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SPHERICALLY MEAN p-VALENT QUASIREGULAR MAPPINGS

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ABSTRACT
We introduce spherically mean p-valent quasiregular mappings. Using the method of modulus of path families we prove a distortion theorem and describe the boundary behaviour of this class of mappings.

1. Introduction

We derive first a lower bound for the modulus of a certain path family in the unit ball B^n in $\mathbf{R}^n$. We also introduce spherically mean p-valent quasiregular mappings (s.m. p-valent). The latter class reduces to the classical circumferentially mean p-valent for $n = 2$. Then using the lower bound for the modulus of a certain path family, we get a distortion theorem for s.m. p-valent quasiregular mappings in B^n.

Finally we show that if $f: B^n \to \mathbf{R}^n$ is s.m. p-valent and quasiregular then, as for $n = 2$, $|f(x)|$ can not grow too rapidly near too many points of ∂B^n.

2. Notation and terminology

Notation and terminology are in general as in [4]. When writing $f: D \to \mathbf{R}^n$, we assume throughout that D is a domain in $\mathbf{R}^n$, f is continuous, and $n \geqq 2$. If $A \subset D$, $y \in \mathbf{R}^n$, we define the multiplicity (possibly infinite) functions

$$N(y, f, A) = \operatorname{card}\{f^{-1}(y) \cap A\},$$

$$N(B, f, A) = \sup_{y \in B} N(y, f, A),$$

$$N(f, A) = N(\mathbf{R}^n, f, A),$$

$$N(f) = N(\mathbf{R}^n, f, D).$$

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If $f: D \to \mathbf{R}^n$ is sense-preserving, discrete and open then every $x \in D$ has arbitrarily small normal neighborhoods U (i.e. domains U with $U \subset D$, $f(\partial(U)) = \partial(f(U))$ and $U \cap f^{-1}(f(x)) = \{x\}$) with connected complement in $\mathbf{R}^n$ [4, 2.9]. The local topological index of f at a point $x \in D$, denoted $i(x, f)$, may be defined as

$$i(x, f) = N(f, U) \tag{2.1}$$

where U is any normal neighborhood of x [4, theorem 2.12].

We denote

$$n(y, f, D) = \sum_{x \in f^{-1}(y)} i(x, f). \tag{2.2}$$

We also denote

$$v(s, \tau) = \frac{1}{\omega_{n-1}\tau^{n-1}} \int_{S^{n-1}(\tau)} n(y, f, B^n(s)) d\Lambda(y) \tag{2.3}$$

where $d\Lambda(y)$ is an element of spherical Lebesgue measure on $S^{n-1}(|y|)$, and

$$L(r) = \operatorname*{Max}_{|x|=r} |f(x)|. \tag{2.4}$$

Let Γ be a family of non-constant paths in $\mathbf{R}^n$. The modulus of Γ is denoted by $M(\Gamma)$. $\Gamma(A, B, D)$ denotes the family of all paths which connect A and B in D.

The modulus of a ring domain, i.e. a domain $R \subset \mathbf{R}^n$ such that CR has exactly two connected components D_1 and D_2, is defined as

$$\operatorname{mod} R = \left(\frac{\omega_{n-1}}{M(\Gamma(D_1, D_2, R))} \right)^{1/(n-1)}. \tag{2.5}$$

The conformal capacity of R is

$$\operatorname{cap} R = \inf_u \int_R |\nabla u|^n dm_n$$

where ∇ denotes the gradient, and where the infimum is taken over all real-valued C^1-functions u in R with boundary values 0 on ∂D_1 and 1 on ∂D_2. It can be shown that

$$\operatorname{cap} R = M(\Gamma) \qquad \text{where } \Gamma = \Gamma(\partial D_1, \partial D_2, R).$$

Therefore we obtain that $\operatorname{mod} R = (\omega_{n-1}/\operatorname{cap} R)^{1/(n-1)}$.

