots, j_l\}$, $l = \operatorname{card} S^c > N'$. We then get

(iii′) $$\bar{d}_l\left(\bigvee_{i=1}^{l} T^i(P), \bigvee_{i \in S^c} T^i(P)/\bar{E} \right) < \hat{\delta}$$

for such E.

As

$$\bar{d}_{n'}\left(\bigvee_{i=1}^{n} T^i(P), \bigvee_{i=1}^{n} B_i \right) < \delta,$$

there are copies $\bigvee_{i=1}^{n} P_i$, $\bigvee_{i=1}^{n} \hat{B}_i$ of $\bigvee_{i=1}^{n'} T^i(P)$ and $\bigvee_{i=1}^{n'} B_i$, with

$$d_n\left(\bigvee_{i=1}^{n} P_i, \bigvee_{i=1}^{n} \hat{B}_i \right) < \delta.$$

As $l > (1-\alpha)n$,

(iv′) $$d_l\left(\bigvee_{i \in S^c} P_i/E, \bigvee_{i \in S^c} \hat{B}_i/E \right) < \frac{\sqrt{\delta}}{1-\alpha}$$

for all but $\sqrt{\delta}$ of the $E \in \bigvee_{i \in S} \hat{B}_i$.

We would be done if we could replace $\bar{E}$, in (iii′) with $E \in \bigvee_{i \in S} B_i$.

Consider atoms $E \in \bigvee_{i \in S} P_i$. As $d_{n'}(\bigvee_{i=1}^{n'} P_i, \bigvee_{i=1}^{m'} \hat{B}_i) < \delta$, for all but $\sqrt[3]{\delta}$ of the $\bar{E}$, all but $\sqrt[3]{\delta}$ of the mass of $\bar{E}$ is in at most $k^{\sqrt{\delta n}}$ atoms $E \in \bigvee_{i \in S} B_i$.

Let $\bar{E}$ be such a good atom of $\bigvee_{i \in S} P_i$. Then all but $\sqrt[3]{\delta} + k^{-\sqrt{\delta n}}$ of $\bar{E}$ is in sets $E \cap \bar{E} \in \bigvee_{i \in S}(P_i \vee \hat{B}_i)$ of size at least $k^{-2\sqrt{\delta n}}$.

Now if δ is small enough, and $l > (1-\alpha)n$ large enough, then by Lemma 2

$$\bar{d}_l\Big(\bigvee_{i\in S^c} P_i / E \cap \bar{E},\ \bigvee_{i=1}^{l} P_i\Big) < \frac{\varepsilon^2}{100}$$

for all but $\varepsilon^2/100$ of the $E \cap \bar{E} \subset \bar{E}$. Hence for all but $\sqrt{\varepsilon + (\varepsilon/100)^2} < \varepsilon/5$ of the atoms $E \in \bigvee_{i\in S} B_i$,

$$d_l\Big(\bigvee_{i\in S^c} T^i(P)/E,\ \bigvee_{i=1}^{l} T^i(P)\Big) < \frac{\varepsilon}{5}, \tag{v$'$}$$

and combining (iv′) and (v′), if $\sqrt{\delta}/(1-\alpha) < \varepsilon/5$,

$$\bar{d}_l\Big(\bigvee_{i\in S^c} B_i / E,\ \bigvee_{i=1}^{l} T^i(P)\Big) < \varepsilon. \qquad \blacksquare$$

This last lemma tells us that a f.d. process has future distributions on which the notion of nesting makes sense, as it tells us that we can condition on the name across a set of indices S, and still have the distribution on the remaining indices looking close in $\bar{d}$ to the original distribution.

With these tools we return to the question of two point extensions of a Bernoulli shift. So from now on $(T, \bar{P})$, with uncolored factor (T, P) is as we described earlier.

Suppose we have any sequence of partitions $\bigvee_{i=1}^n B_i$, whose symbols are the same as those naming the sets in P. A "coloring" $\bigvee_{i=1}^n C_i$ of $\bigvee_{i=1}^n B_i$ will be a sequence of partitions C_i whose sets are labeled $\{r_i, b_i\}$. C_1 is arbitrary, but for $i > 1$, a point $\omega \in r_i$ iff $\omega \in r_{i-1}$ and the set ω belongs to in B_{i-1} has the label of a set in P on which $g = I$, or $\omega \in b_{i-1}$ and the set ω belongs to in B_{i-1} has the label of a set in P on which $g = f$. Notice we could in fact choose any C_i arbitrarily, and use the above rule to construct the other C_j of the coloring. Also notice that $\bigvee_{i=0}^n T^i(\bar{P})$ is $\bigvee_{i=0}^n T^i(P) \vee \bar{C}_i$, where $\bar{C}_i$ is the coloring with $\bar{C}_0 = \{r, b\}$.

By the flip, $\bigvee_{i=1}^n C_n^f$ of the coloring $\bigvee_{i=1}^n C_i$ we mean the coloring where C_1^f is made of the same sets as C_1, but with the labeling inverted. This, then inverts the labeling down the entire sequence.

We now write down a statement which, in the logic of our argument, provides the needed requirement to actually do a nesting argument and prove $(T, \bar{P})$ is v.w.b., and whose negation will provide the starting point for a proof that $(T^2, \bar{P})$ is nonergodic.

(*) The process $(T, \bar{P})$ has the property that there is a fixed constant $a > 0$, so that for any $\varepsilon > 0$, there is a δ, and an N, and for any sequence of partitions $\bigvee_{i=1}^n B_i$, $n \geqq N$ with

$$\bar{d}_n\left(\bigvee_{i=1}^{n} B_i, \bigvee_{i=1}^{n} T^i(P)\right) < \delta$$

and any coloring $\bigvee_{i=1}^{n} C_i$ of $\bigvee_{i=1}^{n} B_i$, there are copies $\bigvee_{i=1}^{n} \bar{B}_i \vee \bar{C}_i$ and $\bigvee_{i=1}^{n} \hat{B}_i \vee \hat{C}_i$ of $\bigvee_{i=1}^{n} B_i \vee C_i$ and $\bigvee_{i=1}^{n} B_i \vee C_i^f$ with

$$d_n\left(\bigvee_{i=1}^{n} \bar{B}_i, \bigvee_{i=1}^{n} \hat{B}_i\right) < \varepsilon$$

and

$$d_n\left(\bigvee_{i=1}^{n} \bar{C}_i, \bigvee_{i=1}^{n} \hat{C}_i\right) < 1 - a.$$

Our next Lemma describes how (*) fits into a nesting scheme on $T, \bar{P}$ future distributions.

