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THE BANACH–MAZUR DISTANCE BETWEEN SYMMETRIC SPACES

BY
NICOLE TOMCZAK-JAEGERMANN†

ABSTRACT
We show that the Banach–Mazur distance between N-dimensional symmetric spaces E and F satisfies $d(E,F) \leqq c\sqrt{N}$, where c is a numerical constant. If E is a symmetric space, then $d(E, l_2^{\dim E}) \leqq 2\sqrt{2}\max(M^{(2)}(E), M_{(2)}(E))$, where $M^{(2)}(E)$ (resp. $M_{(2)}(E)$) denotes the 2-convexity (resp. the 2-concavity) constant of E. We also give an example of a space F with an 1-unconditional basis and enough symmetries that satisfies $d(F, l_2^{\dim F}) = M^{(2)}(F)M_{(2)}(F)$.

§0. Introduction

In this paper we investigate the Banach–Mazur distance within the class of finite-dimensional symmetric spaces. Our main theorem says that

$$(*) \qquad \operatorname{diam} S_k = \max\{d(E,F) \;\; E, F \in S_k\} \sim \sqrt{k},$$

where S_k denotes the class of all k-dimensional symmetric spaces. The role of the symmetry assumption can be seen comparing $(*)$ with the recent result of E. D. Gluskin [4], which says that $\max\{d(E,F) \mid \dim E = k = \dim F\} \sim k$. Our estimate improves earlier, independently obtained results of E. D. Gluskin [3] and the author [10], where estimates $\operatorname{diam} S_k \leqq c\sqrt{k}(\log(k+1))^{\alpha}$, where c is a positive constant, were shown (with $\alpha = 4$ in [3] and $\alpha = 2$ in [10]).

In the case $k = 2^n$ our proof of $(*)$ is constructive. In §2 we construct a certain family $\mathcal{R}$ of orthogonal $2^n \times 2^n$ matrices. In §3 we show that given spaces E, $F \in S_{2^n}$ one has the estimate $\min \| T \| \, \| T^{-1} \| \leqq 2^{12+n/2}$, where the minimum is taken over all operators $T : E \to F$, such that T is determined by a matrix from $\mathcal{R}$ or T is a formal identity operator. This shows the main estimates in $(*)$ for $k = 2^n$. The case of general k follows by a formal argument. Our proof also

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formally implies a version of (*) in terms of p-convexity and p'-concavity constants of spaces involved.

In §4 we investigate the distance from a normed space to an euclidean space. We show that if $E \in S_k$, then $d(E, l_2^k) \leqq 2\sqrt{2}\max(M^{(2)}(E), M_{(2)}(E))$, where $M^{(2)}(E)$ and $M_{(2)}(E)$ denote respectively the 2-convexity and 2-concavity constants of E, thus essentially strengthening in this case the classical estimate of Kwapień [6]. To conclude §4 we give an example of a finite-dimensional normed space F, which has 1-unconditional basis and enough symmetries and for which Kwapień's estimate is, up to a numerical factor, the best possible.

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§1. Notation and preliminary results

Let us recall some notation from the theory of finite-dimensional normed spaces. If E, F are finite-dimensinal real normed spaces and $\dim E = \dim F$, the Banach–Mazur distance $d(E,F)$ is defined by

$$d(E,F) = \inf\{\|T\|\,\|T^{-1}\| \mid T \text{ an isomorphism from } E \text{ onto } F\}.$$

Let X be a k-dimensional real normed space. A basis $\{e_i\}_{i=1}^k$ in X is called 1-unconditional, if

$$\left\|\sum_{i=1}^k x(i)e_i\right\| = \left\|\sum_{i=1}^k \varepsilon_i x(i)e_i\right\|,$$

for every sequence $\{x(i)\}_{i=1}^k$ of real numbers and every $\varepsilon_i = \pm 1$, for $i = 1,\cdots,k$. Let $1 < p < \infty$. The p-convexity (resp. p-concavity) constant of X, denoted by $M^{(p)}(X)$ (resp. $M_{(p)}(X)$) is the least M (resp. $M' > 0$) such that

$$\frac{1}{M'}\left(\sum_m \|x_m\|^p\right)^{1/p} \leqq \left\|\left(\sum_m |x_m|^p\right)^{1/p}\right\| \leqq M\left(\sum_m \|x_m\|^p\right)^{1/p},$$

for all finite sequences $\{x_m\}$ in X. Here $(\Sigma_m |x_m|^p)^{1/p} \in X$ is a vector whose i-th coordinate with respect to the basis $\{e_i\}_{i=1}^k$ is equal to $(\Sigma_m |x_m(i)|^p)^{1/p}$ for $i = 1,\cdots,k$. For further details concerning convexity and concavity the reader is referred to [7], section 1.d.

A k-dimensional real normed space X is called symmetric, if there is a normalized basis $\{e_i\}_{i=1}^k$ in X such that

$$\left\| \sum_{i=1}^k x(i)e_i \right\| = \left\| \sum_{i=1}^k \varepsilon_i x(i) e_{\pi(i)} \right\|,$$

for every sequence $\{x(i)\}_{i=1}^k$ of real numbers, every $\varepsilon_i = \pm 1$ for $i = 1, \cdots, k$ and every permutation π of the set $\{1, \cdots, k\}$. A symmetric space X will be always identified with R^k, endowed with a norm $\|\cdot\|$, and the basis $\{e_i\}_{i=1}^k$ with the standard unit vectors basis in R^k. In particular, if it is not specified otherwise,

$$l_2^k = (R^k, \|\cdot\|_2), \qquad \text{where} \quad \|x\|_2 = \left(\sum_{i=1}^k x(i)^2 \right)^{1/2} \quad \text{for } x = (x(1), \cdots, x(k)) \in R^k.$$

The symmetry group of X is denoted by $\mathcal{S}_k$, or by $\mathcal{S}$, if no confusion on the dimension of X can occur. The orbit of a vector $x \in R^k$ under the symmetry group is denoted by $\{Sx\}_{S \in \mathcal{S}}$. If $x \in R^k$, then x^* is defined as the unique element in $\{Sx\}_{S \in \mathcal{S}}$ such that $x^*(1) \geqq x^*(2) \geqq \cdots \geqq x^*(k) \geqq 0$.

Let $2 \leqq m_0 \leqq k$ and $a(1) \geqq \cdots \geqq a(m_0) > 0$. Define a norm $|||\cdot|||$ on R^k by

$$|||x||| = \sum_{i=1}^{m_0} x^*(i)a(i) \qquad \text{for } x \in R^k. \tag{1.1}$$

Then $(R^k, |||\cdot|||)$ is a symmetric space, in fact it is the Lorentz space $d(1, a)$. We shall need information about a form of extreme points of the unit ball in this space. To formulate the result, let us define vectors

$$f_m = \left(\sum_{i=1}^m a(i) \right)^{-1} \sum_{i=1}^m e_i \qquad \text{for } 1 \leqq m < m_0.$$

$$f_{m_0} = \left(\sum_{i=1}^{m_0} a(i) \right)^{-1} \sum_{i=1}^k e_i.$$

We have the following

LEMMA 1. *Every extreme point of the unit ball $\{x \in R^k \mid |||x||| \leqq 1\}$ is of a form Sf_m for some $S \in \mathcal{S}$ and $m = 1, \cdots, m_0$.*

PROOF. Let $x \in R^k$ with $|||x||| = 1$. Assume that $x \notin \mathcal{F} = \{Sf_m \mid S \in \mathcal{S}$ and $m = 1, \cdots, m_0\}$. We shall show that x can be written as a non-trivial convex combination of vectors from $\mathcal{F}$. Without loss of generality we may assume that $x = x^*$. It is easy to check that

$$x = \sum_{m=1}^{k-1} (x(m) - x(m+1)) \sum_{j=1}^{m} e_j + x(k) \sum_{j=1}^{k} e_j$$

$$= \sum_{m=1}^{m_0-1} (x(m) - x(m+1)) \left(\sum_{i=1}^{m} a(i) \right) f_m + \frac{1}{2} \sum_{j=m_0}^{k-1} (x(j) - x(j+1)) \left(\sum_{i=1}^{m_0} a(i) \right) f_{m_0}$$

$$+ \frac{1}{2} \sum_{j=m_0}^{k-1} (x(j) - x(j+1)) \left(\sum_{i=1}^{m_0} a(i) \right) \phi_j + x(k) \left(\sum_{i=1}^{m_0} a(i) \right) f_{m_0},$$

where

$$\phi_j = \left(\sum_{i=1}^{m_0} a(i) \right)^{-1} \left(\sum_{i=1}^{j} e_i - \sum_{i=j+1}^{k} e_i \right) \qquad \text{for } j = m_0, \cdots, k-1.$$

Since $\phi_j \in \mathscr{F}$ for $j = m_0, \cdots, k-1$, this shows that x is a combination of elements of $\mathscr{F}$ with sum of the coefficients equal to

$$\sum_{m=1}^{m_0-1} (x(m) - x(m+1)) \left(\sum_{i=1}^{m} a(i) \right) + \sum_{j=m_0}^{k-1} (x(j) - x(j+1)) \left(\sum_{i=1}^{m_0} a(i) \right)$$

$$+ x(k) \left(\sum_{i=1}^{m_0} a(i) \right) = |||\, x \,||| = 1.$$

Finally, since $x \notin \mathscr{F}$, it is easy to check that at least two coefficients in the convex combination are different from zero. This concludes the proof.

§2. Main construction

In this section we shall construct a family $\mathscr{R}$ of orthogonal $2^n \times 2^n$ matrices, which will be used to prove the distance estimate. Each matrix in this family is determined by three sequences of non-negative integers $d = (d_1, \cdots, d_{k_0})$, $\alpha = (\alpha_1, \cdots, \alpha_{k_0})$ and $b = (b_1, \cdots, b_{l_0})$ such that

$$d_1 + \alpha_1 \geqq \cdots \geqq d_{k_0} + \alpha_{k_0}, \quad b_1 \geqq \cdots \geqq b_{l_0} \quad \text{and} \quad \sum_{k=1}^{k_0} 2^{d_k + \alpha_k} = 2^n = \sum_{l=1}^{l_0} 2^{b_l}.$$

The construction is done in three steps.