3. Distortion theorem for spherically mean p-valent quasiregular mappings

LEMMA 3.1. *Let $f: B^n \to \mathbf{R}^n$ be a non-constant K-quasiregular mapping. Let $r_0 > 0$, $r_0 < r < 1$ and $r < s < 1$. Let*

$$F = f^{-1}C(B^n(L(r))) \tag{3.1}$$

where $L(r)$ is defined in (2.4), and let

$$\Gamma = \Gamma(B^n(r_0), F, B^n(s)). \tag{3.2}$$

Then

$$M(\Gamma) \geqq \frac{\omega_{n-1}}{2\left(\log \dfrac{\lambda_n r(s^2 - r_0^2)}{r_0(s^2 - r^2)}\right)^{(n-1)}} \tag{3.3}$$

where λ_n is a positive constant that depends only on n.

PROOF. Denote by F^* the symmetric image of $F \cap B^n(s)$ by reflection in $S^{n-1}(s)$, and denote by B^* the symmetric image of $B^n(r_0)$ by reflection in $S^{n-1}(s)$. Then by a Lemma of Gehring's [2, lemma 1], it follows that

$$M(\Gamma) = \tfrac{1}{2} M(\Gamma^*)$$

where $\Gamma^* = \Gamma(B^n(r_0) \cup B^*, (F \cap B^n(s)) \cup F^*, \overline{\mathbf{R}^n})$.

Consider the condenser

$$(P, Q) = (\overline{CB^n(r_0)}, \overline{(F \cap B^n(s)) \cup F^*}).$$

From the definitions it follows that

$$\operatorname{cap}(P, Q) = M(\Gamma(S^{n-1}(r_0), \overline{(F \cap B^n(s)) \cup F^*}, CB^n(r_0))),$$

and therefore

$$M(\Gamma^*) \geqq \operatorname{cap}(P, Q).$$

Let Sym be a cap symmetrization with center at the origin. Then by [6, 7.5]

$$\operatorname{cap}(P, Q) \geqq \operatorname{cap}(\operatorname{Sym}(P), \operatorname{Sym}(Q)).$$

As a K-quasiregular mapping satisfies the maximum principle, for every $r \leqq t \leqq s$ there exists an $x_0 \in S^{n-1}(t)$ such that $|f(x_0)| = R$ and $R \geqq L(r)$, therefore $F \cap S^{n-1}(t) \neq \varnothing$ for every $r \leqq t \leqq s$. The last fact implies that $\operatorname{Sym}(Q)$ contains the line segment $E = \{x \in \mathbf{R}^n;\ x = ue_1,\ r \leqq u \leqq r_1'\}$ where e_1 is a unit vector in the direction of x_1 and $r_1' = s^2/r$. But as $\Gamma(\partial(\mathrm{Sym}(P)), \mathrm{Sym}(Q), \mathrm{Sym}(P)) < \Gamma(\partial(\mathrm{Sym}(P)), E, \mathrm{Sym}(P))$ ($\Gamma_1 < \Gamma_2$ means that Γ_2 is minorized by Γ_1, see [8, 6.3]), it follows that

$$\mathrm{cap}(\mathrm{Sym}(P), \mathrm{Sym}(Q)) \geqq M(\Gamma_2)$$

where $\Gamma_2 = \Gamma(S^{n-1}(r_0), E, C(\overline{B^n(r_0)}))$.

As the modulus of a path family is a conformal invariant, it follows that

$$M(\Gamma_2) = M(\tilde{\Gamma}_2)$$

where $\tilde{\Gamma}_2 = \Gamma(T, S^{n-1}, B^n)$ and $T = \{x \in \mathbf{R}^n;\ x = ue_1,\ 0 \leqq u \leqq r_1\}$ and

$$r_1 = \frac{r_0}{r}\left(\frac{s^2 - r^2}{s^2 - r_0^2}\right).$$

Thus

$$M(\Gamma_2) = M(\tilde{\Gamma}_2) = \mathrm{cap}\, R_G\left(\frac{1}{r_1}\right) = \frac{\omega_{n-1}}{\left[\mathrm{mod}\, R_G\left(\frac{1}{r_1}\right)\right]^{n-1}}$$

where $R_G(a)$ is a Grötzsch' ring domain with complementary components $\overline{B^n}$, and

$$\{x \in \mathbf{R}^n;\ x = ue_1,\ a \leqq u < \infty\} \cup \{\infty\}, \qquad a > 1.$$