LEMMA 6. *If (T, P) is f.d., and $(T, \bar{P})$ satisfies (*) and if there is a number A, so that*

(i) *given any ε, there is a δ, N so that for $n \geqq N$, if $\bigvee_{i=1}^{n} B_i$, and $\bigvee_{i=1}^{n} B'_i$ have*

$$\bar{d}_n\left(\bigvee_{i=1}^{n} B_i, \bigvee_{i=1}^{n} T^i(P)\right) < \delta,$$

$$\bar{d}_n\left(\bigvee_{i=1}^{n} B'_i, \bigvee_{i=1}^{n} T^i(P)\right) < \delta$$

and $\bigvee_{i=1}^{n} C_i$ and $\bigvee_{i=1}^{n} C'_i$ are any colorings of these two distributions, then there are copies $\bigvee_{i=1}^{n} \bar{B}_i \vee \bar{C}_i$ and $\bigvee_{i=1}^{n} \bar{B}'_i \vee \bar{C}'_i$ of $\bigvee_{i=1}^{n} B_i \vee C_i$ and $\bigvee_{i=1}^{n} B'_i \vee C'_i$ with

$$d_n(\bar{B}_i, \bar{B}'_i) < \varepsilon$$

and

$$d_n(\bar{C}_i, \bar{C}'_i) < A$$

then

(ii) *given any $\bar{\varepsilon}$, there is $\bar{\delta}$, $\bar{N}$ so that for $n > \bar{N}$, if $\bigvee_{i=1}^{n} B_i$ and $\bigvee_{i=1}^{n} B'_i$ have*

$$\bar{d}\left(\bigvee_{i=1}^{n} B_i, \bigvee_{i=1}^{n} T^i(P)\right) < \bar{\delta}$$

and

$$\bar{d}\left(\bigvee_{i=1}^{n} B'_i, \bigvee_{i=1}^{n} T^i(P)\right) < \bar{\delta}$$

and $\bigvee_{i=1}^{n} C_i$ and $\bigvee_{i=1}^{n} C'_i$ are any colorings of these two distributions, then there are copies $\bigvee_{i=1}^{n} \hat{B}_i \vee \hat{C}_i$ and $\bigvee_{i=1}^{n} \hat{B}'_i \vee \hat{C}'_i$ of $\bigvee_{i=1}^{n} B_i \vee C_i$ and $\bigvee_{i=1}^{n} B'_i \vee C'_i$ with

$$d_n(\bar{B}_i, \bar{B}'_i) < \varepsilon$$

and

$$d_n(\bar{C}_i, \bar{C}'_i) < \left(1 - \frac{a}{2}\right) A.$$

Proof. We are given $\bar{\varepsilon}$. Pick $\hat{\varepsilon}$, even smaller, whose size we determine later. Pick $\bar{N}, \bar{\delta}$ so that by (i) we have copies $\bigvee_{i=1}^n \bar{B}_i \vee \bar{C}_i$ and $\bigvee_{i=1}^n \bar{B}'_i \vee \bar{C}'_i$ of $\bigvee_{i=1}^n B_i \vee C_i$ and $\bigvee_{i=1}^n B'_i \vee C'_i$ with

$$d_n\left(\bigvee_{i=1}^n \bar{B}_i, \bigvee_{i=1}^n \bar{B}'_i\right) < \hat{\varepsilon}$$

and

$$d_n\left(\bigvee_{i=1}^n \bar{C}_i, \bigvee_{i=1}^n \bar{C}'_i\right) < A.$$

If in fact

$$d_n\left(\bigvee_{i=1}^n \bar{C}_i, \bigvee_{i=1}^n \bar{C}'_i\right) < \left(1 - \frac{a}{2}\right) A,$$

we are done for these partitions. If not, $> (1 - a/2)A$.

Pick maximal subsets $F_1, \cdots, F_m$ of $[0,1]$ on which $\bar{B}_i / F_j = \bar{B}'_i / F_j$ except on a subset of indices $i_{1,j} \cdots i_{l(j),j}$, and $\bar{B}_i \vee \bar{C}_i / F_j$ and $\bar{B}'_i \vee \bar{C}'_i / F_j$ both consist of a single set for $i = i_{l,j}, l \leq l(j)$.

As $d_n(\bigvee_{i=1}^n \bar{B}_i, \bigvee_{i=1}^n \bar{B}'_i) < \hat{\varepsilon}$, for all but a collection of F_j's of total mass at most $\sqrt{\hat{\varepsilon}}$, we have $l(j) < \sqrt{\hat{\varepsilon}} n$, and hence, for all but a set of F_j of total mass $\sqrt{\hat{\varepsilon}} + 2^{-2h(\sqrt{\hat{\varepsilon}})n}$, $\mu(F_j) \geq 2^{-2h(\sqrt{\hat{\varepsilon}})n} \cdot (2k)^{\sqrt{\hat{\varepsilon}} n}$, where P has k sets. This is an application of Stirling's formula.

Now choose $\hat{\hat{\varepsilon}}$, whose size we set later. For $\hat{\hat{\varepsilon}}$ in Lemma 2, we get a δ, N. Now choose $\hat{\varepsilon}$ so that $(2h(\hat{\varepsilon}) + \sqrt{\hat{\varepsilon}} \log_2(2k)) < \delta$ and $\sqrt{\hat{\varepsilon}} \bar{\delta} < \delta$. Then for all but a set of F_j of total mass $(1 - \hat{\varepsilon})/(1 + \sqrt{\hat{\varepsilon}})$,

$$\bar{d}_n\left(\bigvee_{i=1}^n \bar{B}_i / F_j, \bigvee_{i=1}^n T^i(P)\right) \leq \hat{\hat{\varepsilon}}$$

and

$$\bar{d}_n\left(\bigvee_{i=1}^n \bar{B}'_i / F_j, \bigvee_{i=1}^n T^i(P)\right) \leq \hat{\hat{\varepsilon}}$$

Now consider some fixed F_j which satisfies these conditions. Let $S_j \subset \{1, \cdots, n\}$ be the subset of indices where $\bar{C}_i / F_j$ and $\bar{C}'_i / F_j$ are identical.

If $d_n(\bar{C}_i / F_j, \bar{C}'_i / F_j) < aA/4$, stop now on this F_j. Otherwise it is $> aA/4$. For $\alpha = \bar{\varepsilon}$, and a number $\hat{\hat{\varepsilon}}$ we again will specify later, Lemma 5 gives us a δ, m, N. Pick $\hat{\hat{\varepsilon}} < \delta^2(1-\alpha)/m$. Then it follows that S_j is an (α, δ, m) subset of $\{1, \cdots, n\}$. Hence by Lemma 5, for all but $\hat{\hat{\varepsilon}}$ of the atoms $E \in \vee_{i \in S} B_i / F_j$ (which is also an atom of $\vee_{i \in S_j} \bar{B}_i \vee \bar{C}_i / F_j$),

$$\bar{d}\left(\bigvee_{i \in S_j^c} \bar{B}_i / F_j \cap E, \bigvee_{i=1}^{\operatorname{card} S^c} T^i(P)\right) < \hat{\hat{\varepsilon}}.$$

Noticing that $\vee_{i \in S_j^c} \bar{C}_i / F_j \cap E = \vee_{i \in S_j^c} (\bar{C}'_i / F_j \cap E)^f$ and that these are colorings for $\vee_{i \in S_j^c} \bar{B}_i / F_j \cap E$, we are ready to apply (*).