First, given a non-negative integer n_0, define

$$W_{n_0} = [\omega_{ij}^{(n_0)}]_{i,j=1}^{2^{n_0}}$$

as a multiple of the usual $2^{n_0} \times 2^{n_0}$ Walsch matrix. Namely, put $W_0 = [1]$ and, by induction, if $n_0 \geqq 1$,

$$(2.1)\ \omega_{ij}^{(n_0)} = \begin{cases} \omega_{ij}^{(n_0-1)}, & 1 \leqq i,\ j \leqq 2^{n_0-1}, \\ \omega_{il}^{(n_0-1)}, & j = 2^{n_0-1}+l,\quad 1 \leqq i,\quad l \leqq 2^{n_0-1}, \\ \omega_{mk}^{(n_0-1)}, & i = 2^{n_0-1}+m,\quad 1 \leqq m,\quad j \leqq 2^{n_0-1}, \\ -\,\omega_{ml}^{(n_0-1)}, & i = 2^{n_0-1}+m,\quad j = 2^{n-1}+l,\quad 1 \leqq m,\quad l \leqq 2^{n_0-1}. \end{cases}$$

Graphically,

$$W_{n_0} = \left[\begin{array}{c|c} W_{n_0-1} & W_{n_0-1} \\ \hline W_{n_0-1} & -\,W_{n_0-1} \end{array}\right].$$

Next, given sequences of non-negative integers, $d = (d_1, \cdots, d_{k_0})$ and $\alpha = (\alpha_1, \cdots, \alpha_{k_0})$ such that $d_1 + \alpha_1 \geqq \cdots \geqq d_{k_0} + \alpha_{k_0}$ and $\Sigma_{k=1}^{k_0} 2^{d_k+\alpha_k} = 2^n$, define a $2^n \times 2^n$ matrix $T = T(d,\alpha) = [\tau_{ij}]_{i,j=1}^{2n}$ by induction with respect to k_0. If $k_0 = 1$, then

$$(2.2) \qquad \tau_{ij} = \begin{cases} 2^{d_1/2}\omega_{ml}^{(\alpha_1)}, & i = s2^{\alpha_1}+m,\quad j = s2^{\alpha_1}+l,\quad 1 \leqq m, \\ & l \leqq 2^{\alpha_1},\quad 0 \leqq s \leqq 2^{d_1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

Graphically,

$$T = \left[\begin{array}{c|c|c|c} A & & & \\ \hline & A & & \\ \hline & & \ddots & \\ \hline & & & A \end{array}\right]$$

where $A = 2^{d_1/2} W_{\alpha_1}$.

Let $k_0 > 1$ and assume that one can construct the matrices $T(d,\alpha)$ for sequences d and α of length smaller than k_0. Let d and α be sequences of length k_0 such that $d_1 + \alpha_1 \geqq \cdots \geqq d_{k_0} + \alpha_{k_0}$ and $\Sigma_{k=1}^{k_0} 2^{d_k+\alpha_k} = 2^n$. Since $k_0 > 1$ and a sum of at least two powers of 2 can be equal to a power of 2 only if some of the summands are equal, it follows that there is $1 \leqq k_1 < k_0$ such that

$$\sum_{k=1}^{k_1} 2^{d_k+\alpha_k} = 2^{n-1} = \sum_{k=k_1+1}^{k_0} 2^{d_k+\alpha_k}.$$

(This is also a consequence of Lemma 3 in §3.) Denote $\bar{d} = (d_1, \cdots, d_{k_1})$, $\bar{\bar{d}} = (d_{k_1+1}, \cdots, d_{k_0})$ and $\bar{\alpha} = (\alpha_1, \cdots, \alpha_{k_1})$, $\bar{\bar{\alpha}} = (\alpha_{k_1+1}, \cdots, \alpha_{k_0})$. Let

$$T_1 = T(\bar{d},\bar{\alpha}) = [\tau'_{ml}]_{m,l=1}^{2^{n-1}} \quad\text{and}\quad T_2 = T(\bar{\bar{d}},\bar{\bar{\alpha}}) = [\tau''_{ml}]_{m,l=1}^{2^{n-1}}$$

be matrices constructed by the inductive hypothesis. Then define $T = [\tau_{ij}]_{i,j=1}^{2^n}$ by

$$(2.3) \qquad \tau_{ij} = \begin{cases} \tau'_{ij}, & 1 \leqq i, \quad j \leqq 2^{n-1}, \\ \tau''_{il}, & j = 2^{n-1} + l, \quad 1 \leqq i, \quad l \leqq 2^{n-1}, \\ \tau'_{mj}, & i = 2^{n-1} + m, \quad 1 \leqq m, \quad j \leqq 2^{n-1}, \\ -\tau''_{ml}, & i = 2^{n-1} + m, \quad j = 2^{n-1+l}, \quad 1 \leqq m, \quad l \leqq 2^{n-1}. \end{cases}$$

Graphically,

$$T = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_1 & -T_2 \end{array} \right].$$

The next lemma states some simple properties of the matrix T.

Lemma 2. *Let $d = (d_1, \cdots, d_{k_0})$, $\alpha = (\alpha_1, \cdots, \alpha_{k_0})$ be sequences of non-negative integers such that $d_1\alpha_1 \geqq \cdots \geqq d_{k_0}\alpha_{k_0}$ and $\sum_{k=1}^{k_0} 2^{d_k + \alpha_k} = 2^n$. Let $T = T(d, \alpha)$. Then*

(1) *T is an orthogonal matrix;*

(2) *$\sum_{j=1}^{2^n} \tau_{ij}^2 = 2^n$ for $i = 1, \cdots, 2^n$;*

(3) *for every $1 \leqq i \leqq 2^n$ the vector $\sum_{j=1}^{2^n} \tau_{ij} e_j \in R^{2^n}$ belongs to the orbit $\{Su_0\}_{S \in \mathcal{S}}$, where*

$$u_0 = (\underbrace{2^{d_1/2}, \cdots, 2^{d_1/2}}_{2^{\alpha_1} \text{ times}}, \cdots, \underbrace{2^{d_{k_0}/2}, \cdots, 2^{d_{k_0}/2}}_{2^{\alpha_{k_0}} \text{ times}}, 0, \cdots, 0).$$

(4) *If $\alpha \leqq \min \alpha_k$, then, for every s and s' with $0 \leqq s$, $s' < 2^{n-\alpha}$, the $2^\alpha \times 2^\alpha$ matrix $[\tau'_{ij}]_{i,j=1}^{2^\alpha}$ defined by*

$$\tau'_{ij} = \tau_{s2^\alpha + i, s'2^\alpha + j} \qquad \text{for } i, j = 1, \cdots, 2^\alpha$$

is a multiple of the matrix W_α.

Proof. Obvious induction.

Finally, let $b = (b_1, \cdots, b_{l_0})$ be a sequence of non-negative integers such that $b_1 \geqq \cdots \geqq b_{l_0}$ and $\sum_{l=1}^{l_0} 2^{b_l} = 2^n$. A matrix $R = R(d, \alpha, b) = [\rho_{ij}]_{i,j=1}^{2^n}$ will be defined by crossing out some entries of the matrix $T(d, \alpha)$ and multiplying rows by appropriate factors. Namely, for $i = \sum_{s=1}^{l-1} 2^{b_s} + m$, for some $m = 1, \cdots, 2^{b_l}$ and $1 \leqq l \leqq l_0$, put

$$J_i = \{m, 2^{b_l} + m, 2^{b_l+1} + m, 2^{b_l+2} + m, \cdots, 2^n - 2^{b_l} + m)$$

and define

$$\rho_{ij}=\begin{cases}2^{b_l/2}\tau_{ij} & \text{for } j\in J_i,\\ \\ 0 & \text{otherwise.}\end{cases}\tag{2.4}$$

In the case when $b_1 \leqq \min_{1\leqq k\leqq k_0}\alpha_k$, the matrix R has many useful properties, which are formulated in the next proposition.

PROPOSITION 1. *Let $d=(d_1,\cdots,d_{k_0})$, $\alpha=(\alpha_1,\cdots,\alpha_{k_0})$ and $b=(b_1,\cdots,b_{l_0})$ be sequences of non-negative integers such that $d_1+\alpha_1\geqq\cdots\geqq d_{k_0}+\alpha_{k_0}$, $b_1\geqq\cdots\geqq b_{l_0}$ and $\sum_{k=1}^{k_0}2^{d_k+\alpha_k}=\sum_{l=1}^{l_0}2^{b_l}=2^n$ for some non-negative integer n. Assume that $b_1\leqq\min_{1\leqq k\leqq k_0}\alpha_k$. Let $R=R(d,\alpha,b)$ be the matrix defined in* (2.4). *Then*

(1) *R is orthogonal matrix*;

(2) *$\sum_{j=1}^{2^n}\rho_{ij}^2=2^n$ for $i=1,\cdots,2^n$*;

(3) *for every $1\leqq l\leqq l_0$, if $I_l=\{\sum_{s=1}^{l-1}2^{b_s}+1,\cdots,\sum_{s=1}^{l}2^{b_s}\}$, the vector $\sum_{i\in I_l}(\sum_{j=1}^{2^n}\rho_{ij}e_j)$ belongs to the orbit $2^{b_l/2}\{Su_0\}_{S\in\mathcal{S}}$, where*

$$u_0=(\underbrace{2^{d_1/2},\cdots,2^{d_1/2}}_{2^{\alpha_1}\text{ times}},\cdots,\underbrace{2^{d_{k_0}/2},\cdots,2^{d_{k_0}/2}}_{2^{\alpha_{k_0}}\text{ times}},0\cdots0).\tag{2.5}$$

Before we prove Proposition 1 let us state an important consequence.

COROLLARY 1. *Let d, α and b be as in Proposition* 1. *Let $u_0\in R^{2^n}$ be defined by* (2.5) *and let $v_0=(2^{b_1/2},\cdots,2^{b_{l_0}/2},0\cdots0)\in R^{2^n}$. Let $U:R^{2^n}\to R^{2^n}$ be the operator such that $Ue_i=2^{-n/2}\sum_{j=1}^{2^n}\rho_{ij}e_j$ $(i=1,\cdots,2^n)$. Then U acts as a unitary operator on $l_2^{2^n}$. Moreover, if X and Y are 2^n-dimensional symmetric spaces then*

$$\|Ux\|_Y\leqq 2^{-n/2}\|u_0\|_Y\|v_0\|_{X^*}\|x\|_X\qquad \textit{for } x\in R^{2^n}.\tag{2.6}$$

PROOF. Consider a norm $|||\cdot|||$ on R^{2^n} defined by

$$|||x|||=\sum_{l=1}^{l_0}\sup_{i\in I_l}|x(i)|2^{b_l/2}\qquad\text{for } x\in R^{2^n},$$

where I_l are the subsets defined in condition (3) of the proposition. Since the extreme points of the unit ball in $|||\cdot|||$ are of the form $2^{-b_l/2}\sum_{i\in I_l}\pm e_i$ $(l=1,\cdots,l_0)$, it follows from the proposition that if y is an extreme point, then $\|Uy\|_Y=2^{-n/2}\|u_0\|_Y$. Therefore,

$$\|Ux\|_Y\leqq 2^{-n/2}\|u_0\|_Y\,|||x|||,\qquad\text{for } x\in R^{2^n}.$$

On the other hand, for every $x\in R^{2^n}$ there is $\tilde v$ in the orbit of v_0 such that $|||x|||=\langle x,\tilde v\rangle$. Therefore,

$$|||x|||=\langle x,\tilde v\rangle\leqq\|v_0\|_{X^*}\|x\|_X,\qquad\text{for } x\in R^{2^n}.$$

Combining these two inequalities together one derives (2.6).