Using an estimate for $R_G(a)$, see [1, p. 235], we have that

$$\mathrm{mod}\, R_G(a) \leqq \log \lambda_n a$$

where λ_n is a positive constant that depends only on n. Therefore we have

$$M(\Gamma) \geqq \frac{1}{2}\frac{\omega_{n-1}}{\left(\log \frac{\lambda_n}{r_1}\right)^{n-1}}$$

and (3.3) follows by substituting the value of r_1.

COROLLARY 3.1. *Under the same assumptions as in Lemma* 3.1, *if* $s = (1 + r)/2$ *then*

$$M(\Gamma) \geqq \frac{\omega_{n-1}}{2\left[\log \lambda_n \frac{r(1+r)}{r_0(1-r)}\right]^{n-1}} \tag{3.4}$$

where λ_n *is a positive constant that depends only on* n.

THEOREM 3.1. *Let $f : B^n \to \mathbf{R}^n$ be a non-constant K-quasiregular mapping. If $r_0 > 0$, then for $r_0 < r < 1$*

$$\left(\log \frac{L(r)}{L(r_0)}\right)^{-n} \int_{L(r_0)}^{L(r)} \frac{v(s,\tau)}{\tau} d\tau \geqq \frac{1}{2K_0(f)} \left\{ \log \lambda_n \frac{r}{r_0} \cdot \frac{(1+r)}{(1-r)} \right\}^{1-n}$$

where $s = (1+r)/2$, λ_n is a positive constant that depends only on n, $v(s,\tau)$ is defined in (2.3) *and $L(r)$ is defined in* (2.4).

PROOF. Let $r_0 < r < 1$, denote $L_0 = L(r_0)$, and $L = L(r)$. Let $x \in S^{n-1}(r)$ be a point such that $|f(x)| = L$ and define

$$\Gamma = \Gamma(B^n(r_0), f^{-1}C(B^n(L)), B^n(s)).$$

From Corollary 3.1 it follows that

$$M(\Gamma) \geqq \frac{\omega_{n-1}}{2} \left\{ \log \lambda_n \frac{r}{r_0} \cdot \frac{(1+r)}{(1-r)} \right\}^{1-n}$$

where λ_n is a positive constant that depends only on n. In order to find an upper bound for $M(\Gamma)$, define

$$\rho(y) = \begin{cases} \left[\left(\log \frac{L}{L_0} \right) |Y| \right]^{-1} & \text{if } y \in B^n(L) \backslash \overline{B^n(L_0)} = D, \\ 0 & \text{elsewhere.} \end{cases}$$

One can easily show that ρ is an admissible function for $f(\Gamma)$. As in the proof of theorem 3.2 [4] we can get

$$\begin{aligned} M(\Gamma) &\leqq K_0(f) \int_D n(y, f, B^n(s)) \rho(y)^n dm(y) \\ &\leqq K_0(f) \int_{L_0}^{L} \int_{S^{n-1}} \frac{n(\tau y, f, B^n(s))}{\left(\log\left(\frac{L}{L_0}\right)\right)^n} \cdot \frac{1}{\tau} d\Lambda(y) d\tau \\ &\leqq K_0(f) \omega_{n-1} \left(\log \left(\frac{L}{L_0} \right) \right)^{-n} \int_{L_0}^{L} \frac{v(s,\tau)}{\tau} d\tau. \end{aligned}$$

Combining the upper and lower bounds we get the desired inequality.

DEFINITION 3.1. [5] Let $f : D \to \mathbf{R}^n$ be a sense-preserving discrete and open mapping; f is said to be spherically mean p-valent ($p > 0$) if

$$p(R) = p(R, f, D) = \frac{1}{\omega_{n-1} R^{n-1}} \int_{S^{n-1}(R)} n(y, f, D) d\Lambda(y) \leqq p$$

for every $0 < R < \infty$.