Given $\bar{\varepsilon}/3$, by (*) there is a δ, N, so that setting $\hat{\hat{\varepsilon}} < \delta$, $\operatorname{card} S^c > \bar{\varepsilon} n > N$, then there are copies

$$\bigvee_{i \in S_j^c} \bar{\bar{B}}{}^E_{i,j} \vee \bar{\bar{C}}{}^E_{i,j} \quad \text{and} \quad \bigvee_{i \in S_j^c} \bar{\bar{B}}{}'^E_{i,j} \vee \bar{\bar{C}}{}'^E_{i,j}$$

of

$$\bigvee_{i \in S_j^c} \bar{B}_i \vee \bar{C}_i / F_j \cap E \quad \text{and} \quad \bigvee_{i \in S_j^c} \bar{B}'_i \vee \bar{C}'_i / F_j \cap E,$$

and

$$d_{\operatorname{card} S_j^c}\left(\bigvee_{i \in S_j^c} \bar{\bar{B}}{}^E_{i,j}, \bigvee_{i \in S_j^c} \bar{\bar{B}}{}'^E_{i,j}\right) < \frac{\bar{\varepsilon}}{3} + \frac{\sqrt{\hat{\varepsilon}}}{\bar{\varepsilon}} < 2\frac{\bar{\varepsilon}}{3}$$

and

$$d_{\operatorname{card} S_j^c}\left(\bigvee_{i \in S_j^c} \bar{\bar{C}}{}^E_{i,j}, \bigvee_{i \in S_j^c} \bar{\bar{C}}{}'^E_{i,j}\right) < (1-a).$$

For all such good j and good atoms $E \in \vee_{i \in S_j} \bar{B}_i \vee \bar{C}_i / F_j$, replace the partitions $\vee_{i \in S_j^c} \bar{B}_i \vee \bar{C}_i / F_j \cap E$ and $\vee_{i \in S_j^c} \bar{B}'_i \vee \bar{C}'_i / F_j \cap E$ by a copy of $\vee_{i \in S_j^c} (\bar{\bar{B}}{}^E_{i,j} \vee \bar{\bar{C}}{}^E_{i,j} \vee \bar{\bar{B}}{}'_{i,j} \vee \bar{\bar{C}}{}'^E_{i,j})$ on the set $F_j \cap E$. Elsewhere do not modify the partitions. Call these, new partitions $\vee_{i=1}^n \hat{B}_i \vee \hat{C}_i$ and $\vee_{i=1}^n \hat{B}'_i \vee \hat{C}'_i$. It is clear they are copies of $\vee_{i=1}^n B_i \vee C_i$ and $\vee_{i=1}^n B'_i \vee C'_i$.

Reading back through the steps we get

$$d_n\left(\bigvee_{i=1}^n \hat{B}_i, \bigvee_{i=1}^n \hat{B}'_i\right) < \sqrt{\hat{\varepsilon}} + \hat{\varepsilon} + \frac{\bar{\varepsilon}}{3} < \bar{\varepsilon}$$

and

$$d_n\Big(\bigvee_{i=1}^n \hat{C}_i, \bigvee_{i=1}^n \hat{C}'_i\Big) < \sqrt{\hat{\varepsilon}} + \hat{\varepsilon} + \Big(A - \frac{aA}{4}\Big)(1-a)$$
$$< \Big(1 - \frac{a}{2}\Big)A. \qquad \blacksquare$$

THEOREM 1. *If (T, P) is f.d. and $(T, \bar{P})$ satisfies $(*)$, then $(T, \bar{P})$ is v.w.b.*

PROOF. In Lemma 6, statements (i) and (ii) are identical, except that in (ii) A is replaced by $(1 - a/2)A$. Thus iterating Lemma 6 i times, we can replace A by $(1 - a/2)^i A \to 0$. As (i) always holds with $A = 1$, choose i so that $(1 - a/2)^i < \varepsilon$, and we conclude that there is a δ, N so that if

$$\bar{d}\Big(\bigvee_{i=1}^n B'_i, \bigvee_{i=1}^n T^i(P)\Big) < \delta \quad \text{and} \quad \bar{d}\Big(\bigvee_{i=1}^n B'_i, \bigvee_{i=1}^n T^i(P)\Big) < \delta,$$

$\bigvee_{i=1}^n C_i$ and $\bigvee_{i=1}^n C'_i$ are any two colorings of these distributions, then

$$\bar{d}_n\Big(\bigvee_{i=1}^n B_i \vee C_i, \bigvee_{i=1}^n B'_i \vee C'_i\Big) < \varepsilon.$$

Now it follows that as (T, P) is v.w.b., $(T, \bar{P})$ must be. ■

The second half of our argument is to investigate what it means for $(*)$ to be false. To begin we introduce the following idea. Let $\bigvee_{i=1}^n B_i$ and $\bigvee_{i=1}^n B'_i$ be two sequences of partitions whose sets are labeled with the names of sets in P, and $\bigvee_{i=1}^n C_i$ and $\bigvee_{i=1}^n C'_i$ are colorings for these two. We will say $\bigvee_{i=1}^n B_i \vee C_i$ and $\bigvee_{i=1}^n B'_i \vee C'_i$ are (ε, a)-rigidly close, if for *any* copies $\bigvee_{i=1}^n \bar{B}_i \vee \bar{C}_i$ and $\bigvee_{i=1}^n \bar{B}'_i \vee \bar{C}'_i$ of $\bigvee_{i=1}^n B_i \vee C_i$ and $\bigvee_{i=1}^n B'_i \vee C'_i$ with $\bar{d}_n(\bigvee_{i=1}^n \bar{B}'_i, \bigvee_{i=1}^n \bar{B}_i) < \varepsilon$, we must have $\bar{d}_n(\bigvee_{i=1}^n \bar{C}'_i, \bigvee_{i=1}^n \bar{C}_i) < a$, which is equivalent to saying $\bar{d}_n(\bigvee_{i=1}^n \bar{C}'_i, \bigvee_{i=1}^n \bar{C}^f_i) > 1 - a$.

With this notation, the negation of $(*)$ can be written as:

$(**)$ The process $(T, \bar{P})$ has the property that for any $a > 0$, there is an $\varepsilon(a)$, and a sequence of sequences of partitions $\bigvee_{i=1}^{n_j(a)} B_{i,j}(a)$ such that $n_j(a) \underset{j}{\to} \infty$ and

$$\bar{d}_{n_j(a)}\Big(\bigvee_{i=1}^{n_j(a)} B_{i,j}(a), \bigvee_{i=1}^{n_j(a)} T^i(P)\Big) \underset{j}{\to} 0,$$

with colorings $\bigvee_{i=1}^{n_j(a)} C_{i,j}(a)$, so that for all j, $\bigvee_{i=1}^{n_j(a)} B_{i,j}(a) \vee C_{i,j}(a)$ is $(\varepsilon(a), a)$-rigidly close to itself.

Using Lemmas 2 and 5 we want to eliminate all the dependencies in $(**)$ and end up with a splitting of $\bigvee_{i=1}^\infty T^i(\bar{P})$ into two arbitrarily rigid subdistributions, which, as we shall see, will force $(T^2, \bar{P})$ to be nonergodic.

We begin by using $(**)$ to write a version of Lemma 5 for $\bigvee_{i=1}^{n_j(a)} C_j(a)$.

