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A NEW COUNTABLY DETERMINED BANACH SPACE

BY

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ABSTRACT

We construct a Banach space which is weak*-countably determined in its second dual, but which is not K-analytic for its weak topology.

I. Introduction

We briefly recall some definitions. For more details, the reader is referred to [1] or [2].

If A and B are Hausdorff topological spaces, a map f from A into the compact sets of B is said to be upper semi-continuous if for each neighbourhood V of $f(a)$ there is a neighbourhood U of x such that $f(b) \subset V$ for $b \in U$.

A topological space A is called K-analytic (resp. countably determined) if it is the image of a Polish space (resp. a separable metrisable space) under a compact valued upper semi-continuous map. (The reason for the name *countably determined* is that if A is countably determined and is a subset of a compact space K, there exists a sequence (K_n) of closed subsets of K such that for $x \in A$ and $y \notin A$ there is n with $x \in K_n$, $y \notin K_n$.)

A Banach space E is called WCG if it contains a weak compact set which is total.

Consider the following properties of a Banach space E.

(1) E is subspace of a WCG space.

(2) E is a $K_{\sigma\delta}$ of (E^{**}, w^*).

(3) (E, weak) is K-analytic.

(4) (E, weak) is countably determined.

It has been shown in [4] that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4). These properties are very close, and they define classes of Banach spaces with very similar properties.

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However, an example was provided to show that (2) $\not\Rightarrow$ (1). In this paper, we will construct an example to show that (4) $\not\Rightarrow$ (3). We still do not know whether (3) $\Rightarrow$ (2). The construction will use a general technique, which was introduced in [4], and which we describe in the next section.

II. A general method of constructing Banach spaces

Consider a topological space (T, τ). In this paper, T will be a subspace of a Polish space, with the induced topology.

A family $\mathcal{A}$ of subsets of T will be called *adequate* if it satisfies the following conditions:

(a) Each $A \in \mathcal{A}$ is closed.

(b) For each $t \in T$, $\{t\} \in \mathcal{A}$.

(c) If $A \in \mathcal{A}$ and $B \subset A$, then $B \in \mathcal{A}$.

(d) If $B \subset T$ is such that for each finite subset F of B, $F \in \mathcal{A}$, then $B \in \mathcal{A}$.

Note that (a) and (c) together imply that each $A \in \mathcal{A}$ is discrete.

Consider a point $\omega \notin T$, and let $\tilde{T} = T \cup \{\omega\}$. Provide $\tilde{T}$ with the topology which makes each point of T open, and such that a basis of neighbourhoods of ω are the sets $\tilde{T} \setminus B$, where B is a finite union of elements of $\mathcal{A}$.

The main idea of this construction is that the map $t \to \{\omega, t\}$ from (T, τ) into the compact sets of $\tilde{T}$ is upper semi-continuous, as is implied by condition (a). Hence, $\tilde{T}$ is k-analytic if (T, τ) is Polish, and is countably determined if (T, τ) is a subspace of a Polish space.

Consider now the set $K = \{x_A : A \in \mathcal{A}\}$. Each $g \in K$ is a continuous function on $\tilde{T}$. Moreover, condition (d) means that K, provided with the pointwise convergence topology (that is, the product topology of $\{0,1\}^T$), is compact.

Consider the evaluation map $\delta : T \to C(K)$, which sends t to $\delta(t) \in C(K)$, where for $g \in K$, $\delta(t)(g) = g(t)$. Then, as is easily seen (and shown in [4]), $\{0\} \cup \delta(T)$, provided with the topology of pointwise convergence on K, is homeomorphic to $\tilde{T}$. Moreover, $\delta(T)$ separates the points of K. So, we have, as shown in [4]:

PROPOSITION 1. (a) *$(C(K), weak)$ is K-analytic if and only if $\tilde{T}$ is K-analytic;*

(b) *$(C(K), weak)$ is countably determined if and only if $\tilde{T}$ is countably determined.*

III. A countably determined space which is not K-analytic

We shall construct a space $\tilde{T}$ as above which is countably determined but not

K-analytic. The construction relies on a simple and deep idea of classical descriptive theory.