PROOF OF PROPOSITION 1. First note that property (3) follows directly from Lemma 2, (3) and (4). To prove (1) and (2), fix sequences d and α. Denote by $\mathcal{B}$ the set of all sequences of integers, $b=(b_1,\cdots,b_{l_0})$, such that

$$\min_{1\leqq k\leqq k_0}\alpha_k\geqq b_1\geqq b_2\geqq\cdots\geqq b_{l_0}\geqq 0,\qquad \sum_{l=1}^{l_0}2^{b_l}=2^n.$$

For $b\in\mathcal{B}$ as above, denote $l_0=|b|$.

We shall proceed by downward induction on $|b|$, i.e. starting with $|b|=2^n$. Clearly, there is only one element $b\in\mathcal{B}$ with $|b|=2^n$, namely

$$\mathbf{0}=\underbrace{(0,\cdots,0)}_{2^n\text{ times}}.$$

Since $R(d,\alpha,\mathbf{0})=T(d,\alpha)$, (1) and (2) are satisfied by Lemma 2.

Let now $b=(b_1,\cdots,b_{l_0})\in\mathcal{B}$ with $l_0<2^n$ and assume that for all $b'\in\mathcal{B}$ with $|b'|>|b|$ the conditions (1) and (2) are satisfied. Let $l_1=\max\{l\mid b_l>0\}$ and let

$$b'=(b_1,b_2,\cdots,b_{l_1}-1,b_{l_1}-1,b_{l_1+1},\cdots,b_{l_0}).$$

Thus $|b'|=|b|+1$ and, by the inductive hypothesis, $R'=R(d,\alpha,b')$ satisfies (1) and (2). Let

$$R=R(d,\alpha,b)=[\rho_{ij}]_{i,j=1}^{2^n}\quad\text{and}\quad R'=[\rho'_{ij}]_{i,j=1}^{2^n}.$$

We claim that, for every $1\leqq k,\ m\leqq 2^n$,

$$\sum_{i=1}^{2^n}\rho_{ik}\rho_{im}=\sum_{i=1}^{2^n}\rho'_{ik}\rho'_{im}. \tag{2.7}$$

This, obviously, implies that R satisfies (1) and (2) since R' does. Set $B=\sum_{s=1}^{l_1-1}2^{b_s}$ and $\beta=b_{l_1}$. Since $\rho_{ij}=\rho'_{ij}$, unless $B<i\leqq B+2^\beta$ $(1\leqq i,j\leqq 2^n)$, (2.7) is equivalent to

$$\sum_{i=B+1}^{B+2^\beta}\rho_{ik}\rho_{im}=\sum_{i=B+1}^{B+2^\beta}\rho'_{ik}\rho'_{im}. \tag{2.8}$$

Let $T(d,\alpha)=[\tau_{ij}]_{i,j=1}^{2^n}$. For $j=1,\cdots,2^n$ define vectors C_j, A_j, $A'_j\in R^{2^\beta}$ by

$$C_j=(\tau_{B+1,j},\tau_{B+2,j},\cdots,\tau_{B+2^\beta,j}),$$

$$A_j=(\rho_{B+1,j},\rho_{B+2,j},\cdots,\rho_{B+2^\beta,j}),$$

$$A'_j=(\rho'_{B+1,j},\rho'_{B+2,j},\cdots,\rho'_{B+2^\beta,j}).$$

Fix $1 \leqq k,\ m \leqq 2^n$. By Lemma 2, (4), there exist integers $1 \leqq i_1,\ i_2 \leqq 2^\beta$ (actually, $i_1 = k \bmod 2^\beta$, $i_2 = m \bmod 2^\beta$) and positive numbers K_1 and K_2 such that

$$C_k = K_1(\omega^{(\beta)}_{1,i_1}, \omega^{(\beta)}_{2,i_1}, \cdots, \omega^{(\beta)}_{2^\beta,i_1}),$$

$$C_m = K_2(\omega^{(\beta)}_{1,i_2}, \omega^{(\beta)}_{2,i_2}, \cdots, \omega^{(\beta)}_{2^\beta,i_2}).$$

By the definition of matrices $R(d,\alpha,b)$,

$$A_k = 2^{\beta/2}(0,\cdots,0,\tau_{B+i_1,k},0,\cdots,0),$$

$$A_m = 2^{\beta/2}(0,\cdots,0,\tau_{B+i_2,m},0,\cdots,0).$$

Consequently, $\langle A_k, A_m\rangle = 0$ if $i_1 \neq i_2$ and $\langle A_k, A_m\rangle = 2^\beta K_1K_2$ if $i_1 = i_2$.

The case of A'_1, A'_m is slightly more complicated. Let $i'_1 = i_1 \bmod 2^{\beta-1}$ and $i'_2 = i_2 \bmod 2^{\beta-1}$. Then

$$A'_k = 2^{(\beta-1)/2}(0,\cdots,0,\tau_{B+i'_1,k},0,\cdots,0,\tau_{B+2^{\beta-1}+i'_1,k},0,\cdots,0),$$

$$A'_m = 2^{(\beta-1)/2}(0,\cdots,0,\tau_{B+i'_2,m},0,\cdots,0,\tau_{B+2^{\beta-1}+i'_2,m},0,\cdots,0).$$

Thus $\langle A'_k, A'_m\rangle = 0$ if $i'_1 \neq i'_2$. If $i_1 = i_2$, then

$$\langle A'_k, A'_m\rangle = 2^{\beta-1}2K_1K_2 = \langle A_k, A_m\rangle.$$

Finally, if $i'_1 = i'_2$ but $i_1 \neq i_2$, then $|i_2 - i_1| = 2^{\beta-1}$. Assume, without loss of generality, that $i_1 < i_2$, thus $i_2 = i_1 + 2^{\beta-1}$. By the definition of the Walsh matrices one has

$$\tau_{B+i'_1,k} = \tau_{B+2^{\beta-1}+i'_1,k} \quad \text{and} \quad \tau_{B+i'_2,m} = -\tau_{B+2^{\beta-1}+i'_2,m}.$$

Therefore, $\langle A'_k, A'_m\rangle = 0$.

It shows that for every $1 \leqq k,\ m \leqq 2^n$,

$$\begin{aligned}\sum_{i=B+1}^{B+2^\beta} \rho_{ik}\rho_{im} &= \langle A_k, A_m\rangle \\ &= \langle A'_k, A'_m\rangle \\ &= \sum_{i=B+1}^{B+2^\beta} \rho'_{ik}\rho'_{im},\end{aligned}$$

concluding the proof.

§3. The distance between symmetric spaces

The main theorem in this section says

THEOREM 1. *Let n be a non-negative integer: Let E, F be symmetric spaces with* $\dim E = \dim F = 2^n$. *Then*

$$(3.1) \qquad d(E,F) \leqq 2^{12+n/2}.$$

The case of general dimension is a formal consequence of Theorem 1.

THEOREM 2. *Let E, F be symmetric spaces,* $\dim E = \dim F = N$. *Then*

$$(3.2) \qquad d(E,F) \leqq 2^{25/2}(2^{1/4}-1)^{-2}\sqrt{N}.$$

PROOF OF THEOREM 2. Write $N = \sum_{i=1}^{i_0} 2^{k_i}$ with $k_1 > k_2 > \cdots > k_{i_0} \geqq 0$. Let $\{1,2,\cdots,N\} = I_1 \cup \cdots \cup I_{i_0}$ be a decomposition into disjoint subsets such that $|I_i| = 2^{k_i}$ $(i = 1,\cdots,i_0)$. Define

$$E_i = (\operatorname{span}(e_m)_{m\in I_i}, \|\cdot\|_E) \quad \text{and} \quad F_i = (\operatorname{span}(e_m)_{m\in I_i}, \|\cdot\|_F)$$

and let $T_i : E_i \to F_i$ be an isomorphism such that $\|T_i\| = \|T_i^{-1}\| = \sqrt{d(E_i,F_i)}$, for $i = 1,\cdots,i_0$. Then define an isomorphism $T: E \to F$ by

$$Tx = \sum_{i=1}^{i_0} T_i\Big(\sum_{m\in I_i} x(m)e_m\Big),$$

for $x = \sum_{k=1}^N x(k)e_k \in R^N$. Obviously, for $y = \sum_{k=1}^N y(k)e_k \in R^N$,

$$T^{-1}y = \sum_{i=1}^{i_0} T_i^{-1}\Big(\sum_{m\in I_i} y(m)e_m\Big).$$

Therefore, by (3.1),

$$\|T\| \leqq \sum_{i=1}^{i_0} \|T_i\| \leqq 2^6 \sum_{i=1}^{i_0} 2^{k_i/4} \leqq 2^{25/4}(2^{1/4}-1)N^{1/4}.$$

Similarly,

$$\|T^{-1}\| \leqq 2^{25/4}(2^{1/4}-1)^{-1}N^{1/4}.$$

So

$$d(E,F) \leqq \|T\|\,\|T^{-1}\| \leqq 2^{25/2}(2^{1/4}-1)^{-2}\sqrt{N}.$$

Before we prove Theorem 1 we need some notation and a few technical lemmas on sequences of positive numbers.