LEMMA 7. *If $(T, \bar{P})$ is f.d. and satisfies (**), then given ε, α and a, there is an $\bar{\varepsilon} < \varepsilon$, J, δ and m so that for $j \geqq J$, if δ is an $\{\alpha, \delta, m\}$ subset of $\{1, \cdots, n_j(a)\}$, then for all but $\bar{\varepsilon}$ of the atoms $E \in \bigvee_{i \in S} B_{i,j}(a) \vee C_{i,j}(a)$,*

$$\bar{d}_{\operatorname{card} S^c}\Big(\bigvee_{i \in S^c} B_{i,j}(a)/E, \bigvee_{i \in S^c} B_{i,j}(a)\Big) < \bar{\varepsilon},$$

and for all but $\sqrt{a}/(1-\alpha)$ of these atoms E, $\bigvee_{i \in S^c} B_{i,j}(a) \vee C_{i,j}(a)/E$, and $\bigvee_{i \in S^c} B_{i,j}(a) \vee C_{i,j}(a)$ are $(\bar{\varepsilon}, \sqrt{a}/(1-\alpha))$-rigidly close.

PROOF. We are given ε, α and a. Using $\hat{\varepsilon} = \inf(\varepsilon/10, \varepsilon(a)/10)$ (as ε) in Lemma 2 we get $\hat{\delta}, \hat{N}$.

Using $\hat{\delta}^2(\hat{\varepsilon}/10)$ (as ε) again in Lemma 2 we get $\hat{\hat{\delta}}$ and $\hat{\hat{N}}$.

Finally using $\inf(\hat{\hat{\delta}}, \hat{\delta}\hat{\varepsilon}/20)$ (as ε) and $\inf(\alpha, \hat{\varepsilon}/10)$ (as α) in Lemma 5 we get $\tilde{\delta}$, $\tilde{m}$ and $\tilde{N}$. Set J so large that for $j \geqq J$,

$$n_j(a) \geqq \sup\Big(\hat{N}\frac{10}{\hat{\varepsilon}}, \hat{\hat{N}}, \tilde{N}\Big),$$

and

$$\bar{d}_{n_j(a)}\Big(\bigvee_{i=1}^{n_j(a)} B_{i,j}(a), \bigvee_{i=1}^{n_j(a)} T^i(P)\Big) < \hat{\hat{\delta}}$$

and pick m and δ so that $\delta < \tilde{\delta}$, $m > \tilde{m}$ and $((\delta + 1)/m)(10/\hat{\varepsilon}) < \inf(\hat{\hat{\delta}}, \tilde{\delta})$.

This now gives us J, δ, m. Let $j > J$ and S an (α, δ, m)-subset of $\{1 \cdots n_j(a)\}$.

Let $\bar{E}$ be an atom of $\bigvee_{i \in S} B_{i,j}(a)$. By our use of Lemma 5, for all but $\inf(\hat{\hat{\delta}}, \hat{\delta}^2\hat{\varepsilon}/20)$ of the atoms $\bar{E}$,

$$\bar{d}\Big(\bigvee_{i \in S^c} B_{i,j}(a)/\bar{E}, \bigvee_{i \in S^c} B_{i,j}(a)\Big) < \inf\Big(\hat{\hat{\delta}}, \frac{\hat{\delta}^2\hat{\varepsilon}}{20}\Big).$$

An $\bar{E}$ contains at most

$$2^{(\delta + 1/m)\alpha n_j(a)} < 2^{\hat{\hat{\delta}}\alpha n_j(a)} \text{ atoms } E \subset \bigvee_{i \in S} B_{i,j}(a) \vee C_{i,j}(a).$$

Hence, by Lemma 2, in its second application, for all but

$$\inf\Big(\hat{\hat{\delta}}, \frac{\hat{\delta}^2\hat{\varepsilon}}{20}\Big) + \frac{\hat{\delta}^2\hat{\varepsilon}}{20} < \frac{\hat{\delta}^2\hat{\varepsilon}}{10} \text{ of the atoms}$$

$$E \subset \bigvee_{i \in S} B_{i,j}(a) \vee C_{i,j}(a),$$

$$\bar{d}\Big(\bigvee_{i \in S^c} B_{i,j}(a)/E, \bigvee_{i \in S^c} B_{i,j}(a)\Big) < \frac{\hat{\delta}^2\hat{\varepsilon}}{10}.$$

Set $\bar{\varepsilon} = \hat{\delta}^2\hat{\varepsilon}/10$.

This means that for all but $\bar{\varepsilon}$ of the atoms $E \subset \bigvee_{i\in S} B_{i,j}(a) \vee C_{i,j}(a)$, there are copies $\bigvee_{i\in S^c} \bar{B}^E_{i,j} \vee \bar{C}^E_{i,j}$ and $\bigvee_{i\in S^c} \bar{B}_{i,j} \vee \bar{C}_{i,j}$ of $\bigvee_{i\in S^c} B_{i,j}(a) \vee C_{i,j}(a)/E$ and $\bigvee_{i\in S^c} B_{i,j}(a) \vee C_{i,j}(a)$ with

$$\bar{d}_{\operatorname{card} S^c}\Big(\bigvee_{i\in S^c} \bar{B}^E_{i,j}, \bigvee_{i\in S^c} \bar{B}_{i,j}\Big) < \bar{\varepsilon}.$$

Let these be *any* such copies. Paste them together, the E'th with weight $\mu(E)$, adding in the $\bar{\varepsilon}$ "bad" E's with any joint distribution, to get two copies $\bigvee_{i\in S^c} \bar{\bar{B}}{}^*_{i,j}$ and $\bigvee_{i\in S^c} \bar{\bar{B}}_{i,j}$ of $\bigvee_{i\in S^c} B_{i,j}(a)$, with

$$\bar{d}_{\operatorname{card}(S^c)}\Big(\bigvee_{i\in S^c} \bar{\bar{B}}{}^*_{i,j}, \bigvee_{i\in S^c} \bar{\bar{B}}_{i,j}\Big) < 2\bar{\varepsilon}.$$

We want to extend this to a match for $i \in S$. There are two possible cases

Case 1. If $\operatorname{card}(S) < \frac{1}{10}\hat{\varepsilon} n_j(a)$, extend the matching to $\bar{\bar{B}}{}^*_{i,j}$ and $\bar{\bar{B}}_{i,j}$, $i \in S$, by any joining.

Case 2. If $\operatorname{card}(S) > \frac{1}{10}\hat{\varepsilon} n_j(a)$ then S^c is an $(\hat{\varepsilon}/10, \hat{\hat{\delta}}, m)$ partition of $\{1, \cdots, n_j(a)\}$, hence the application of Lemma 5, and the first application of Lemma 2 tell us we can extend the joining to copies $\bar{\bar{B}}{}^*_{i,j}$ and $\bar{\bar{B}}_{i,j}$, $i \in S$, with

$$\bar{d}_{\operatorname{card} S}\Big(\bigvee_{i\in S} \bar{\bar{B}}{}^*_{i,j}, \bigvee_{i\in S} \bar{\bar{B}}_{i,j}\Big) < \frac{\hat{\varepsilon}}{10}.$$

Notice that now $\bar{\bar{C}}{}^*_{i,j}$ and $\bar{\bar{C}}_{i,j}$ automatically extend to $i \in S$.