COROLLARY 3.2. *Let $f: B^n \to \mathbf{R}^n$ be a spherically mean p-valent quasiregular mapping. Then*

$$|f(x)-f(0)| \leqq C|x|^{\beta}\left\{\frac{1+|x|}{1-|x|}\right\}^{\gamma}$$

where C is a positive constant that depends on f, $\beta = (N/K_I(f))^{1/(n-1)}$ with $N = i(0,f)$ and $\gamma = (2pK_0(f))^{1/(n-1)}$.

PROOF. By theorem 5.2 in [7], there exists a $r_0 > 0$ such that

$$|f(x)-f(0)| \leqq A|x|^{\beta}; \qquad x \in \overline{B^n(r_0)}$$

and therefore it is easy to see that

$$|f(x)-f(0)| \leqq B|x|^{\beta}\left(\frac{1+|x|}{1-|x|}\right)^{\gamma}; \qquad x \in \overline{B^n(r_0)}.$$

In particular $L(r_0) \leqq Ar_0^{\beta} + |f(0)|$.

Since f is spherically mean p-valent, $v(s,\tau) \leqq p$, and by Theorem 3.1

$$p\left\{\log \frac{L(r)}{L(r_0)}\right\}^{1-n} \geqq \frac{1}{2K_0(f)}\left\{\log \lambda_n \frac{|x|}{r_0}\frac{(1+|x|)}{(1-|x|)}\right\}^{1-n}$$

for $x \in B^n \setminus \overline{B^n(r_0)}$, simple calculation then gives the desired inequality.

REMARK. Corollary 3.2 may be viewed as a generalization of a classical result about circumferentially mean p-valent analytic functions [3, theorem 5.1].

4. Boundary behaviour of spherically mean p-valent quasiregular mappings

LEMMA 4.1. *Let $f: \Delta \to \mathbf{R}^n$ be a K-quasiregular and spherically mean p-valent mapping ($p > 0$). Suppose Δ contains k disjoint balls $B^n(x_j, r_j)$ $1 \leqq j \leqq k$ and in every ball there exists a point x_j' such that $|f(x_j)| \leqq R_1$ and $|f(x_j')| \geqq R_2$; $j = 1, 2, \cdots, k$ where $0 < eR_1 < R_2 < \infty$. If $f(x) \neq 0$ for $x \in \bigcup_{j=1}^{k} B^n(x_j, r_j/2)$, then*

$$\sum_{j=1}^{k}\left[\log\left(\frac{2\lambda_n^2}{\delta_j}\right)\right]^{(1-n)} \leqq 2pK_0(f)\{\log R_2/R_1\}^{(1-n)}$$

where $\delta_j = 1 - |x_j' - x_j|/r_j$, $1 \leqq j \leqq k$.

PROOF. Define $B_j = B^n(x_j, r_j)$, $E_j = f^{-1}(B^n(R_1)) \cap B_j$, $F_j = f^{-1}(C(B^n(R_2))) \cap B_j$, $\Gamma_j = \Gamma(E_j, F_j, B_j)$, $1 \leqq j \leqq k$ and $\Gamma = \bigcup_{j=1}^{k} \Gamma_j$.

As the Γ_j lie in disjoint Borel sets it follows that $M(\Gamma) = \Sigma_{j=1}^{k} M(\Gamma_j)$. In order to find an upper bound for $M(\Gamma)$, define

$$\rho(y)=\begin{cases}\{[\log(R_2/R_1)]\,|y|\}^{-1} & \text{if } y\in B^n(R_2)\backslash\overline{B^n(R_1)},\\ 0 & \text{elsewhere.}\end{cases}$$

As in the proof of Theorem 3.1 we get

$$M(\Gamma)\leqq pK_0(f)\omega_{n-1}\{\log(R_2/R_1)\}^{(1-n)}.$$

In order to find a lower bound for $M(\Gamma)$ we shall prove that

$$M(\Gamma_j)\geqq\frac{\omega_{n-1}}{2\left(\log\left(\frac{2\lambda_n^2}{\delta_j}\right)\right)^{(n-1)}},\qquad 1\leqq j\leqq k, \tag{4.1}$$

and therefore

$$M(\Gamma)\geqq\frac{\omega_{n-1}}{2}\sum_{j=1}^{k}\left(\log\frac{2\lambda_n^2}{\delta_j}\right)^{(1-n)}.$$

Combining the upper and lower bounds we obtain the desired inequality. Now we turn to the proof of (4.1).