LEMMA 3. *Let k be a positive integer and let $b_1 \geqq \cdots \geqq b_M \geqq 0$ be integers such that $\sum_{l=1}^M 2^{b_l} \geqq 2^k$ and $b_1 \leqq k$. Then there exists $1 \leqq l_0 \leqq M$ such that $\sum_{l=1}^{l_0} 2^{b_l} = 2^k$. In particular, if $\sum_{l=1}^M 2^{b_l} = 2^{k+1}$ and $b_1 \leqq k$, then there exists $1 \leqq l_0 < M$ such that $\sum_{l=1}^{l_0} 2^{b_l} = 2^k = \sum_{l=l_0+1}^M 2^{b_l}$.*

PROOF. Obviously, it is enough to prove only the first part of the lemma. Put $l_0 = \max\{l \mid \sum_{j=1}^{l} 2^{b_j} \leqq 2^k\}$. If one had $\sum_{j=1}^{l_0} 2^{b_j} < 2^k$, then $2^{b_{l_0+1}}$ would divide

$2^k - \Sigma_{j=1}^{l_0} 2^{b_j}$. In particular, $2^{b_{l_0+1}} \leqq 2^k - \Sigma_{j=1}^{l_0} 2^{b_j}$, which would contradict the definition of l_0. Therefore $\Sigma_{j=1}^{l_0} 2^{b_j} = 2^k$.

In the sequel we fix a non-negative integer n. We define

$$\mathcal{D} = \left\{ u = (2^{b_1/2}, \cdots, 2^{b_M/2}, 0, \cdots, 0) \in R^{2^n} \;\middle|\; b_1 \geqq \cdots \geqq b_M \geqq 0 \text{ are integers} \right.$$
$$\left. \text{and } \sum_{l=1}^{M} 2^{b_l} = 2^n \right\}.$$

For $u = (u(1), \cdots, u(2^n)) \in \mathcal{D}$ let $h(u)$ be the positive integer such that $\Sigma_{i=1}^{h(u)} u(i)^2 = 2^{n-1}$. Then we define vectors u' and u'' by

$$\begin{aligned} u' &= \sqrt{2}((u(1)), \cdots, u(h(u)), 0, \cdots, 0) \in R^{2^n}, \\ u'' &= \sqrt{2}(u(h(u)+1), \cdots, u(2^n), 0, \cdots, 0) \in R^{2^n}. \end{aligned} \tag{3.3}$$

Obviously, u' and u'' belong to $\mathcal{D}$.

Finally, we introduce on $\mathcal{D}$ a relation $\succ$. If $u = (2^{b_1/2}, \cdots, 2^{b_M/2}) \in \mathcal{D}$ and $v = (2^{c_1/2}, \cdots, 2^{c_N/2}, 0 \cdots 0) \in \mathcal{D}$, we say that $u \succ v$, if $b_M \geqq c_1$.

LEMMA 4. *Let u, $v \in \mathcal{D}$. Then either $u' \succ v''$ or $v' \succ u''$.*

PROOF. It is easy to see that the condition $u(h(u)) \geqq v(h(v))$ implies $u' \succ v''$ while the condition $u(h(u)) \leqq v(h(v))$ implies $v' \succ u''$.

An importance of the set $\mathcal{D}$ lies in the following easy lemma.

LEMMA 5. *Let $x(1) \geqq \cdots \geqq x(2^n) \geqq 0$ satisfy $\Sigma_{i=1}^{2^n} x(i)^2 = 2^n$. Then there exists $u = (u(1), \cdots, u(2^n)) \in \mathcal{D}$ such that $u(i) \leqq 2x(i)$ for $i = 1, \cdots, 2^n$.*

PROOF. Put $i_0 = \max\{i \mid x(i) \geqq 1/\sqrt{2}\}$. Then $\Sigma_{i=1}^{i_0} x(i)^2 \geqq 2^{n-1}$. For every $1 \leqq i \leqq i_0$ let c_i be the integer such that $x(i)^2 < 2^{c_i} \leqq 2x(i)^2$. Since $\Sigma_{i=1}^{i_0} 2^{c_i} \geqq 2^{n-1}$, there exists i_0' such that $\Sigma_{i=1}^{i_0'} 2^{c_i} = 2^{n-1}$. Put $u(i) = 2^{(c_i+1)/2}$ for $1 \leqq i \leqq i_0'$ and $u(i) = 0$ for $i_0' < i \leqq 2^n$. Obviously $u = (u(1), \cdots, u(2^n))$ satisfies the hypothesis of the lemma.

Now we are prepared to prove Theorem 1.

PROOF OF THEOREM 1. Pick vectors $\tilde{x}_0$, $\tilde{y}_0$, $\tilde{z}_0$, $\tilde{w}_0 \in R^{2^n}$ such that $\|\tilde{x}_0\|_2 = \|\tilde{y}_0\|_2 = \|\tilde{z}_0\|_2 = \|\tilde{w}_0\|_2 = 2^{n/2}$ and

$$\begin{aligned} \|\tilde{x}_E\|_E &= \min\{\|x\|_E \mid x \in R^{2^n}, \|x\|_2 = 2^{n/2}\}, \\ \|\tilde{y}_0\|_{E^*} &= \min\{\|y\|_{E^*} \mid y \in R^{2^n}, \|y\|_2 = 2^{n/2}\}; \end{aligned} \tag{3.4}$$

$$\| \tilde{z}_0 \|_F = \min\{\| z \|_F \mid z \in R^{2^n}, \| z \|_2 = 2^{n/2}\},$$
$$(3.5) \qquad \| \tilde{w}_0 \|_{F^*} = \min\{\| w \|_{F^*} \mid w \in R^{2^n}, \| w \|_2 = 2^{n/2}\}.$$

These vectors will be used to estimate the norms of some operators.

LEMMA 6. *Let* $\tilde{x}_0, \tilde{y}_0, \tilde{z}_0, \tilde{w}_0 \in R^{2^n}$ *satisfy* $\| \tilde{x}_0 \| = \| \tilde{y}_0 \| = \| \tilde{z}_0 \| = \| \tilde{w}_0 \| = 2^{n/2}$ *and* (3.4) *and* (3.5). *Let* $V : R^{2^n} \to R^{2^n}$ *act as a unitary operator on* $l_2^{2^n}$. *Then*

$$\| V : F \to E \| \leqq 2^n / \| \tilde{y}_0 \|_{E^*} \| \tilde{z}_0 \|_F,$$
$$\| V : E \to F \| \leqq 2^n / \| \tilde{x}_0 \|_E \| \tilde{w}_0 \|_{F^*}.$$

PROOF. Obviously it is enough to prove the first estimate only. The second will follow by changing the role of E and F.

Clearly, if Id denotes the formal identity operator, then

$$\| \mathrm{Id} : E^* \to l_2^{2^n} \| = 2^{n/2} / \| \tilde{y}_0 \|_{E^*},$$
$$\| \mathrm{Id} : F \to l_2^{2^n} \| = 2^{n/2} / \| \tilde{z}_0 \|_F.$$

Therefore,

$$\begin{aligned} \| V : F \to E \| &\leqq \| \mathrm{Id} : F \to l_2^{2^n} \| \, \| V : l_2^{2^n} \to l_2^{2^n} \| \, \| \mathrm{Id} : l_2^{2^n} \to E \| \\ &\leqq (2^{n/2} / \| \tilde{y}_0 \|_{E^*})(2^{n/2} / \| \tilde{z}_0 \|_F) \\ &= 2^n / \| \tilde{y}_0 \|_{E^*} \| \| \tilde{z}_0 \|_F. \end{aligned}$$

This concludes the proof.

By Lemma 5 there exist vectors $\bar{x}_0, \bar{y}_0, \bar{z}_0, \bar{w}_0$ in $\mathcal{D}$ such that

$$\bar{x}_0(i) \leqq 2\tilde{x}_0(i), \qquad \bar{y}_0(i) \leqq 2\tilde{y}_0(i),$$
$$(3.6) \qquad \bar{z}_0(i) \leqq 2\tilde{z}_0(i), \qquad \bar{w}_0(i) \leqq 2\tilde{w}_0(i) \quad \text{for } i = 1, 2, \cdots, 2^n.$$

By Lemma 4 one has $\bar{x}_0' > \bar{y}_0''$ or $\bar{y}_0' > \bar{x}_0''$. Since $d(E, F) = d(E^*, F^*)$ and one may consider, if necessary, the pair E^*, F^* instead of E, F, then without loss of generality one may assume that $\bar{x}_0' > \bar{y}_0''$.

LEMMA 7. *Let* $\bar{x}_0, \bar{y}_0, \bar{z}_0, \bar{w}_0$ *be vectors in* $\mathcal{D}$ *such that* $\bar{x}_0' > \bar{y}_0''$. *Then there exist vectors* $x_0, y_0, z_0, w_0 \in \mathcal{D}$ *such that*

$$x_0(i) \leqq 2\bar{x}_0(i), \qquad y_0(i) \leqq 2\bar{y}_0(i) \qquad \textit{for } i = 1, \cdots, 2^n,$$
$$(3.7) \qquad z_0(i) \leqq 2\bar{z}_0(i), \qquad w_0(i) \leqq 2\bar{w}_0(i),$$

and satisfying at least one of the following conditions:

(3.8.i) $$x_0 > y_0 \quad and \quad z_0 > w_0 \quad and \quad y_0 > w_0,$$

(3.8.ii) $$x_0 > y_0 \quad and \quad z_0 > w_0 \quad and \quad w_0 > y_0,$$

(3.8.iii) $$x_0 > y_0 \quad and \quad w_0 > z_0.$$

PROOF. If $\bar{w}_0(h(\bar{w}_0)) \geqq \bar{z}_0(h(\bar{z}_0))$, then the vectors $x_0 = \bar{x}'_0$, $y_0 = \bar{y}''_0$, $z_0 = \bar{z}''_0$, $w_0 = \bar{w}'_0$ obviously satisfy (3.7) and (3.8.iii). Assume now that $\bar{z}_0(h(\bar{z}_0)) \geqq \bar{w}_0(h(\bar{w}_0))$, then $\bar{z}'_0 > \bar{w}''_0$. Consider the vectors $u_0 = \bar{y}''_0$ and $v_0 = \bar{w}''_0$. If $u_0(h(u_0)) \geqq v_0(h(v_0))$, then $u'_0 > v''_0$. In this case define $x_0 = \bar{x}'_0$, $z_0 = \tilde{z}'_0$, where $\tilde{z}_0 = \bar{z}'_0$, and $y_0 = u'_0$, $w_0 = v''_0$. Clearly, these vectors satisfy (3.7) and (3.8.i). If $v_0(h(v_0)) \geqq u_0(h(u_0))$, then the vectors $x_0 = x'_0$, $z_0 = z'_0$ and $y_0 = u''_0$, $w_0 = v'_0$ satisfy (3.7) and (3.8.ii). This concludes the proof of the lemma.

Case I.