In either case we get

$$\bar{d}_{n_j(a)}\Big(\bigvee_{i=1}^{n_j(a)} \bar{\bar{B}}{}^*_{i,j}, \bigvee_{i=1}^{n_j(a)} \bar{\bar{B}}_{i,j}\Big) < \hat{\varepsilon} < \varepsilon(a).$$

Hence, by (**),

$$\bar{d}_{n_j(a)}\Big(\bigvee_{i=1}^{n_j(a)} C^*_{i,j}, \bigvee_{i=1}^{n_j(a)} C_{i,j}\Big) < a.$$

Hence

$$\bar{d}_{\operatorname{card}(S^c)}\Big(\bigvee_{i\in S^c} C^*_{i,j}, \bigvee_{i\in S^c} C_{i,j}\Big) < \frac{a}{1-\alpha},$$

and so, for all but $\sqrt{a}/(1-\alpha)$ of the atoms $E \in \bigvee_{i\in S} B_{i,j}(a) \vee C_{i,j}(a)$,

$$\bar{d}_{\operatorname{card} S^c}(\bar{C}^E_{i,j}, \bar{C}_{i,j}) < \frac{\sqrt{a}}{1-\alpha}. \qquad \blacksquare$$

We want to replace $B_{i,j}(a)$ by $T^i(P)$ in (**), that is, get colorings for the T, P-distribution itself which are again rigid. Our first step is the next lemma which tells us that shorter blocks in $\bigvee_{i=1}^{n_j(a)} B_{i,j}(a) \vee C_{i,j}(a)$ must also, by themselves, be rigid.

LEMMA 8. *If (T, P) is f.d. and $(T, \bar{P})$ satisfies (**), then given any ε and $\hat{a}$, there is an A, so that if $a < A$, there is an $\hat{N}$, depending on a, so that if $\hat{n} > \hat{N}$, there is a J depending on a and $\hat{n}$, and an ε, depending on a, so that for all $j > J$, for all but ε of the values*

$$l \in \left\{0, 1, \cdots, \left[\frac{n_j(a)}{\hat{n}}\right] - 1\right\}, \ \bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} B_{i,j}(a) \vee C_{i,j}(a)$$

is $(\hat{\varepsilon}, \hat{a})$-rigidly close to itself.

PROOF. We are given ε and $\hat{a}$. Pick a value ε'' we set later independently of a. Set $A = \varepsilon''$. Choose $a < A$. Choose ε', whose value will depend on a. Use $1 - \sqrt{\varepsilon''}$ (as α) and use ε' (as ε) in Lemma 7 to get $\bar{\varepsilon} < \varepsilon'$, J_1, δ_1, m_1. Use $1 - \varepsilon'$ (as α) and ε' (as ε) in Lemma 5 to get δ_2 and N_2, m_2. Now make sure $2m_i / \hat{N} < \inf(\delta_i)$. Fix $\hat{n} > \hat{N}$, pick $J > J_1$ so large that for $j > J$, $\hat{n} / n_j(a) < \varepsilon$ and

$$\bar{d}_{n_j}\left(\bigvee_{i=1}^{n_j(a)} B_{i,j}(a), \bigvee_{i=1}^{n_j(a)} T^i(P)\right) < \inf(\delta_i).$$

For any subset $L \subset \{0, 1, \cdots, [n_j(a)/\hat{n}] - 1\}$, let

$$S_L = \bigcup_{l \in L} \{l\hat{n} + 1, l\hat{n} + 2, \cdots, (l+1)\hat{n}\}.$$

If $\operatorname{card}(S_L) < (1 - \sqrt{\varepsilon''})n_j(a)$, then S_L is a $(1 - \sqrt{\varepsilon''}, \delta_1, m_1)$-subset of $\{1, \cdots, n_j(a)\}$ and Lemma 7 applies. If $\operatorname{card}(S_L) < (1 - \varepsilon')n_j(a)$, then S_L is a $(1 - \varepsilon', \delta_2, m_2)$ subset of $\{1, \cdots, n_j(a)\}$ and Lemma 5 applies. Thus

(a) if $\operatorname{card}(S_L) < (1 - \sqrt{\varepsilon''})n_j(a)$, for all but $\sqrt[3]{\bar{\varepsilon}} + \sqrt{\varepsilon''}$ of the values $l \notin L$, and for every $E \in \bigvee_{i \in S_L} B_{i,j}(a) \vee C_{i,j}(a)$, there are copies $\bigvee_{i=ln+1}^{(l+1)n} \bar{B}_{i,j}^E \vee \bar{C}_{i,j}^E$ and $\bigvee_{i=ln+1}^{(l+1)n} \bar{B}_{i,j} \vee \bar{C}_{i,j}$ of $\bigvee_{i=ln+1}^{(l+1)n} B_{i,j}(a) \vee C_{i,j}(a)/E$ and $\bigvee_{i=ln+1}^{(l+1)n} B_{i,j}(a) \vee C_{i,j}(a)$, so that for all but $\sqrt{\bar{\varepsilon}}$ of the atoms E,

$$\bar{d}_{\hat{n}}\left(\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} \bar{B}_{i,j}^E, \bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} \bar{B}_{i,j}\right) < \sqrt[3]{\bar{\varepsilon}}$$

and for all but $\sqrt{\varepsilon''}$ of the values E,

$$\bar{d}_{\hat{n}}\left(\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} \bar{C}_{i,j}^E, \bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} \bar{C}_{i,j}\right) < \sqrt{\varepsilon''},$$

and

(b) if $\mathrm{card}(S_L) < 1 - \varepsilon'$, for all but $\sqrt[3]{\varepsilon'}$ of the values $l \not\in L$, there are copies

$$\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} \bar{B}^E_{i,j} \vee \bar{C}^E_{i,j} \quad \text{and} \quad \bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} \bar{B}_{i,j} \vee \bar{C}_{i,j}$$

of

$$\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} B_{i,j} \vee C_{i,j}(a)/E \quad \text{and} \quad \bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} B_{i,j}(a) \vee C_{i,j}(a)$$

so that for all but $\sqrt[3]{\varepsilon'}$ of the E,

$$\bar{d}_n\left(\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} \bar{B}^E_{i,j} \bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} \bar{B}_{i,j}\right) < \sqrt[3]{\varepsilon'}.$$

Now define an auxiliary sequence of partitions $\bigvee_{i=1}^{n_j(a)} P_i \vee C_i$, where, for any l,

$$\mathrm{dist}\left(\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} P_i \vee C_i\right) = \mathrm{dist}\left(\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} B_{i,j}(a) \vee C_{i,j}(a)\right)$$

and each of these blocks of length $\hat{n}$ is independent of all the others. Note, the C_i's are not a coloring for the P_i's.

