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GROWTH IN ENVELOPING ALGEBRAS

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ABSTRACT
It is proven that a finitely generated soluble-by-finite Lie algebra has a subexponential growth. This implies that in its universal envelope every subring is an Ore domain.

1. Let S be an associative or a Lie algebra over a field K generated by the finite subset X, $S(X,n)$ denote the subspace of S spanned by all monomials on X of length less than or equal to n. The growth function of S with respect to X is

$$\gamma_S(n) = \dim S(X,n).$$

We remind the reader (see [8], [9]) that $\lim_{n\to\infty}(\gamma_S(n))^{1/n}$ always exists and does not depend on X. If this limit is greater than 1 then S has exponential growth, otherwise the growth is subexponential. S has polynomially bounded growth if there exists a polynomial p with $\gamma_S(n) \leqq p(n)$ for all n.

Martha K. Smith has shown in [8] that there exists an (infinite dimensional) solvable Lie algebra L whose universal envelope $U(L)$ has subexponential but not polynomially bounded growth. It has been proven too in [8] that $U(L)$ has a subexponential growth if L does and that $U(L)$ has polynomially bounded growth if and only if L is finite dimensional. The existence of subexponential but not polynomially bounded growth in soluble Lie algebras shows that the situation for enveloping algebras is different from the case of groups where the theorem of Milnor–Wolf (see [6], [10]) states that the solvable groups with polynomially bounded growth are precisely the nilpotent-by-finite groups and all the other soluble groups have an exponential growth.

The results of the present paper imply that *the universal envelope of an arbitrary finitely generated infinite dimensional solvable-by-finite Lie algebra has subexponential but not polynomially bounded growth.*

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This follows from the following theorem.

THEOREM 1. *Let L be a finitely generated Lie algebra, $H \lhd L$ be a solvable ideal. If the quotient algebra $\bar{L} = L/H$ has subexponential growth then so does L (and, hence, $U(L)$).*

COROLLARY. *Let L be finitely generated solvable-by-finite Lie algebra. Then L has subexponential growth.* □

Since an associative free algebra of rank 2 has an exponential growth we obtain immediately from Corollary 1 the following fact.

COROLLARY 2. *Let L be a solvable-by-finite Lie algebra. Then $U(L)$ contains no free subalgebras of rank 2.*

COROLLARY 3. *Let L be a solvable-by-finite Lie algebra and R be an arbitrary subring of $U(L)$. Then R is an Ore domain.*

PROOF. Since $U(L)$ is a domain so is R. The theorem of Jategaonkar–Koševoi (see [3], 0.7) states that a domain which is not an Ore ring must contain a free subalgebra of rank 2 and the assertion now follows from Corollary 1. □

REMARK. The proof of the fact that $U(L)$ is an Ore domain when L is solvable-by-finite can be obtained by applying the localization technique. (See [5], §4). The main content of Corollary 3 is therefore in the condition that R is an *arbitrary* subring of $U(L)$. □

In the last section of the paper we prove (see Proposition 3) that *any finitely generated infinite dimensional solvable-by-finite Lie algebra contains a subalgebra which can be mapped homomorphically on the Lie algebra H with basis $x, y_1, y_2, \cdots$ such that*

$$[y_i, x] = y_{i+1}, [y_i, y_j] = 0 \qquad (i, j \in Z). \tag{1}$$

One can then use the formulae for the growth in $U(H)$ (see [9], p. 252 or [2], 2.7) and to obtain some lower bound for $\gamma_{U(L)}(n)$, where L is solvable-by-finite. This bound would be similar to the one obtained by Borho and Kraft for an enveloping algebra of an arbitrary Lie algebra (see [2], 2.9).

I would like to express my gratitude to Martha Smith for interesting conversations on the subject. In fact, the article could not have been written without these conversations.

2. Let S be Lie or associative algebra with a finite system of generators X. We remind the reader (see [8]) that the X-length of the element $s \in S$ is

$$l_S(s) = \min\{n \mid s \in S(X, n)\}.$$

We will need the following relation, which is true in an arbitrary Lie algebra:

$$[S(X, n_1), S(X, n_2)] \subseteq S(X, n_1 + n_2). \tag{2}$$

The proof is easy.