We shall use a similar argument to the one in the proof of Lemma 3.1.

Let E_j^* be the symmetric image of E_j by reflection in $S^{n-1}(x_j,r_j)$ and F_j^* the symmetric image of F_j by reflection in $S^{n-1}(x_j,r_j)$.

By [2, lemma 1] it follows that

$$M(\Gamma_j)=\tfrac{1}{2}M(\Gamma_j^*)$$

where $\Gamma_j^*=\Gamma(E_j\cup E_j^*,F_j\cup F_j^*,\mathbf{R}^n)$.

Consider the condenser

$$(P_j,Q_j)=(C(\overline{E_j\cup E_j^*}),\overline{F_j\cup F_j^*}).$$

Let Sym be a cap symmetrization with center at x_j. Then by [6, 7.5]

$$\operatorname{cap}(P_j,Q_j)\geqq\operatorname{cap}(\operatorname{Sym}(P_j),\operatorname{Sym}(Q_j)).$$

Again by the maximum principle, for every $|x_j'-x_j|<t\leqq r_j$ there exists an $x_{0j}\in S^{n-1}(x_j,t)$ such that $|f(x_{0j})|=R_{(j)}$ and $R_{(j)}\geqq R_2$; thus $\operatorname{Sym}(Q_j)$ contains the line segment

$$\tilde{E}_j=\left\{x\in\mathbf{R}^n;\,x=-ue_1;\,|x_j'|\leqq u\leqq|x_j'|+\frac{r_j^2}{|x_j'-x_j|}\right\}.$$

As $f(x)\neq 0$ for $x\in B^n(x_j,r_j/2)$ and f is K-quasiregular it satisfies the minimum principle, therefore $\operatorname{Sym}(P_j)$ contains the line segments

$$F_j'=\{x\in\mathbf{R}^n;\,x=(|x_j|+u)e_1;\,0\leqq u\leqq r_j/2\text{ or }2r_j\leqq u<\infty\}.$$

Therefore $\Gamma(\partial(\mathrm{Sym}(P_j)), \mathrm{Sym}(Q_j), \mathrm{Sym}(P_j)) < \bar{\Gamma}_j$, where $\bar{\Gamma}'_j = \Gamma(\bar{E}_j, \bar{F}'_j, \mathbf{R}^n)$. But $\bar{\Gamma}'_j \supset \bar{\Gamma}_j$ where $\bar{\Gamma}_j = \Gamma(\bar{E}_j, \bar{F}_j, \mathbf{R}^n)$ and $\bar{F}_j = \{x \in \mathbf{R}^n; x = (|x_j| + u)e_1; 2r_j \leqq u < \infty\}$. As the modulus of a path family is a conformal invariant it follows that

$$M(\Gamma_j) \geqq \frac{1}{2}\mathrm{cap}\, R_T(b_j) = \frac{\omega_{n-1}}{[\mathrm{mod}\, R_T(b_j)]^{(n-1)}}$$

where $R_T(b_j)$, $b_j > 0$, is the Teichmüller ring bounded by the segment $\{x \in \mathbf{R}^n; -1 \leqq x_1 \leqq 0,\ x_2 = \cdots = x_n = 0\}$ and the ray $\{x \in \mathbf{R}^n;\ b_j \leqq x_1 < \infty,\ x_2 = \cdots = x_n = 0\}$, with

$$b_j = \left(2 + \frac{|x_j - x'_j|}{r_j}\right) \Big/ \left(\frac{r_j}{|x_j - x'_j|} - \frac{|x_j - x'_j|}{r_j}\right).$$

The modulus of Teichmüller's ring domain and the modulus of Grötzsch' ring domain are related; see for example [1, p. 232]. Using the relation between the moduli, see [1, p. 235], and the estimate for Grötzsch' ring domain one obtains (4.1).