(3.9) $$x_0(h(x_0))w_0(h(w_0)) \leqq 2^{(n-4)/2}$$

or

(3.10) $$z_0(h(z_0))y_0(h(y_0)) \leqq 2^{(n-4)/2}.$$

Observe first that by relabelling we may assume that (3.9) holds. We shall construct an operator $U: R^{2^n} \to R^{2^n}$, which acts as a unitary operator on $l_2^{2^n}$, such that

(3.10) $$\| U : F \to E \| \leqq 2^{(11-n)/2} \| \tilde{x}_0 \|_E \, \| \tilde{w}_0 \|_{F^*}.$$

Then we shall complete the proof applying Lemma 6 to the operator $V = U^{-1}$:

$$\begin{aligned} d(E,F) &\leqq \| U : F \to E \| \, \| U^{-1} : E \to F \| \\ &\leqq (2^{(11-n)/2} \| \tilde{x}_0 \|_E \, \| \tilde{w}_0 \|_{F^*})(2^n / \| \tilde{x}_0 \|_E \, \| \tilde{w}_0 \|_{F^*}) \\ &= 2^{(n+11)/2}. \end{aligned}$$

The required operator U will be constructed by means of Corollary 1. To apply this corollary, we need the following lemma.

LEMMA 8. *Let $\tilde{u} \in \mathcal{D}$ and assume that $\tilde{u}(1) \leqq 2^{(n-2)/2}$. Then there exists a subset $I \subset \{1, \cdots, 2^n\}$ such that*

(i) $\Sigma_{i \in I} \tilde{u}(i)^2 = 2^{n-1}$;

(ii) *if $\tilde{u}_I \in R^{2^n}$ is defined by*

$$\tilde{u}_I(i)=\begin{cases}\sqrt{2}\,\tilde{u}(i) & \text{if } i\in I,\\ 0 & \text{if } i\notin I,\end{cases}$$

then a certain rearrangement u_0 *of* $\tilde{u}_I$ *can be written in a form*

$$u_0=(\underbrace{2^{d_1/2},\cdots,2^{d_1/2}}_{2^{\alpha_1}\text{ times}},\cdots,\underbrace{2^{d_N/2},\cdots,2^{d_N/2}}_{2^{\alpha_N}\text{ times}},0\cdots0),$$

with $d_1+\alpha_1\geqq\cdots\geqq d_N+\alpha_N$ *and*

$$2^{n-2}/\tilde{u}(1)^2\leqq\min_{1\leqq\nu\leqq N}2^{\alpha_\nu}. \tag{3.11}$$

Assuming the truth of Lemma 8 we complete the construction of U as follows. Let $u=x_0''$ and let $I\subset\{1,\cdots,2^n\}$ and u_0 be as in Lemma 8. Put $v_0=w_0''$. By (3.9) one has

$$v_0(1)=\sqrt{2}\,w_0(h(w_0))\leqq 2^{(n-2)/2}/\tilde{u}(1).$$

It follows from (3.11) that u_0 and v_0 satisfy the assumptions of Corollary 1. Let U be the operator defined in this corollary. Notice that since

$$\tilde{u}_I(i)\leqq\sqrt{2}\,x_0''(i)\leqq 2x_0(i)\quad\text{and}\quad v_0(i)\leqq\sqrt{2}\,w_0(i)\qquad\text{for all } i=1,\cdots,2^n,$$

it follows that

$$\|u_0\|_E=\|\tilde{u}_0\|_E\leqq 2\|x_0\|_E\quad\text{and}\quad\|v_0\|_{F^*}\leqq\sqrt{2}\|w_0\|_{F^*},$$

hence, by (3.7) and (3.6),

$$\|u_0\|_E\leqq 8\|\tilde{x}_0\|_E\quad\text{and}\quad\|v_0\|_{F^*}\leqq 4\sqrt{2}\|\tilde{w}_0\|_{F^*}.$$

Therefore, by (2.6).

$$\begin{aligned}\|Ux\|_E&\leqq 2^{-n/2}\|u_0\|_E\|v_0\|_{F^*}\|x\|_F\\&\leqq 2^{(11-n)/2}\|\tilde{x}_0\|_E\|\tilde{w}_0\|_{F^*}\|x\|_F,\end{aligned}$$

for all $x\in R^{2^n}$. This shows (3.10).

PROOF OF LEMMA 8. Write $\tilde{u}$ in a form

$$\tilde{u}=(\underbrace{2^{s_1/2},\cdots,2^{s_1/2}}_{2^{\beta_1}\text{ times}},\cdots,\underbrace{2^{s_M/2},\cdots,2^{s_M/2}}_{2^{\beta_M}\text{ times}},0,\cdots,0),$$

with $s_1 \geqq \cdots \geqq s_M \geqq 0$ and if $s_\mu = s_{\mu+1}$, then $\beta_\mu > \beta_{\mu+1}$ for all $1 \leqq \mu \leqq M$. Put $\mathcal{M}' = \{1 \leqq \mu \leqq M \mid 2^{\beta_\mu} \geqq 2^{n-2}/\tilde{u}(1)^2\}$. We shall show that

$$\sum_{\mu \in \mathcal{M}'} 2^{s_\mu + \beta_\mu} > 2^{n-1}. \tag{3.12}$$

Therefore, by Lemma 3 applied to $b_l = s_\mu + \beta_\mu$, there exists $\mathcal{M} \subset \mathcal{M}'$ such that $\sum_{\mu \in \mathcal{M}} 2^{s_\mu + \beta_\mu} = 2^{n-1}$. It is easy to see that the set

$$I = \bigcup_{\mu \in \mathcal{M}} \left\{ \sum_{\nu=1}^{\mu-1} 2^{\beta_\nu} + 1, \sum_{\nu=1}^{\mu-1} 2^{\beta_\nu} + 2, \cdots, \sum_{\nu=1}^{\mu} 2^{\beta_\mu} \right\}$$

satisfies the conclusion of the lemma.

It remains to prove (3.12). Let $K \in \{1, \cdots, M\} \setminus \mathcal{M}'$ be a set such that:

(1) for every $\nu \in \{1, \cdots, M\} \setminus \mathcal{M}'$ there exists $\mu \in K$ such that $s_\mu = s_\nu$,

(2) if $\mu_1, \mu_2 \in K$ and $\mu_1 \neq \mu_2$, then $s_{\mu_1} \neq s_{\mu_2}$.

For every $\mu \in K$ let $\mathcal{N}_\mu = \{\nu \notin \mathcal{M}' \mid s_\nu = s_\mu\}$. Notice that $\{1, \cdots, M\} \setminus \mathcal{M}' = \bigcup_{\mu \in K} \mathcal{N}_\mu$ and that if $\nu_1, \nu_2 \in \mathcal{N}_\mu$ and $\nu_1 \neq \nu_2$, then $\beta_{\nu_1} \neq \beta_{\nu_2}$ and both β_{ν_1} and β_{ν_2} are smaller than $2^{n-2}/\tilde{u}(1)^2$ ($\mu \in K$). Thus, since $2^{n-2}/\tilde{u}(1)^2$ is a diadic number, one has

$$\begin{aligned}
\sum_{\mu \notin \mathcal{M}'} 2^{s_\mu + \beta_\mu} &= \sum_{\mu \in K} 2^{s_\mu} \sum_{\nu \in \mathcal{N}_\mu} 2^{\beta_\nu} \\
&< \sum_{\mu \in K} 2^{s_\mu} (2^{n-2}/\tilde{u}(1)^2) \\
&< \left(2 \max_{\mu \in K} 2^{s_\mu}\right) (2^{n-2}/\tilde{u}(1)^2) \\
&\leqq (2\tilde{u}(1)^2)(2^{n-2}/\tilde{u}(1)^2) \\
&= 2^{n-1}.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{\mu \in \mathcal{M}'} 2^{s_\mu + \beta_\mu} &= \sum_{\mu=1}^{M} 2^{s_\mu + \beta_\mu} - \sum_{\mu \notin \mathcal{M}'} 2^{s_\mu + \beta_\mu} \\
&> \| \tilde{u} \|_2^2 - 2^{n-1} \\
&= 2^{n-1}.
\end{aligned}$$

This shows (3.12) and concludes the proof of Lemma 8.

Case II. Assume that

$$x_0(h(x_0))w_0(h(w_0)) > 2^{(n-4)/2} \quad \text{and} \quad z_0(h(z_0))y_0(h(y_0)) > 2^{(n-4)/2}. \tag{3.13}$$

LEMMA 9. *Let $u, v \in \mathscr{D}$. Assume that $v > u$. Then for any 2^n-dimensional symmetric space X one has*

$$\frac{\| v'' \|_X}{\| u' \|_X} \leqq v(h(v))/u(h(u)),$$

where u' and v'' are defined by formula (3.3) *applied to the vectors u and v respectively.*

PROOF. Notice that since $v > u$ then, given $1 \leqq m \leqq 2^n$, if $v(m) \neq 0$, then $v(m) \geqq u(i)$ for all $1 \leqq i \leqq 2^n$. It follows that u' has more non-zero coordinates than v'' has. Moreover, for all $1 \leqq i,\ m \leqq 2^n$, if $u'(i) \neq 0$, then

$$u'(i) \geqq \frac{u(h(u))}{v(h(v))} v''(m).$$

It follows that if X is a 2^n-dimensional symmetric space, then

$$\| u' \|_X \geqq \frac{u(h(u))}{v(h(v))} \| v'' \|_X.$$

This concludes the proof.

We are ready now to prove the theorem in Case II. We shall show that in this case the estimate for the distance $d(E, F)$ can be obtained by considering the formal identity operators. Recall that at least one of conditions (3.8) of Lemma 7 is satisfied. Assume first that either (3.8.i) or (3.8.ii) holds. Then, by relabelling, one may assume that (3.8.i) holds.