Consider a Polish space (T_0, τ), and a family $\mathscr{A}_0$ of subsets of T_0, which satisfies conditions (b) to (d). Let T_1 be the set of points t of T such that there exists $A \in \mathscr{A}_0$ which clusters at t. The quantificator "there exists $A \in \mathscr{A}_0$" gives hope to be able to construct A_0 such that T_1 is an analytic subset of T_0, but not a Borel subset of T_0. Consider then $T = T_0 \setminus T_1$ and $\mathscr{A} = \{A \cap T : A \in \mathscr{A}_0\}$. Then $\mathscr{A}$ satisfies (a) to (d). Moreover, there is some hope that $\check{T}$ is not K-analytic, since (T, τ) is not analytic.

To implement this idea, one needs an explicit example of an analytic non-Borel set. The standard example is the set of "trees with an infinite branch" which we describe below.

We denote by I the set of all finite (strictly) increasing sequences on $\mathbf{N}$. If $s = (s_1, \cdots, s_n) \in I$ and $u = (u_1, \cdots, u_m) \in I$, we say $s \leqq u$ if $n \leqq m$ and if $s_i = u_i$ for $i \leqq n$.

A tree X on I is a subset of I which is hereditary, that is, such that if $u \in X$ and $s \leqq u$, then $s \in X$. Trees will be denoted by the letters X, Y, Z. We denote by T_0 the set of trees on I. It is a closed subset of $\{0,1\}^I$, hence a compact metric space. We denote its topology by τ.

We say that X has an infinite branch if it contains an increasing sequence s^n with the length of s^n going to infinity. We denote by T_1 the set of trees with an infinite branch. It is a classical result that T_1 is analytic non-Borel. (We shall not need this result explicitly.)

Given a tree X, we denote by $V_n(X)$ the set of trees Y such that $X \cap I_n = Y \cap I_n$, where I_n denotes the set of finite increasing sequences of integers less than or equal to n. The sets $V_n(X)$ form a basis of neighborhoods of X.

We denote by $\mathscr{A}_0$ the set of finite subsets B of T, which are of the following type: B can be expressed as $\{Y_1, \cdots, Y_n\}$, where, for some $X \in T_0$ and $(s_1, \cdots, s_n) \in X$, we have $Y_i \in V_{s_i}(X)$ for all $i \leqq n$.

We denote by $\mathscr{A}_1$ the smallest set of subsets of T_0 which contains $\mathscr{A}_0$, and satisfies (c), (d). (One sees easily that $\mathscr{A}_0$ satisfies (b).) The following lemma contains the crucial fact.

LEMMA 1. *Let $A \in \mathscr{A}_1$. Then, each cluster point of A belongs to T_1.*

PROOF. Let Z be a cluster point of A. Let (Y_n) be a sequence in A which converges to Z, with $Y_n \neq Z$. For each n there is an integer $p(n)$ such that $Y_n \in V_{p(n)}(Z) \setminus V_{p(n)+1}(Z)$. We can assume $p(n) \geqq n$ and the sequence $(p(n))$ increasing.

Let us fix n. Then, there is $B \in \mathscr{A}_0$ such that $Y_1, \cdots, Y_n \in B$. By definition of $\mathscr{A}_0$, we can write $B = \{Y'_1, \cdots, Y'_k\}$, where there exists $X \in T$, and a sequence $(s_1, \cdots, s_k) \in X$ such that for $i \leqq k$, we have $Y'_i \in V_{s_i}(X)$. We can write $Y_i = Y'_{q(i)}$, where $q(i) \leqq k$. For $i \neq i'$, we have $q(i) \neq q(i')$. Since $s_i \geqq i$ and $p(i) \geqq i$, there exists $j \leqq n$ such that $p(j), s_{q(j)} \geqq n/2$. Let $m = \mathrm{Inf}(p(j), s_{q(j)})$. We have $Y_j \in V_m(Z)$ and $Y_j = Y'_{q(j)} \in V_m(X)$. This forces $V_m(X) = V_m(Z)$. Let $r(n) = \sup\{i : p(i) \leqq n/2 - 1\}$. Let $i \leqq r(n)$. We have

$$Y'_{q(i)} = Y_i \notin V_{p(i)+1}(Z) = V_{p(i)+1}(X).$$

This implies $s_{q(i)} \leqq p(i)$.