Definition 4.1. Let $f: B^n \to \mathbf{R}^n$ be a K-quasiregular mapping. Let $a \in S^{n-1}$, if there exists a path $\gamma: [0,1] \to \overline{B^n}$ such that $\gamma([0,1]) \subset B^n$ and $\gamma(1) = a$, and a positive δ such that

$$\varliminf_{t \to 1} (1 - |\gamma(t)|)^\delta |f(\gamma(t))| > 0.$$

Then define the lower order $\alpha(a)$ of f at a point a as

$$\mathrm{Sup}\left\{\delta > 0; \varliminf_{t \to 1} (1 - |\gamma(t)|)^\delta |f(\gamma(t))| > 0\right\}.$$

If no such path γ and a positive δ exist, we put $\alpha(a) = 0$.

Theorem 4.1. *Let $f: B^n \to \mathbf{R}^n$ be a K-quasiregular spherically mean p-valent ($p > 0$) mapping. Let E be the set defined as $E = \{x \in S^{n-1}; \alpha(x) > 0\}$. Then*

$$\sum_{a \in E} \alpha(a)^{n-1} \leqq 2pK_0(f). \tag{4.2}$$

Proof. It is enough to show that if $a_1, \cdots, a_k$ are disjoint points on S^{n-1} then

$$\sum_{j=1}^{k} \alpha(a_j)^{n-1} \leqq 2pK_0(f). \tag{4.3}$$

Letting $k \to \infty$, in (4.3) this yields (4.2).

Suppose the theorem is false. Then we can find a finite number of points $a_1, a_2, \cdots, a_k$ and $\varepsilon > 0$ such that

$$\sum_{j=1}^{k} \alpha(a_j)^{n-1} = 2K_0(f)(p + k\varepsilon).$$

For every a_j there exists a path γ_j such that $\gamma_j : [0,1) \rightarrow B^n$, $\gamma_j(1) = a_j$, $a_j \in S^{n-1}$ and $(1 - |\gamma(t)|)^{\eta_j} |f(\gamma(t))| > 1$, where

$$\eta_j^{n-1} = \alpha(a_j)^{n-1} - \varepsilon, \qquad 1 \leqq j \leqq k.$$

Therefore $\Sigma_{j=1}^{k} \eta_j^{n-1} > 2K_0(f)p$, and there exists $R_0 > 0$ such that for every $R_2 > R_0$ we can find x_j' on $\gamma_j(t)$, $0 \leqq t < 1$ such that

$$|f(x_j')| = R_2 > \left(\frac{1}{1 - |x_j'|}\right)^{\eta_j}; \qquad 1 \leqq j \leqq k.$$

Choose δ so that the following two conditions are satisfied:

(i) $f(x)$ is free of zeros in $B^n \backslash \overline{B^n(1 - 2\delta)}$,

(ii) $4\delta < \mathrm{Min}_{1 \leqq m,j \leqq k} |a_m - a_j|$; $m \neq j$.

Take $r_0 = 1 - \delta$. If R_2 is sufficiently large $|x_m' - x_j'| > 4\delta$, $1 \leqq m, j \leqq k$. As $x_j' \rightarrow a_j$ when $R_2 \rightarrow \infty$, the balls $B^n((r_0/|x_j'|)x_j', \delta)$ are disjoint for $1 \leqq j \leqq k$. Also $|f(x_j)| \leqq R_1 = \max_{|x|=r_0} |f(x)|$ and $|f(x_j')| = R_2$ thus by using Lemma 4.1 with $\delta_j = (1 - |x_j'|)/\delta$ we can complete the proof as in [3, theorem 2.7].

Remark. This chapter contains known results for areally mean p-valent analytic functions. The proofs of the results are different in major parts from that in [3, theorems 2.6 and 2.7] mainly by using Lemma 3.1.

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