We want to use (a) and (b) to show $\bigvee_{i=1}^{n_j(a)} B_{i,j}(a) \vee C_{i,j}(a)$ and $\bigvee_{i=1}^{n_j(a)} P_i \vee C_i$ are close in $\bar{d}$. To do this, set $L_0 = \varnothing$, and use $L = L_0$ in (a) to select $l_1 \not\in L_0$. Set $L_1 = \{l_1\}$ and use (a) to select $l_2 \not\in L_1$. Set $L_2 = \{l_1, l_2\}$, and use (a) to select $l_3 \not\in L_2$. Continue until $\mathrm{card}(S_L) > (1 - \sqrt{\varepsilon''})n_j(a)$. Then set $L_k = L$ and use (b) to select $l_{k+1} \not\in L_k$. Continue using (b) until $\mathrm{card}(S_{L_{k'}}) > (1 - \varepsilon')n_j(a)$. Now adding blocks $\{l_j n + 1, \cdots, l_j(n+1)\}$ one j at a time and filling in the remainder arbitrarily, we can build copies $\bigvee_{i=1}^{n_j(a)} \hat{B}_{i,j} \vee \hat{C}_{i,j}$ and $\bigvee_{i=1}^{n_j(a)} \hat{P}_i \vee \hat{C}_i$ of $\bigvee_{i=1}^{n_j(a)} B_{i,j}(a) \vee C_{i,j}(a)$ and $\bigvee_{i=1}^{n_j(a)} P_i \vee C_i$ with

$$\bar{d}_{n_j(a)}\left(\bigvee_{i=1}^{n_j(a)} \hat{B}_{i,j}, \bigvee_{i=1}^{n_j(a)} \hat{P}_i\right) < \varepsilon' + 2\sqrt[3]{\varepsilon'}$$

and

$$\bar{d}_{n_j(a)}\left(\bigvee_{i=1}^{n_j(a)} \hat{C}_{i,j}, \bigvee_{i=1}^{n_j(a)} \hat{C}_i\right) < \sqrt{\varepsilon''} + 2\sqrt[3]{\varepsilon''}.$$

(c) Now if for more than ε of the l we had $\bigvee_{i=ln+1}^{(l+1)n} B_{i,j}(a) \vee C_{i,j}(a)$ not $(\varepsilon', \hat{a})$-rigidly close to itself, we could build a copy $\bigvee_{i=1}^{n_j(a)} \hat{P}'_i \vee \hat{C}'_i$ with

$$\bar{d}_{n_j(a)}\left(\bigvee_{i=1}^{n_j(a)} \hat{P}_i, \bigvee_{i=1}^{n_j(a)} \hat{P}'_i\right) < \varepsilon'$$

and

$$\bar{d}_{n_j(a)}\left(\bigvee_{i=1}^{n_j(a)} \hat{C}_i, \bigvee_{i=1}^{n_j(a)} \hat{C}'_i\right) > \varepsilon \hat{a}.$$

Join to $\bigvee_{i=1}^{n_j(a)} \hat{P}_i \vee \hat{C}'_i$ a copy $\bigvee_{i=1}^{n_j(a)} \hat{B}'_{i,j} \vee \hat{C}'_{i,j}$ exactly as $\bigvee_{i=1}^{n_j(a)} \hat{B}_{i,j} \vee \hat{C}_{i,j}$ joins $\bigvee_{i=1}^{n_j(a)} \hat{P}_i \vee \hat{C}_i$. We conclude

$$\bar{d}_{n_j(a)}\left(\bigvee_{i=1}^{n_j(a)} \hat{B}_{i,j}, \bigvee_{i=1}^{n_j(a)} \hat{B}'_{i,j}\right) < 3\varepsilon' + 4\sqrt[3]{\varepsilon'}$$

and

$$\bar{d}_{n_j(a)}\left(\bigvee_{i=1}^{n_j(a)} \hat{C}_{i,j}, \bigvee_{i=1}^{n_j(a)}\right) > \varepsilon \hat{a} - 2\sqrt{\varepsilon''} - 4\sqrt{\varepsilon''}.$$

If $3\varepsilon' + 4\sqrt[3]{\varepsilon'} < \varepsilon(a)$ and $2\sqrt{\varepsilon''} - 4\sqrt{\varepsilon''} < \varepsilon\hat{a}/2$, we have a conflict with (**). Hence (c) is false. Setting $\hat{\varepsilon} = \varepsilon'$, we are done. ∎

Using this we can now replace (**) by a stronger statement.

LEMMA 9. *If (T, P) is f.d. and satisfies (**) then given any $\hat{a}$, there is an $\hat{\varepsilon}(\hat{a})$ and an $\hat{N}(\hat{a})$ so that for $\hat{n} > \hat{N}(\hat{a})$, there is a coloring $\bigvee_{i=1}^{\hat{n}} C_i^{\hat{n}}(\hat{a})$ of $\bigvee_{i=1}^{\hat{n}} T^i(P)$ with $C_i^{\hat{n}}(\hat{a}) \subset \bigvee_{i=1}^{\hat{n}} T^i(P)$, and $\bigvee_{i=1}^{\hat{n}} T^i(P) \vee C_i^{\hat{n}}(\hat{a})$ is $(\hat{\varepsilon}(\hat{a}), \hat{a})$-rigidly close to itself.*

PROOF. Use $\frac{1}{2}(\hat{a}^2/4)$ (as $\hat{a}$) and $\varepsilon = \frac{1}{2}$ in Lemma 8 to pick A. Set $a = A$, to get $\hat{N} = \hat{N}(a)$ and pick any $\hat{n} > \hat{N}(a)$. We now get $\hat{\varepsilon}$ and J. Pick $j > J$ so large that

(i) $$\bar{d}_{n_j(a)}\left(\bigvee_{i=1}^{n_j(a)} B_{i,j}(a), \bigvee_{i=1}^{n_j(a)} T^i(P)\right) < \frac{\hat{\varepsilon}_2}{3}\frac{1}{\hat{n}}.$$

By Lemma 8, then, we can pick l so that (ii) $\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} B_{i,j}(a) \vee C_{i,j}(a)$ is $(\hat{\varepsilon}, (\hat{a}/4)^2\frac{1}{2})$-rigidly close to itself, and by (i)

$$\left|\operatorname{dist}\left(\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} B_{i,j}(a)\right) - \operatorname{dist}\left(\bigvee_{i=1}^{\hat{n}} T^i(P)\right)\right| < \frac{\hat{\varepsilon}}{3}.$$

Use this to define $\bigvee_{i=1}^{\hat{n}} \bar{C}_i^{\hat{n}}$ as a coloring of $\bigvee_{i=1}^{\hat{n}} T^i(P)$ so that

$$\left|\operatorname{dist}\left(\bigvee_{i=l\hat{n}+1}^{(l+1)\hat{n}} B_{i,j}(a) \vee C_{i,j}(a)\right) - \operatorname{dist}\left(\bigvee_{i=1}^{\hat{n}} T^i(P) \vee \bar{C}_i^{\hat{n}}\right)\right| < \frac{\hat{\varepsilon}}{3}.$$

From this and (ii) we can conclude that $\bigvee_{i=1}^{\hat{n}} T^i(P) \wedge \bar{C}_i^{\hat{n}}$ is $(\hat{\varepsilon}/3, (\hat{a}/4)^2\frac{1}{2} + 2\hat{\varepsilon}/3)$-rigidly close to itself, and hence $(\hat{\varepsilon}/3, (\hat{a}/4)^2)$-rigidly close to itself.