Let x_0' and w_0' be vectors defined by formula (3.3) applied to the vectors x_0 and w_0 respectively. Consider a Lorentz norm $||| \cdot |||$ defined on R^{2^n} by

$$||| x ||| = \sum_{k=1}^{2^n} x^*(k) w_0'(k) \qquad \text{for } x \in R^{2^n}.$$

Since $w_0'(k) \leqq \sqrt{2}\, w_0(k) \leqq 4\sqrt{2}\, \tilde{w}_0(k)$ for $k = 1, \cdots, 2^n$, one obviously has

$$||| x ||| \leqq \| w_0' \|_{F^*} \| x \|_F \leqq 4\sqrt{2} \| \tilde{w}_0 \|_{F^*} \| x \|_F \qquad \text{for } x \in R^{2^n}. \tag{3.14}$$

We shall show that

$$\| x \|_E \leqq 2^{(11-n)/2} \| \tilde{x}_0 \|_E\, ||| x ||| \qquad \text{for } x \in R^{2^n}. \tag{3.15}$$

Assuming the truth of (3.15) and combining it with (3.14) we shall get

$$\| \mathrm{Id} : F \to E \| \leqq 2^{(16-n)/2} \| \tilde{x}_0 \|_E \, \| \tilde{w}_0 \|_{F^*}.$$

Therefore, by Lemma 6,

$$\begin{aligned} d(E,F) &\leqq \| \mathrm{Id}: F \to E \| \, \| \mathrm{Id}: E \to F \| \\ &\leqq (2^{(16-n)/2} \| \tilde{x}_0 \|_E \, \| \tilde{w}_0 \|_{F^*})(2^n / \| \tilde{x}_0 \|_E \, \| \tilde{w}_0 \|_{F^*}) \\ &= 2^{(16+n)/2}. \end{aligned}$$

In the proof of (3.15) one needs an estimate for the norm $\| \cdot \|_E$ in terms of a new Lorentz norm $| \cdot |$. Define

$$| u | = \sum_{j=1}^{2^n} u^*(j) x_0'(j) \qquad \text{for } u \in R^{2^n}.$$

Analogously to (3.14) one has

$$| u | \leqq 4 \sqrt{2} \| \tilde{x}_0 \|_E \, \| u \|_{E^*} \qquad \text{for } u \in R^{2^n}.$$

Therefore

(3.16) $\quad \| x \|_E \leqq 4 \sqrt{2} \| \tilde{x}_0 \|_E \sup \{ | \langle x, u \rangle | \mid u \in R^{2^n}, | u | \leqq 1 \} \quad \text{for } x \in R^{2^n}.$

The advantage of using norms $||| \cdot |||$ and $| \cdot |$ lies in the fact that the form of extreme points of the unit balls in these norms is known. Observe that

$$i_0 = \max \{ i \mid x_0'(i) \neq 0 \} = h(x_0) \quad \text{and} \quad m_0 = \max \{ m \mid w_0'(m) \neq 0 \} = h(w_0),$$

and define

$$f_m = \left(\sum_{k=1}^{m} w_0'(k) \right)^{-1} \sum_{k=1}^{m} e_k \qquad \text{for } 1 \leqq m < m_0$$

and

$$f_{m_0} = \left(\sum_{k=1}^{m_0} w_0'(k) \right)^{-1} \sum_{k=1}^{2^n} e_k.$$

Similarly, define

$$g_i = \left(\sum_{j=1}^{i} x_0'(j) \right)^{-1} \sum_{j=1}^{i} e_j \qquad \text{for } 1 \leqq i < i_0$$

and

$$g_{i_0} = \left(\sum_{j=1}^{i_0} x'(j) \right)^{-1} \sum_{j=1}^{2^n} e_j.$$

Fix m with $1 \leqq m < m_0$. Let $1 \leqq i \leqq i_0$. Then, since $w_0'(m) \geqq \sqrt{2}\, w_0(m_0)$ and $x_0'(i) \geqq \sqrt{2}\, x_0(i_0)$, one has

$$
\begin{aligned}
|\langle f_m, g_i\rangle| &= \left(\sum_{k=1}^{m} w_0'(k)\right)^{-1}\left(\sum_{j=1}^{i} x_0'(i)\right)^{-1}\min(m,i)\\
&\leqq \left(m\Big/\sum_{k=1}^{m} w_0'(k)\right)\left(\sum_{j=1}^{i} x_0'(i)\right)^{-1}\\
&\leqq \tfrac{1}{2} i w_0(m_0) x_0(i_0)\\
&\leqq 2^{1-n/2}.
\end{aligned}
$$

Therefore, from (3.16) and Lemma 1 it follows that

$$\| f_m \|_E \leqq 2^{(7-n)/2} \| \tilde{x}_0 \|_E \qquad \text{for } 1 \leqq m < m_0. \tag{3.17}$$

Notice now that by the definition (3.4) of $\tilde{y}_0$, one has

$$\| x \|_2 \leqq \frac{2^{n/2}}{\| \tilde{y}_0 \|_{E^*}} \| x \|_{E^*} \qquad \text{for all } x \in R^{2^n}.$$

Thus, by duality,

$$\| x \|_E \leqq \frac{2^{n/2}}{\| \tilde{y}_0 \|_{E^*}} \| x \|_2.$$

Therefore, if y_0' is defined by formula (3.3) applied to the vector y_0, then $y_0'(j) \leqq 4\sqrt{2}\, \tilde{y}_0(j)$ for all $j = 1, \cdots, 2^n$ and

$$\| f_{m_0} \|_E \leqq \frac{2^{(n+5)/2}}{\| y_0' \|_{E^*}} \| f_{m_0} \|_2.$$

On the other hand, if $x_0'' \in \mathcal{D}$ is defined by (3.3) applied to the vector x_0, then $\| x_0'' \|_X \leqq \sqrt{2} \| x_0 \|_X \leqq 4\sqrt{2} \| \tilde{x}_0 \|_X$, for any 2^n-dimensional symmetric space X. Therefore,

$$2^n = \| x_0'' \|_2^2 \leqq \| x_0'' \|_{E^*} \| x_0'' \|_E \leqq 4\sqrt{2} \| x_0'' \|_{E^*} \| \tilde{x}_0 \|_E. \tag{3.18}$$

Combining this inequality with the estimate for $\| f_{m_0} \|_E$ and with Lemma 9 one gets

$$
\begin{aligned}
\| f_{m_0} \| &\leqq 2^{(5-n)/2} \| f_{m_0} \| \times 4\sqrt{2} \| x_0'' \|_{E^*} \| \tilde{x}_0 \|_E / \| y' \|_{E^*}\\
&\leqq 2^{5-n/2} \| \tilde{x}_0 \|_E \left(\sum_{k=1}^{m_0} w_0'(k)\right)^{-1} \times 2^{n/2} \frac{x_0(h(x_0))}{y_0(h(y_0))}.
\end{aligned}
$$

Since

$$\sum_{k=1}^{m_0} w_0'(k) \geqq \sum_{k=1}^{m_0} w_0'(k)^2 / w_0'(1) = 2^n / \sqrt{2} w_0(1) \quad \text{and} \quad w_0(1) \leqq y_0(h(y_0))$$

(because $y_0 > w_0$), the last expression is smaller than or equal to

$$2^{5-n/2}\|\tilde{x}_0\|_E 2^{1/2-n} w_0(1) 2^{n-2}\frac{x_0(h(x_0))}{y_0(h(y_0))}2^{(11-n)/2}\|\tilde{x}_0\|_E.$$

This shows that $\|f_{m_0}\|_E \leqq 2^{(11-n)/2}\|\tilde{x}_0\|_E$ and, combined with (3.17) and Lemma 1, concludes the proof of (3.15).

Finally, to complete Case II assume that condition (3.8.ii) holds. It follows from Lemma 6 that

$$\begin{aligned} d(E,F) &\leqq \|\mathrm{Id}: F\to E\|\,\|\mathrm{Id}: E\to F\| \\ &\leqq 2^{2n}/\|\tilde{y}_0\|_{E^*}\|\tilde{z}_0\|_F\,\|\tilde{x}_0\|_E\,\|\tilde{w}_0\|_{F^*}. \end{aligned}$$

By (3.18) one has $1/\|\tilde{x}_0\|_E \leqq 2^{5/2-n}\|x_0''\|_{E^*}$ and, similarly, $1/\|\tilde{w}_0\|_{F^*} \leqq 2^{5/2-n}\|w_0''\|_F$. Since $\|\tilde{y}_0\|_{E^*} \geqq 2^{-5/2}\|y_0'\|_{E^*}$ and $\|\tilde{z}_0\|_F \geqq 2^{-5/2}\|z_0'\|_F$, then combining all these estimates with Lemma 9 and conditions (3.8.ii) and (3.13), one obtains

$$\begin{aligned} d(E,F) &\leqq 2^{2n}2^{5-n}(\|x_0''\|_{E^*}/\|y'\|_{E^*})2^{5-n}(\|w_0''\|_F/\|z_0'\|_F) \\ &\leqq 2^{10}x_0(i_0)w_0(m_0)/y_0(j_0)z_0(k_0) \\ &\leqq 2^{10}2^{n/2}2^{n/2}/2^{(n-4)/2} \\ &= 2^{12+n/2}. \end{aligned}$$

This shows Case II and concludes the proof of Theorem 1.

To conclude this section let us prove a "p-convex, p'-concave version" of Theorems 1 and 2.

Corollary 2. *Let $1 < p \leqq 2$, let $p' = p/(p-1)$. Let E, F be symmetric spaces. If $\dim E = \dim F = 2^n$, with a non-negative integer n, then*

$$d(E,F) \leqq M2^{(24+n)(1/p-1/2)}, \tag{3.19}$$

where

$$M = M^{(p)}(E)M_{(p')}(E)M^{(p)}(F)M_{(p')}(F).$$

In general, if $\dim E = \dim F = N$, then

$$d(E,F) \leqq 2^{24(1/p-1/2)}(2^{1/2p-1/4}-1)^{-2}MN^{1/p-1/2}. \tag{3.20}$$

Proof. The proof of (3.19) is a slight modification of the proof of Theorem 1. First we need some renorming. It is well-known (see [7], proposition 1.d.8) that there exist norms $|||\cdot|||_E$ and $|||\cdot|||_F$ on R^{2^n} such that $\tilde{E}=(R^{2^n},|||\cdot|||_E)$ and $\tilde{F}=(R^{2^n},|||\cdot|||_F)$ are symmetric spaces such that

$$M^{(p)}(\tilde{E})=M_{(p')}(\tilde{E})=M^{(p)}(\tilde{F})=M_{(p')}(\tilde{F})=1$$

and

$$d(E,\tilde{E})\leqq M^{(p)}(E)M_{(p')}(E) \quad \text{and} \quad d(F,\tilde{F})\leqq M^{(p)}(F)M_{(p')}(F).$$