If S is an arbitrary Lie algebra then E_1 is defined as a subset of X, which gives a basis of $S(X,1)$, E_2 is an arbitrary subset of the set of the products $[X, X]$, which gives a basis of $S(X,2)$ modulo $S(X,1)$, etc. Let

$$E^{(n)} = \bigcup_{j=1}^{(n)} E_j.$$

Clearly, the set $E^{(n)}$ forms a basis of $S(X, n)$ and it is worth remarking that simultaneously for arbitrary $n_1 \leqq n$ the subset

$$E^{(n_1)} = \bigcup_{j=1}^{n_1} E_j$$

forms a basis for $S(X, n_1)$. Finally, the basis of L is given by the set

$$E = \bigcup_{j=1}^{\infty} E_j.$$

We order the elements of E_j in arbitrary way and then extend the order to E assuming that the elements of E_{j_1} precede the elements of E_{j_2} if $j_1 < j_2$. The standard monomials on the set E

$$e_{i_1} e_{i_2} \cdots e_{i_r} \qquad (i_1 \leqq i_2 \leqq \cdots \leqq i_r)$$

form a basis of the universal envelope $U(S)$. *It is proven in* [8], *section* 4, *that the length of such a standard monomial with respect to any system of generators of* $U(S)$ *is given by*

$$l_{U(S)}(e_{i_1} e_{i_2} \cdots e_{i_r}) = \sum_{\beta=1}^{r} l_L(e_{i_\beta}). \tag{3}$$

LEMMA 1. *Let L be a Lie algebra, generated by the elements $x, y_1, y_2, \ldots, y_k$, S be a subalgebra, generated by the system of elements $Y = \{y_1, y_2, \ldots, y_k\}$ and let u_j be an arbitrary element of $S(Y, n_j)$ $(j = 1, 2, \ldots, s)$ where*

$$n_1 + n_2 + \cdots + n_s \leqq n.$$

Then the element

$$[x, u_1, u_2, \ldots, u_s] \tag{4}$$

is a linear combination of the elements

$$[x, e_{i_1}, e_{i_2}, \ldots, e_{i_r}] \qquad (1 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_r \leqq m) \tag{5}$$

where $\{e_1, e_2, \ldots, e_m\} = E^{(n)}$ *is a basis of* $S(Y, n)$ *and*

$$\sum_{\beta=1}^{r} l_l\,(e_{i_\beta}) \leqq n. \tag{6}$$

Proof. First of all, since the basis of $S(Y, n_j)$ is given by the set $E^{(j)}$ $(j = 1, 2, \ldots, s)$ we can assume that for every j the element u_j is a monomial of length $\leqq n_j$ in the set $E^{(j)} \subseteq E^{(n)}$.

If $u_1 \leqq u_2 \leqq \cdots \leqq u_s$ (in particular, when $s = 1$) the assertion is obvious. We apply induction on s, using the same type of arguments as in the proof of the Poincare–Birkhoff–Witt Theorem (see [4], V.2).

Let s_1 be the first natural number such that

$$u_{s_1} > u_{s_1} + 1.$$

We say that the part $[x, u_1, u_2, \ldots, u_{s_1}]$ of the element $[x, u_1, u_2, \ldots, u_{s_1}, u_{s_1+1}, \ldots, u_s]$ is tame and apply the relation

$$\begin{aligned}[x, u_1, u_2, \ldots, u_{s_1}, u_{s_1+1}, \ldots, u_s] &= [x, u_1, u_2, \ldots, [u_{s_1}, u_{s_1+1}], \ldots, u_s] \\ &\quad + [x, u_1, u_2, \ldots, u_{s_1+1}, u_{s_1}, \ldots, u_s].\end{aligned} \tag{7}$$

We will consider separately every term in the right side of (7).