We would be done if $\bar{C}_1 \subset \bigvee_{i=1}^{\hat{n}} T^i(P)$. But as $\bigvee_{i=1}^{\hat{n}} T^i(P) \vee C_i^{\hat{n}}$ is $(\hat{\varepsilon}/3, (\hat{a}/2)^2)$-rigidly close to itself, we must have all but $\hat{a}/4$ of the atoms $E \in \bigvee_{i=1}^{\hat{n}} T^i(P)$, are all but $\hat{a}/4$ in one set of $\bar{C}_i^{\hat{n}}$. Hence define $C_i^{\hat{n}}(\hat{a})$ as the partition putting all of E into that color most of E is in $\bar{C}_i^{\hat{n}}$. This gives us a coloring $\bigvee_{i=1}^{\hat{n}} C_i^{\hat{n}}(\hat{a})$, and we must have

$$d_n\left(\bigvee_{i=1}^{\hat{n}} \bar{C}_i^{\hat{n}}, \bigvee_{i=1}^{\hat{n}} C_i^{\hat{n}}(\hat{a})\right) < \frac{\hat{a}}{4}.$$

But this forces $\bigvee_{i=1}^{\hat{n}} T^i(P) \vee C_i^{\hat{n}}(\hat{a})$ to be $(\hat{\varepsilon}/3, (\hat{a}/4)^2 + \hat{a}/2)$-rigidly close to itself, hence $(\hat{\varepsilon}/3, a)$-rigidly close to itself. Set $\hat{\varepsilon}(\hat{a}) = \hat{\varepsilon}/3$ and we are done. ∎

The distributions $\bigvee_{i=1}^{\hat{n}} T^i(P) \vee C_i^{\hat{n}}(\hat{a})$, and $\bigvee_{i=1}^{\hat{n}} T^i(P) \vee (C_i^{\hat{n}}(\hat{a}))^f$ split $\bigvee_{i=1}^{\hat{n}} T^i(\bar{P})$ into two equal sized sets of atoms. We want to use their rigidity, shown in Lemma 9, to force $(T^2, \bar{P})$ to be nonergodic.

The sequences $\bigvee_{i=1}^{\hat{n}} C_i^{\hat{n}}(\hat{a})$ depend on $\hat{a}$. Our next Lemma, though, will tell us if $\hat{n}$ is large enough, there are essentially only two colorings of $\bigvee_{i=1}^{\hat{n}} T^i(P)$ rigidly close to themselves, one the flip of the other.

Lemma 10. *Given any ε, there are ε' and a', so that for any $\bar{\varepsilon}' < \varepsilon'$, there is an N', so that if $\hat{n} > N'$ and both $\bigvee_{i=1}^{\hat{n}} T^i(P) \vee C_i$ and $\bigvee_{i=1}^{\hat{n}} T^i(P) \vee C'_i$ are $(\bar{\varepsilon}', a')$-rigidly close to themselves, then either*

$$\bar{d}_{\hat{n}}\left(\bigvee_{i=1}^{\hat{n}} C_i, \bigvee_{i=1}^{\hat{n}} C'_i\right) < \varepsilon$$

or

$$\bar{d}_{\hat{n}}\left(\bigvee_{i=1}^{\hat{n}} C_i^f, \bigvee_{i=1}^{\hat{n}} C'_i\right) < \varepsilon.$$

Proof. For some $\varepsilon < \frac{1}{2}$, assume this is false for arbitrarily small $\bar{\varepsilon}'$ and a' and arbitrarily large $\hat{n}$. Suppose $\bigvee_{i=1}^{\hat{n}} C_i$ and $\bigvee_{i=1}^{\hat{n}} C'_i$ are two colorings of $\bigvee_{i=1}^{\hat{n}} T^i(P)$ for which the result fails.

Let $\{F_1, F_2, F_3\}$ be a partition of $\bar{\Omega}$ into three sets so that C_i and C'_i C'_i on F_1 and disagree on F_2, and $\varepsilon \leqq \mu(F_1) = \mu(F_2) \leqq \frac{1}{2}$. By Lemma 1, if $\hat{n}$ is large enough,

$$\bar{d}\left(\bigvee_{i=1}^{\hat{n}} T^i(P)/F_1, \bigvee_{i=1}^{\hat{n}} T^i(P)/F_2\right) < \bar{\varepsilon}'.$$

Using this $\bar{d}$ match, build copies $\bigvee_{i=1}^{\hat{n}} \bar{P}_i \vee \bar{C}_i \vee \bar{C}'_i$ and $\bigvee_{i=1}^{\hat{n}} \bar{\bar{P}}_i \vee \bar{\bar{C}}_i \vee \bar{\bar{C}}'_i$ of $\bigvee_{i=1}^{\hat{n}} T^i(P) \vee C_i \vee C'_i$, where the F_1 part of $\bigvee_{i=1}^{\hat{n}} \bar{P}_i \vee \bar{C}_i \vee \bar{C}'_i$ is matched to the F_2 part of $\bigvee_{i=1}^{\hat{n}} \bar{\bar{P}}_i \vee \bar{\bar{C}}_i \vee \bar{\bar{C}}'_i$ and vice versa, and F_3 is matched to itself by the identity match. We get

$$\bar{d}_{\hat{n}}\left(\bigvee_{i=1}^{\hat{n}} \bar{P}_i, \bigvee_{i=1}^{\hat{n}} \bar{\bar{P}}_i\right) < \bar{\varepsilon}',$$

so

$$\bar{d}_{\hat{n}}\left(\bigvee_{i=1}^{\hat{n}} \bar{C}_i, \bigvee_{i=1}^{\hat{n}} \bar{\bar{C}}_i\right) < a'.$$

But then

$$\bar{d}_{\hat{n}}\left(\bigvee_{i=1}^{\hat{n}} \bar{C}'_i, \bigvee_{i=1}^{\hat{n}} \bar{\bar{C}}'_i\right) > 2\varepsilon - a' > a',$$

if $a' < \varepsilon$. This is a conflict. ∎

Corollary 11. *For any ε, if a' and a are small enough and $\hat{n}$ is large enough then*

$$\bar{d}_{\hat{n}}\left(\bigvee_{i=1}^{\hat{n}} C_i^{\hat{n}}(a), \bigvee_{i=1}^{\hat{n}} C_i^{\hat{n}}(a')\right) < \varepsilon$$

or

$$\bar{d}_{\hat{n}}\left(\bigvee_{i=1}^{\hat{n}} (C_i^{\hat{n}}(a))^f, \bigvee_{i=1}^{\hat{n}} C_i^{\hat{n}}(a')\right) < \varepsilon.$$

Proof. Follows from Lemmas 9 and 10. ∎

This corollary allows us to remove the dependency of $C_i^{\hat{n}}(\hat{a})$ on $\hat{a}$. Let $\varepsilon_j = 1/2j$, and for this ε_j, let the bound for $\hat{n}$ in Corollary 11 be $\hat{N}_j$, and for $\hat{a}$ be $\hat{a}_j$. Now for $\hat{N}_j \leqq \hat{n} \leqq \hat{N}_{j+1}$, define

$$\bigvee_{i=1}^{\hat{n}} C_i^{\hat{n}} = \bigvee_{i=1}^{\hat{n}} C_i^{\hat{n}}(\hat{a}_j).$$

Corollary 12. *Given any a', there is an $N'(a')$ and an $\varepsilon'(a')$ so that for $n > N'(a')$, $\bigvee_{i=1}^{n} T^i(P) \vee C_i^{n'}$ is $(\varepsilon'(a'), a')$-rigidly close to itself.*

Proof. We are given a'. Pick $\varepsilon_j < a'/4$, set $\varepsilon'(a') = \hat{\varepsilon}(\hat{a}_j), N'(a') = \hat{N}(\hat{a}_j)$, and the result follows from Lemma 9 and Corollary 11. ∎

Notice Corollary 12 reads just like (**) but with the uncolored distributions exact, and the colorings independent of a'. Hence we can simply copy over Lemmas 7 and 8 into this context.