It follows from the interpolation theorem of Pisier [8] that there exist spaces with 1-unconditional basis, say E_0 and F_0, such that $\tilde{E}$ is isometric to the Calderon interpolation space $[E_0, l_2^{2^n}]_\theta$ and $\tilde{F}$ is isometric to $[F_0, l_2^{2^n}]_\theta$, where $\theta=2-2/p$. Since $\tilde{E}$ and $\tilde{F}$ are symmetric, E_0, F_0 are symmetric too, so we can repeat the proof of Theorem 1 for the spaces E_0 and F_0. In particular, let $\tilde{x}_0$, $\tilde{y}_0$, $\tilde{z}_0$, $\tilde{w}_0$ be vectors in R^{2^n} such that $\|\tilde{x}_0\|_2=\|\tilde{y}_0\|_2=\|\tilde{z}_0\|_2=\|\tilde{w}_0\|_2=2^{n/2}$ which satisfy conditions (3.4) and (3.5) for the spaces E_0, F_0 and let $x_0, y_0, z_0, w_0\in\mathcal{D}$ be defined by Lemmas 5 and 7. Assume that E_0, F_0 satisfy the assumptions of Case I and let $U: R^{2^n}\to R^{2^n}$ be the operator constructed in this case. In particular,

$$\|U: F_0\to E_0\|\leqq 2^{(11-n)/2}\|\tilde{x}_0\|_{E_0}\|\tilde{w}_0\|_{F_0^*},$$

$$\|U^{-1}: E_0\to F_0\|\leqq 2^n/\|\tilde{x}_0\|_{E_0}\|\tilde{w}_0\|_{F_0^*}.$$

Therefore, since $\tilde{E}$ and $\tilde{F}$ are interpolation spaces,

$$\|U:\tilde{F}\to\tilde{E}\|\leqq(2^{(11-n)/2}\|\tilde{x}_0\|_{E_0}\|\tilde{w}_0\|_{F^*})^{1-\theta},$$

$$\|U^{-1}:\tilde{E}\to\tilde{F}\|\leqq(2^n/\|\tilde{x}_0\|_{E_0}\|\tilde{w}_0\|_{F_0^*})^{1-\theta}.$$

It follows that

$$d(\tilde{E},\tilde{F})\leqq(2^{(11+n)/2})^{1-\theta}=2^{(11+n)(1/p-1/2)}.$$

Assume now that E and F satisfy the assumptions of Case II. The same interpolation argument shows that

$$d(\tilde{E},\tilde{F})\leqq 2^{(24+n)(1/p-1/2)}.$$

Therefore,

$$d(E,F)\leqq d(E,\tilde{E})d(\tilde{E},\tilde{F})d(\tilde{F},F)\leqq M2^{(24+n)(1/p-1/2)}.$$

This concludes the proof of (3.19). Inequality (3.20) follows from (3.19) in exactly the same way as in the proof of Theorem 2.

§4. The distance from a symmetric space to an euclidean space

If X is a k-dimensional real space with an 1-unconditional basis, then it is well-known that

$$d(X, l_2^k) \leqq M^{(2)}(X) M_{(2)}(X). \tag{4.1}$$

Indeed, for $x = \sum_{i=1}^k x(i) e_i \in X$, define $x_m = x(m) e_m$ $(m = 1, \cdots, k)$. Then

$$\left(\sum_m x_m^2\right)^{1/2}(i) = |x(i)| \qquad \text{for } i = 1, \cdots, k, \qquad \text{so} \left\| \left(\sum_m x_m^2\right)^{1/2} \right\| = \| x \|.$$

It follows from the definitions of $M^{(2)}(X)$ and $M_{(2)}(X)$ that

$$\frac{1}{M_{(2)}(X)} \left(\sum_{m=1}^k x(m)^2 \| e_m \|^2 \right)^{1/2} \leqq \| x \|$$

$$\leqq M^{(2)}(X) \left(\sum_{m=1}^k x(m)^2 \| e_m \|^2 \right)^{1/2},$$

for every $x \in X$. This obviously implies (4.1).

Estimate (4.1) can be viewed as a particular case of the well known theorem of Kwapień ([6] cf. also [2], chapter 6), which says that for any k-dimensional normed space X one has

$$d(X, l_2^k) \leqq \tilde{T}_{(2)}(X) \tilde{C}_{(2)}(X), \tag{4.2}$$

where $\tilde{T}_{(2)}(X)$ and $\tilde{C}_{(2)}(X)$ denote the Gaussian type 2 and cotype 2 constants of X (the definition of these constants can be found, e.g., in [2], chapter 6, where they are denoted by $\sup_n \tilde{\alpha}_n(X)$ and $\sup_n \tilde{\beta}_n(X)$ respectively).

As we shall show in this section, for symmetric spaces inequality (4.1) can be essentially improved, while in the general case, both estimates (4.2) and (4.1) are asymptotically, as $\dim X \to \infty$, the best possible .

The positive result states

THEOREM 3. *Let E be a k-dimensional symmetric space. Then*

$$d(E, l_2^k) \leqq 2\sqrt{2} \max(M^{(2)}(E), M_{(2)}(E)). \tag{4.3}$$

In the proof we shall need the following lemma, which is a generalization of Lemma 4.

LEMMA 10. *Let $u(1) \geqq \cdots \geqq u(k) \geqq 0$, $v(1) \geqq \cdots \geqq v(k) \geqq 0$ be such that $\sum_{i=1}^k u(i) = 1 = \sum_{j=1}^k v(j)$. Then at least one of the following conditions holds:*

(i) *there exist positive integers* i_0, j_0 *such that* $u(i_0) \geqq v(j_0)$ *and* $\Sigma_{i=1}^{i_0} u(i) \geqq \frac{1}{2}$, $\Sigma_{j=j_0+1}^{k} v(j) < \frac{1}{2} \leqq \Sigma_{j=j_0}^{k} v(j)$,

(ii) *there exist positive integers* i_0, j_0 *such that* $v(j_0) \geqq u(i_0)$ *and* $\Sigma_{j=1}^{j_0} v(j) \geqq \frac{1}{2}$, $\Sigma_{i=i_0+1}^{k} u(i) < \frac{1}{2} \leqq \Sigma_{i=i_0}^{k} u(i)$.

PROOF. Set $i_0 = \inf\{k \mid \Sigma_{i=1}^{k} u(i) > \frac{1}{2}\}$ and $j_0 = \inf\{k \mid \Sigma_{j=1}^{k} v(j) \geqq \frac{1}{2}\}$. Then either $u(i_0) \geqq v(j_0)$, which gives (i), or $v(j_0) \geqq u(i_0)$, which gives (ii).

PROOF OF THEOREM 3. Pich x_0, $y_0 \in R^k$ such that $\| x_0 \|_2 = 1 = \| y_0 \|_2$ and $\| x_0 \|_E = \min\{\| x \|_E \mid x \in R^k, \| x \|_2 = 1\}$, $\| y_0 \|_{E^*} = \min\{\| y \|_{E^*} \mid y \in R^k, \| y \|_2 = 1\}$. Then similarly, as in Lemma 6, $\| \mathrm{Id} : E \to l_2^k \| = 1/\| x_0 \|_E$ and $\| \mathrm{Id} : l_2^k \to E \| = \| \mathrm{Id} : E^* \to l_2^k \| = 1/\| y_0 \|_{E^*}$. Therefore,

$$d(E, l_2^k) \leqq 1/\| x_0 \|_E \| y_0 \|_{E^*}. \tag{4.4}$$

Put $u(i) = x_0^*(i)^2$ and $v(j) = y_0^*(j)^2$ for $i, j = 1, \cdots, k$ and apply Lemma 10. Assume first that condition (i) of the lemma is satisfied. Put $\tilde{x}_0 = (x_0^*(1), \cdots, x_0^*(i_0), 0, \cdots, 0) \in R^k$ and $\tilde{y}_0 = (y_0^*(j_0), \cdots, y_0^*(k), 0, \cdots, 0) \in R^k$. Then $\tilde{x}_0(i) \leqq x_0^*(i)$ and $\tilde{y}_0(j) \leqq y_0^*(j)$ for $i, j = 1, \cdots, k$, hence $\| \tilde{x}_0 \|_E \leqq \| x_0 \|_E$ and $\| \tilde{y}_0 \|_{E^*} \leqq \| y_0 \|_{E^*}$. Notice that

$$\tfrac{1}{2} \leqq \| \tilde{y}_0 \|_2^2 \leqq \| \tilde{y}_0 \|_E \| \tilde{y}_0 \|_{E^*} \leqq \| \tilde{y}_0 \|_E \| y_0 \|_{E^*}.$$

It follows that $1/\| y_0 \|_{E^*} \leqq 2 \| \tilde{y}_0 \|_E$. Combining the obtained inequalities with (3.4) one gets

$$d(E, l_2^k) \leqq 2 \| \tilde{y}_0 \|_E / \| \tilde{x}_0 \|_E. \tag{4.5}$$

Define now vectors $\tilde{u} \in R^k$ and $\tilde{v} \in R^k$ by

$$\tilde{u}(i) = \tilde{x}_0(i)^2 \quad \text{and} \quad \tilde{v}(i) = \tilde{y}_0(i)^2 \quad \text{for } i = 1, \cdots, k. \tag{4.6}$$

The conclusion of (i) implies, in particular, that $\Sigma_{j=1}^{m} \tilde{v}(j) \leqq 2 \Sigma_{i=1}^{m} \tilde{u}(i)$ for all $m = 1, \cdots, k$. From the well known theorem of Hardy, Littlewood and Pσlya (cf. [5]) it follows that there exist positive numbers a_n with $\Sigma_n a_n = 1$, permutations π_n of the set $\{1, \cdots, k\}$ and $\varepsilon_n(i) = \pm 1$ such that

$$\tilde{v}(i) = 2 \sum_n a_n \varepsilon_n(i) \tilde{u}(\pi_n(i)) \qquad \text{for } i = 1, \cdots, k.$$

Therefore,

$$\tilde{y}_0(i) \leqq \left(\sum_n a_n x_n(i)^2 \right)^{1/2} \qquad \text{for } i = 1, \cdots, k,$$

where $x_n \in R^k$ is defined by $x_n(i) = \sqrt{2}\,\tilde{x}_0(\pi_n(i))$ for $i = 1, \cdots, k$ and $n = 1, 2, \cdots$. Clearly $\| x_n \|_E = \sqrt{2} \| \tilde{x}_0 \|_E$ for $n = 1, 2, \cdots$, so

$$\begin{aligned} \| \tilde{y} \|_E &\leqq \left\| \left(\sum_n a_n x_n^2 \right)^{1/2} \right\|_E \\ &\leqq M^{(2)}(E) \left(\sum_n a_n \| x_n \|_E^2 \right)^{1/2} \\ &= \sqrt{2}\, M^{(2)}(E) \| \tilde{x}_0 \|_E. \end{aligned}$$

Combining this estimate with (4.5) one gets $d(E, l_2^k) \leqq 2\sqrt{2}\, M^{(2)}(E)$.