Relation (2) implies that $[u_{s_1}, u_{s_1+1}] \in S(Y, 2s_1 + 1)$ and hence $[u_{s_1}, u_{s_1+1}]$ is a linear combination of monomials from $E^{(2s_1+1)}$. We conclude therefore that the first summand in the right side of (7) is a linear combination of elements of the form

$$[x, v_1, v_2, \ldots, v_{s-1}] \tag{8}$$

where $v_1, v_2, \ldots, v_{s-1}$ are elements from $E^{(n)}$ such that the sum of their length is less than or equal to n and the induction hypotheses imply that any element of the form (8) has a representation (5).

The tame part of the second summand in (7) is $[x, u_1, u_2, \ldots, u_{s_1+1}, u_{s_1}]$. We now apply the same type of arguments to this summand and the representation (5) is obtained in a finite number of steps. □

COROLLARY. *Every element*

$$[x, y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_s}] \qquad (1 \leqq \alpha_1, \alpha_2, \dots, \alpha_s \leqq k)$$

is a linear combination of the elements of the form (5).

PROOF. Follows from Lemma 1 via the fact

$$y_{\alpha_i} \in S(Y, 1) \qquad (i = 1, 2, \dots, s).$$

3. We need now one assertion about embedding of some classes of Lie algebras in a wreath product of Lie algebras. This fact is a special case of Šmelkin's theorem 1 in [7] which deals with the embedding of more general classes of Lie algebras into the verbal wreath products.

Let A be an abelian Lie algebra with a basis a_i $(i \in I)$ over a field K, B be an arbitrary Lie algebra over K. Consider a free B-module $\bar{A}$ with the basis a_i $(i \in I)$. *The wreath product* A wr B *is the split extension of the module* $\bar{A}$ *by the algebra* B. It is worth remarking that the B-module $\bar{A}$ can be considered in the usual way as the module over the universal envelope $U(B)$.

LEMMA 2. *Let F be a free Lie algebra with free generators x_i $(i \in I)$, R be an ideal of F and let $\bar{g}$ denote the image of the element $g \in F$ in the quotient algebra F/R. Consider an abelian algebra A with a basis a_i $(i \in I)$ and the wreath product* A wr (F/R). *Then the map $x_i \to \bar{x}_i + a_i$ can be extended to a monomorphism*

$$F/R' \to A \text{ wr } (F/R),$$

where $R' = [R, R]$.

PROOF. Šmelkin considered in [7] (Theorem 1) a more general case: a verbal ideal of R and the embedding of $F/V(R)$ into a verbal wreath product. Although his theorem is proven under a restriction that char $K = 0$ the argument (see lemmas 1 and 2 of [7] and the first half of the proof of Theorem 1) shows that in the case when $V(R) = R'$ the statement is true when F is a free Lie algebra over an *arbitrary* field K. □

4.

PROPOSITION 1. *Let A be an abelian Lie algebra with a basis $a_1, a_2, \dots, a_l$ and B be a finitely generated Lie algebra which has a subexponential growth. Then the wreath product* A wr B *has a subexponential growth.*

PROOF. Let $X_1 = \{b_1, b_2, \dots, b_k\}$ be a system of generators of B. The system of elements

$$X = \{a_1, a_2, \ldots, a_l; b_1, b_2, \ldots, b_k\}$$

generates the algebra A wr B and for arbitrary n we have two types of products of length less than or equal to n:

(I) The products which involve only the elements $b_1, b_2, \ldots, b_k$.

(II) The products which may involve the elements $a_1, a_2, \ldots, a_l$ too. Since these elements commute with each other any product of this type either coincides with one of the elements a_j ($j = 1, 2, \ldots, l$) or has a form

(9) $[a_j, b_{\alpha_1}, b_{\alpha_2}, \ldots, b_{\alpha_s}] \qquad (j = 1, 2, \ldots, l; 1 \leqq \alpha_1 \alpha_2, \ldots, \alpha_s \leqq k; s \leqq n-1).$

The first products belong to the subalgebra B and the number of linearly independent among them thus does not exceed $\gamma_B(X_1, n)$.