Corollary 13. *If (T, P) is f.d. and $(T, \bar{P})$ satisfies (**), and hence Corollary 12, then given ε, α and a, there is an $\bar{\varepsilon} < \varepsilon$, N, δ and m so that for $n' > N$, if S is an $\{\alpha, \delta, m\}$-subset of $\{1, \cdots, n'\}$, then for all but $\bar{\varepsilon}$ of the atoms $E \in \bigvee_{i \in S} T^i(P) \vee C_i^{n'}$,*

$$\bar{d}_{\text{card} S^c}\left(\bigvee_{i \in S_c} T^i(P)/E, \bigvee_{i \in S^c} T^i(P)\right) < \bar{\varepsilon},$$

and for all but $\sqrt{a'}/(1-\alpha)$ of these atoms E, $\bigvee_{i \in S^c} T^i(P) \vee C_i^n/E$ and $\bigvee_{i \in S^c} T^i(P) \vee C_i^{n'}$ are $(\bar{\varepsilon}, \sqrt{a'}/(1-\alpha))$-rigidly close.

Proof. Exactly as Lemma 7, replacing (**) by Corollary 12.

Corollary 14. *If (T, P) is f.d. and $(T, \bar{P})$ satisfies (**) and hence Corollary 12, then given any ε and a'', there is an N'', so that if $n'' > N''$, there is an M'' depending on n'', and an ε'' depending on M'', so that for $m'' > M''$, for all but ε of the values $l \in \{0, 1, \cdots, [m''/n''] - 1\}$, $\bigvee_{i=ln''+1}^{(l+1)n''} T^i(P) \vee C_i^{m''}$ is (ε'', a'')-rigidly close to itself.*

Proof. Exactly like Lemma 8 with Corollary 12 replacing (**), and Corollary 13 replacing Lemma 7. ■

Corollary 15. *Given any ε, there is an $\hat{N}'$, so that for any $\hat{n}' > \hat{N}'$ there is an $\hat{M}'$ so that for any $\hat{m}' > \hat{M}'$, for all but ε of the $l \in \{0, 1, \cdots, [\hat{m}'/\hat{n}'] - 1\}$, either*

$$d_{\hat{n}'}\left(\bigvee_{i=1}^{\hat{n}'} C_i^{n'}, \bigvee_{i=1}^{\hat{n}'} T^{-\hat{n}'l+1}(C_{n'+i}^{\hat{m}'})\right) < \varepsilon$$

or

$$d_{\hat{n}''}\left(\bigvee_{i=1}^{\hat{n}'} (C_i^{\hat{n}'})^f, \bigvee_{i=1}^{\hat{n}'} T^{-\hat{n}'l+1}(C_{\hat{n}'+i}^{\hat{m}'})\right) < \varepsilon.$$

Proof. By Corollary 12 and Corollary 14 both of the above are colorings of $\bigvee_{i=1}^{\hat{n}'} T^i(P)$ which are as rigidly close to themselves as we like, if $\hat{n}'$ and then $\hat{m}'$ are large enough. Lemma 10 finishes the result. ■

Define partitions H_m of $\bar{\Omega}$ by putting $\bar{\omega}$ in $h_{1,n} \in H_n$ if the $T, \bar{P}, n$-name of ω is a name in $\bigvee_{i=1}^n T^i(P) \vee C_i^n$, and in $h_{2,n}$ if it is a name in $\bigvee_{i=1}^n T^i(P) \vee (C_i^n)^f$. The partitions H_n each split $\bar{\Omega}$ into two sets of equal size.

Using this we can rewrite Corollary 15 as follows.

Corollary 16. *Given any ε, there is an $\hat{N}'$ so that for any $\hat{n}' > \hat{N}'$, there is an $\hat{M}'$, so that for any $\hat{m}' > \hat{M}'$, for all but ε of the $l \in \{0, 1, \cdots, [\hat{m}'/\hat{n}'] - 1\}$, either*

$$\mu(h_{1,\hat{n}'} \cap T^{-\hat{n}'l+1}(h_{1,\hat{m}'})) > \frac{1-\varepsilon}{2}$$

or

$$\mu(h_{2,\hat{n}'} \cap T^{-\hat{n}'l+1}(h_{1,\hat{m}'})) > \frac{1-\varepsilon}{2}.$$

PROOF. Clear from Corollary 15. ■

Although it is possible to continue to push explicitly through to a limit partition of the H_n's that is invariant under T^2, a much easier proof presents itself.

THEOREM 2. *If (T, P) is f.d. and $(T, \bar{P})$ satisfies (**), then $(T^2, \bar{P})$ is nonergodic.*

PROOF. By Corollary 16, given ε, if $\hat{n}$ is large enough, there are arbitrarily large values k with either

$$\mu(h_{1,\hat{n}} \cap T^k(h_{1,\hat{n}})) > \frac{1}{2} - \varepsilon$$

or

$$\mu(h_{1,\hat{n}} \cap T^k(h_{1,\hat{n}})) < \varepsilon.$$

Thus $(T, \bar{P})$ is not mixing, and by the Parry result [5], $(T^2, \bar{P})$ is nonergodic.

This completes the result that we set out to prove. There is one easy corollary that must be stated.

COROLLARY 17. *If T is loosely Bernoulli of positive entropy, and T^2 is ergodic, then T^2 is loosely Bernoulli.*

See the original paper of J. Feldman [1] and B. Weiss [8] for a discussion of loosely Bernoulli processes. In fact our entire argument can be viewed as a generalization of Weiss's proof of this result in the 0-entropy case.

The arguments here beg to be generalized, first to finite extensions, and further to skew products with rotations of the circle, and perhaps compact group extensions. For finite extensions and circle extensions, the corresponding results hold and will appear separately. For more general group extensions, something surely can be done, but what exactly is not clear.

The result that there are only three two-point extensions of a given transformation, does not hold outside the class of Bernoulli transformations. This can be seen in the example in [6] where two nonisomorphic, but K, two-point extensions of the same K-automorphism are constructed. It is not too difficult to see that there are in fact uncountably infinitely many such nonisomorphic 2-point extensions of this K-automorphism, through the same technique used by

Ornstein and Shields [4], only now on color sequences instead of spacer sequences.

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