Assume now that condition (ii) of Lemma 10 is satisfied. Put

$$\tilde{x}_0 = (x_0^*(i_0), \cdots, x_0^*(k), 0, \cdots, 0) \in R^k$$

and

$$\tilde{y}_0 = (y_0^*(1), \cdots, y_0^*(j_0), 0, \cdots, 0) \in R^k.$$

A similar argument as before shows that

$$d(E, l_2^k) \leqq 2 \| \tilde{x}_0 \|_{E^*} / \| \tilde{y}_0 \|_{E^*}. \tag{4.7}$$

If $\tilde{u} \in R^k$ and $\tilde{v} \in R^k$ are defined by (4.6), then the conclusion of (ii) implies, in particular, that $\sum_{i=1}^m \tilde{u}(i) \leqq 2 \sum_{j=1}^m \tilde{v}(j)$ for all $m = 1, \cdots, k$. Therefore there exist positive numbers a_n with $\sum_n a_n = 1$ and vectors $y_n \in R^k$ with $\| y_n \|_{E^*} = \sqrt{2} \| \tilde{y}_0 \|_{E^*}$ for $n = 1, 2, \cdots$, such that

$$\tilde{x}_0(i) \leqq \left(\sum_n a_n y_n(i)^2 \right)^{1/2} \qquad \text{for } i = 1, \cdots, k.$$

Thus

$$\begin{aligned} \| \tilde{x}_0 \|_{E^*} &\leqq M^{(2)}(E^*) \left(\sum_n a_n \| y_n \|_{E^*}^2 \right)^{1/2} \\ &= \sqrt{2}\, M_{(2)}(E) \| \tilde{y}_0 \|_{E^*}. \end{aligned}$$

Combining this estimate with (4.7) one obtains the estimate $d(E, l_2^n) \leqq 2\sqrt{2}\, M_{(2)}(E)$. Together with the first part of the proof it shows (4.3).

The next example shows that the symmetry assumption in (4.3) cannot be dropped. Before stating the result, let us recall some generalization of symmetric spaces (cf. e.g. [1], chapter 4). A normed space E is said to have enough symmetries if the group G of isometries of E has the property that for any operator $T : E \to E$ the condition $Tg = gT$ for all $g \in G$ implies $T = \lambda\, \mathrm{Id}$, for some $\lambda \in R$.

PROPOSITION 2. *Let* $1 < p \leqq 2$ *and let* $F = (\bigoplus_1^n l_p^n)_{l_{p'}}$, *where* $1/p' + 1/p = 1$. *Then* F *has an* 1-*unconditional basis and enough symmetries, but still*

$$d(F, l_2^{n^2}) = n^{2/p-1} \geqq M^{(2)}(F) M_{(2)}(F)$$
$$\geqq c\tilde{T}_{(2)}(F)\tilde{C}_{(2)}(F)$$

with some numerical constant $c > 0$.

PROOF. The norm in F is given by the formula

$$\|(a_{ij})\| = \left\{ \sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}|^p \right)^{p'/p} \right\}^{1/p'} \qquad \text{for } (a_{ij}) \in R^{n^2}.$$

Obviously F has an 1-unconditional basis and it is easy to see that it has enough symmetries too. Let $a^m = (a_{ij}^m) \in F$ $(m = 1, 2, \cdots)$, then, by Hölder's inequality,

$$\left\| \left(\sum_m (a^m)^2 \right)^{1/2} \right\| = \left\{ \sum_{j=1}^n \left(\sum_{i=1}^n \left(\sum_m |a_{ij}^m|^2 \right)^{p/2} \right)^{p'/p} \right\}^{1/p'}$$
$$\leqq \left\{ \sum_{j=1}^n n^{p'(1/p-1/2)} \left(\sum_{i=1}^n \left(\sum_m |a_{ij}^m|^2 \right) \right)^{p'/2} \right\}^{1/p'}$$
$$= \left\{ \sum_{j=1}^n n^{p'(1/p-1/2)} \left(\sum_m \sum_{i=1}^n |a_{ij}^m|^2 \right)^{p'/2} \right\}^{1/p'}$$
$$\leqq n^{1/p-1/2} \left\{ \sum_{j=1}^n \left(\sum_m \left(\sum_{i=1}^n |a_{ij}^m|^p \right)^{2/p} \right)^{p'/2} \right\}^{1/p'}$$
$$\leqq n^{1/p-1/2} \left\{ \sum_m \left(\sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}^m|^p \right)^{p'/p} \right)^{2/p'} \right\}^{1/2}$$
$$= n^{1/p-1/2} \left(\sum_m \|a^m\|^2 \right)^{1/2}.$$

This shows that $M^{(2)}(F) \leqq n^{1/p-1/2}$. A similar argument shows that $M_{(2)}(F) \leqq n^{1/p-1/2}$.

It is well-known (cf. e.g. [6]) that there exists a numerical constant $\bar{c}$ such that if $1 < p \leqq 2 \leqq q < \infty$ then $\tilde{C}_{(2)}(l_p^n) \leqq \bar{c}$ and $\tilde{T}_{(2)}(l_q^n) \leqq \bar{c}$. Therefore, by Kahane's inequality (cf. e.g. [2], (5.3)), one has

$$\tilde{C}_{(2)}(F) \leqq d(l_{p'}^n, l_2^n) \tilde{C}_{(2)}\left(\left(\bigoplus_1^n l_p^n \right)_{l_2} \right)$$
$$\leqq n^{1/2-1/p'} \tilde{C}_{(2)}(l_p^n)$$
$$\leqq \bar{c} n^{1/p-1/2}$$

and

$$\tilde{T}_{(2)}(F) \leqq \tilde{c}\tilde{T}_{(2)}(l_p^n) \leqq \tilde{c}d(l_p^n, l_p^n) \leqq \tilde{c}n^{1/p-1/2},$$

where $\tilde{c}$ is a numerical constant.

To calculate the distance $d(F, l_2^{n^2})$, consider an ellipsoid $\mathscr{E}$ of maximal volume contained in the unit ball of F, and let $\|\cdot\|_2$ be the euclidean norm on $F = R^{n^2}$ induced by $\mathscr{E}$. Since F has enough symmetries, it follows that

$$\begin{aligned} d(F, l_2^{n^2}) &= \|\mathrm{Id} : F \to (R^{n^2}, \|\cdot\|_2)\| \times \|\mathrm{Id} : (R^{n^2}, \|\cdot\|_2) \to F\| \\ &= \|\mathrm{Id} : F \to (R^{n^2}, \|\cdot\|_2)\|, \end{aligned}$$

where Id denotes the formal identity operator ([1], lemma 4.6). Since $\mathscr{E}$ is unique, then, obviously, it is invariant with respect to all isometries of F. Hence, in particular, $\|\cdot\|_2$ must be of a form $\|(a_{ij})\|_2 = (\sum_{j=1}^n \sum_{i=1}^n |c_{ij}a_{ij}|^2)^{1/2}$, for some $c_{ij} \in R$ $(i, j = 1, \cdots, n)$. Since, given permutation π of the set $\{1, \cdots, n\}$, the operators $S((a_{ij})) = (a_{\pi(i),j})$ and $\tilde{S}((a_{ij})) = (a_{i,\pi(j)})$ are isometries of F, then $c_{ij} = \cdots = c_{nj}$ for every $j = 1, \cdots, n$ and $c_{i1} = \cdots = c_{in}$ for every $i = 1, \cdots, n$. Therefore,

$$\|(a_{ij})\|_2 = c\left(\sum_{j=1}^n \sum_{i=1}^n |a_{ij}|^2\right)^{1/2} \qquad \text{for some } c > 0$$

and obvious computation shows that $c = n^{1/p-1/2}$. Now observe that

$$\begin{aligned} \|(a_{ij})\|_2 &= n^{1/p-1/2}\left(\sum_{j=1}^n \sum_{i=1}^n |a_{ij}^2|^{1/2}\right) \\ &\leqq n^{1/p-1/2}\left\{\sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}|^p\right)^{p'/p}\right\}^{1/p'} \end{aligned}$$

for all $(a_{ij}) \in R^{n^2}$. Moreover if $(b_{ij}) \in R^{n^2}$ is defined by $b_{ij} = 1$, for $i = 1, \cdots, n$ and $b_{ij} = 0$, for $i, j = 1, \cdots, n$ and $j \neq 1$, then

$$\|(b_{ij})\| = n^{1/p'} = n^{2/p-1}\|(b_{ij})\|_2.$$

It follows that $d(F, l_2^{n^2}) = \|\mathrm{Id} : F \to (R^{n^2}, \|\cdot\|_2)\| = n^{2/p-1}$. This equality, combined with the estimates for $M^{(2)}(F)$, $M_{(2)}(F)$, and $\tilde{T}_{(2)}(F)$ and $\tilde{C}_{(2)}(F)$, concludes the proof of the proposition.

Our Theorems 1 and 2 raise the natural question, how essential in this kind of estimates the symmetry assumptions are. As we mentioned in the introduction, the recent result of Gluskin [3] shows, that some assumptions on the symmetry or unconditional structure of spaces involved are necessary.

PROBLEM 1. What is the order of growth, as $k \to \infty$, of the diameter of the set of all k-dimensional normed spaces with an 1-unconditional basis?

PROBLEM 2. What is the order of growth, as $k \to \infty$, of the diameter of the set of all k-dimensional normed spaces with enough symmetries?

Finally let us mention problems related to Schatten ideals of operators on a Hilbert space. If $E \in S_k$, by C_E^k we denote the space $L(l_2^k)$ of all operators on l_2^k, endowed with the norm $\| A \|_{C_E} = \| \{S_i(A)\} \|_E$, where $\{S_i(A)\}_{i=1}^k$ is the sequence of eigenvalues of the operator $|A| = (A^*A)^{1/2} \in L(l_2^k)$ (cf. e.g. [2], §3).

PROBLEM 3. What is the order of growth, as $k \to \infty$, of $\max\{d(C_E^k, C_F^k \mid E, F \in S_k\}$?

The answer to this problem would be implied by the positive answer to the next problem.

PROBLEM 4. Does there exist a constant $c > 0$ such that for all positive integers k and all E, $F \in S_k$ one has $d(C_E^k, C_F^k) \leqq cd(E, F)$?

For spaces $E = l_p^k$, $F = l_q^k$, $1 \leqq p$, $q \leqq \infty$, the answer to Problem 4 is positive ([9]).

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DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION, TX 77843 USA
AND
INSTITUTE OF MATHEMATICS
WARSAW UNIVERSITY
WARSAW, POLAND