To find the number of linearly independent among the products of type (II) we apply the Corollary of Lemma 1. For arbitrary $n \geqq 1$ the set $E^{(n-1)} = \{e_1, e_2, \ldots, e_m\}$ forms a basis of $B(X, n-1)$ and, hence, the element (9) is a linear combination of the element of the form

$$[a_j, e_{i_1}, e_{i_2}, \ldots, e_{i_r}] \qquad (1 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_r \leqq m)$$

which satisfies the condition $\Sigma_{\beta=1}^{r} l_L(e_{i_\beta}) \leqq n-1$.

Consider now the one-to-one correspondence

(10) $$[a_j, e_{i_1}, e_{i_2}, \ldots, e_{i_r}] \xrightarrow{\theta} e_{i_1} e_{i_2} \cdots e_{i_r}.$$

The element in the right side of (10) is a standard monomial in $U(B)$. The comparison of (10) and (3) shows that

$$e_{i_1} e_{i_2} \cdots e_{i_r} \in U(B)(X_1, n),$$

and hence the number of different products of the form (9) for all $1 \leqq j \leqq l$ does not exceed $l_{\gamma_{U(B)}}(X_1, n)$. We obtain thus that

(11) $$\gamma_{A \text{ wr } B}(X, n) \leqq \gamma_B(X_1, n) + l + l\gamma_{U(B)}(X_1, n).$$

The function $\gamma_B(X_1, n)$ has a subexponential growth by the conditions of the assertion; this implies via section 4 of [8] that the same is true for the growth function $\gamma_{U(B)}(X_1, n)$ of $U(B)$.

One of the ways to obtain the assertion from (11) is the following one. First, remark that it is easy to verify that

$$B(x_1, n) \subseteq U(B)(X_1, n),$$

and hence, the right side of (11) is not greater than

$$l+(l+1)\gamma_{U(B)}(X_1,n)\leqq(l+2)\gamma_{U(B)}(X_1,n).$$

Hence,

$$\lim_{n\to\infty}(\gamma_{A\text{ wr }B}(X,n))^{1/n}\leqq\lim_{n\to\infty}((l+2)\gamma_{U(B)}(X_1,n))^{1/n}=1. \qquad \square$$

5.

PROOF OF THEOREM 1. We remark first of all that it is enough to consider the case when H is abelian: the truth of the assertion would follow from this special case by an obvious induction on the length of the derived series of H.

Let thus H be abelian and let $\bar{L}=L/H$. Take a free Lie algebra F with the same number of generators as $\bar{L}$ and find an ideal $R\lhd F$ such that $F/R\simeq\bar{L}$. It is easy to see that the algebra L is a homomorphic image of the algebra F/R'. (This follows from the observation that all the relations of the algebra F/R' are satisfied in L. This fact is true too in the categories of associative algebras and groups.) It is enough therefore to verify that F/R' has a subexponential growth since a homomorphic image of an algebra with subexponential growth has subexponential growth too.

To prove that F/R' has a subexponential growth we apply Lemma 2 and embed F/R' into the wreath product $A\,\mathrm{wr}(F/R)\simeq A\,\mathrm{wr}\,\bar{L}$, where A is a finite-dimensional abelian Lie algebra. Since $\bar{L}$ has a subexponential growth we obtain Proposition 1 that so does the algebra $A\,\mathrm{wr}(F/R)$ and its subalgebra F/R'. $\square$

6. Let K be a field. We consider the Lie algebra H with basis $x,y_1,y_2,\ldots$ such that

$$[y_1,x]=y_{i+1},[y_i,y_j]=0 \qquad (i,j\in Z). \tag{12}$$

Clearly, $y_{i+1}=[y,x,\ldots,x]$.

PROPOSITION 2. *Let L be a finitely generated infinite dimensional abelian-by-finite Lie algebra over a field K. Then L contains a subalgebra isomorphic to H.*

PROOF. Let L_0 be an abelian ideal of a finite codimension and $\bar{L}=L/L_0$. Since L has an infinite dimension, L_0 can not be finite dimensional: otherwise, L would be a finite-dimensional-by-finite-dimensional Lie algebra which would imply that L is finite dimensional itself.

The Lie operation in L defines on L_0 a structure of a right $\bar{L}$-module. We have thus for $u\in L_0$, $h\in L$,

$$u\cdot h=[u,h].$$

Since the elements of L_0 act in a trivial way on the module L_0 we see that L_0 is in fact a $\bar{L}$-module; this $\bar{L}$-module is finitely generated by corollary 11.1.8 in [1]. If $v_1, v_2, \ldots, v_r$ is a system of generators of the module L_0 then the fact that L_0 has an infinite K-dimension implies that there exists $v \in \{v_1, v_2, \ldots, v_r\}$ such that the cyclic $\bar{L}$-module $v^{\bar{L}}$, generated by v, has an infinite K-dimension.

Let $e_1, e_2, \ldots, e_n$ be a system of elements in L which gives a basis $\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_n$ of $\bar{L}$. By the Poincare–Birkhoff–Witt Theorem the elements

$$\bar{e}_{i_1} \bar{e}_{i_2} \cdots \bar{e}_{i_m} \qquad (1 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_m \leqq i_m \leqq n) \tag{13}$$

form a basis of $U(\bar{L})$.

The $\bar{L}$-module L_0 becomes in a usual way a module over $U(\bar{L})$ and in particular the action of the monomial (13) on the element v is defined by

$$v \cdot (e_{i_1} e_{i_2} \cdots e_{i_m}) = [v, \bar{e}_{i_1}, \bar{e}_{i_2}, \ldots, \bar{e}_{i_m}] = [v, e_{i_1}, e_{i_2}, \ldots, e_{i_m}].$$

Since the elements (13) form a basis of $U(\bar{L})$ the elements

$$[v, e_{i_1}, e_{i_2}, \ldots, e_{i_m}] \qquad (1 \leqq i_1 \leqq i_2 \leqq \cdots \leqq i_m \leqq n) \tag{14}$$

form a system of K-generators for the cyclic module $v^{\bar{l}}$.

Now we shall show that there exist $x \in v^{\bar{L}}$, $e \in \{e_1, e_2, \ldots, e_n\}$ such that the elements

$$x^{(k)} = [\underbrace{x, e, e, \ldots, e}_{k}] \qquad (k = 1, 2, \ldots) \tag{15}$$

are linearly independent. This would imply that the elements x and e generate a subalgebra of L, isomorphic to H.

To show the existence of such elements x, e we take arbitrary $x \in v^{\bar{L}}$ and $e \in \{e_1, e_2, \ldots, e_n\}$. Assume that for this arbitrary pair x, e the elements (15) are linearly dependent. Then for some k_0

$$x^{(k_0+1)} = \lambda_1 x^{(1)} + \lambda_2 x^{(2)} + \cdots + \lambda_{k_0} x^{(k_0)}$$

and hence *every* $x^{(k)}$ is a linear combination of the elements $x^{(1)}, x^{(2)}, \ldots, x^{(k_0)}$. But if this is true for arbitrary elements $x \in v^{\bar{L}}$ $e \in \{e_1, e_2, \ldots, e_n\}$ we can conclude easily that all the elements (14) can be expressed as a linear combination of a finite number of them, which contradicts the choice of the element v.

PROPOSITION 3. *Assume that L is a finitely generated solvable-by-finite infinite dimensional Lie algebra. Then L contains a subalgebra L_1, which can be mapped homomorphically on the algebra H.*

PROOF. Let Q be a solvable ideal of finite codimension in H and k be the solvability class of Q. When $k = 1$ the truth of the assertion follows from Proposition 2. Assume therefore that $k > 1$ and let Q_k be the last nontrivial term of the derived series of Q. The algebra L/Q_k has an ideal Q/Q_k of a finite codimension and of solubility class $k-1$. Applying induction on k we can assume that the algebra L/Q_k either is finite dimensional *or contains a subalgebra* L_1/Q_k *which can be mapped on* H. In the first case we obtain that L is an extension of an abelian ideal Q_k by a finite dimensional algebra L/Q_k and the assertion follows from Proposition 2. In the second case the subalgebra $L_1 \subseteq L$ can be mapped on H. □

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