In particular, there are at least i of the s_j that are $\leqq p(i)$. Since $s_1 < s_2 < \cdots < s_k$, we have shown that for $i \leqq r(n)$, we have $s_i \leqq p(i)$. Moreover, the sequence $(s_1, \cdots, s_{r(n)})$ belongs to X and consists of elements less than or equal to $n/2$, so it belongs to Z. Since the terms of the sequence $(s_1, \cdots, s_{r(n)})$ depend on n, let us denote them by $(s_1^n, \cdots, s_{r(n)}^n)$. Since $s_i^n \leqq p(i)$ for each $i \leqq r(n)$, there exists a sequence n_k such that each $s_i^{n_k}$ is eventually equal to some s_i. And, for each n, the sequence $(s_1, \cdots, s_n)$ belongs to Z, which proves the lemma.

We now set $T = T_0 \setminus T_1$ and $\mathscr{A} = \{A \subset T : A \in \mathscr{A}_1\}$. It follows from Lemma 1 that $\mathscr{A}$ satisfies conditions (a), (b), and (c). We know that the space $\tilde{T}$ is countably determined; it remains to show that it is not K-analytic.

The next idea to be needed is the classical idea of order of a tree. Given a tree X we define its derivative $X^{(1)}$ by:

$$X^{(1)} = \{s \in X, \text{there exists } t \in I, s < t, t \neq s, t \in X\}.$$

In other words, we delete from X the elements which are maximal.

Denote by Ω the first uncountable ordinal. For $\alpha < \Omega$ we define by induction $X^{(\alpha)}$ by $X^{(\gamma+1)} = (X^{(\gamma)})^{(1)}$ and $X^{(\gamma)} = \bigcap_n X^{(\gamma_n)}$ if $\gamma = \sup \gamma_n$. If there is an ordinal γ such that $X^{(\gamma)} = \emptyset$, we denote by $o(X)$ the smallest such ordinal. Otherwise, we set $o(X) = \Omega$. Note that $o(X) = \Omega$ if and only if $X \in T_1$.

Given two sequences $s, t \in I$, with $s = (s_1, \cdots, s_n)$, $t = (t_1, \cdots, t_m)$, and $t_1 > s_n$, we set $s \frown t = (s_1, \cdots, s_n, t_1, \cdots, t_n) \in I$.

Given $s \in I$ and $X \in T_0$, let $s \mid X = \{t : s \frown t \in X\}$.

We now embark on proving that $\tilde{T}$ is not K-analytic. Otherwise, $\tilde{T}$ would be the image of $\mathbf{N}^{\mathbf{N}}$ by a compact-valued upper semi-continuous map f. For a sequence $t \in \mathbf{N}^n$ and $\sigma \in \mathbf{N}^{\mathbf{N}}$, we write $t < \sigma$ if $t(i) = \sigma(i)$ for $i \leqq n$. Let $A_t = \bigcup_{t<\sigma} f(\sigma)$.

The main point of the argument is to construct a sequence $t_1, \cdots, t_m$ of integers, an increasing sequence $s_1, \cdots, s_n$, and a sequence X_n of trees such that if one sets $t^n = (t_1, \cdots, t_n)$ and $s^n = (s_1, \cdots, s_n)$, the following conditions are satisfied for each n:

(a) $(s_1, \cdots, s_{n-1}) \in X_n$;

(b) $V_{s_p}(X_p) = V_{s_p}(X_n)$ for each $p \leqq n$;

(c) $\{o(s \mid X)\colon X \in V_{s_n}(X_n) \cap A_{t^n}\}$ is unbounded (i.e. has supremum Ω).

The first step proceeds as follows. Since $T = \bigcup_n A_n$, there is a t_1 such that o is not bounded on A_{t_1}. For a tree X, it is easily checked that $o(X) \leqq \sup_n o(n \mid X) + 1$. It follows that there exists s_1 such that $\{o(s_1 \mid X)\colon X \in A_{t_1}\}$ is unbounded. Finally, since there are only finitely many sets of the type $V_{s_1}(X)$, we can find X_1 such that the set $\{o(s_1 \mid X)\colon X \in V_{s_1}(X_1) \cap A_{t_1}\}$ is unbounded.

Suppose now the construction has been done up to n. Since $A_{t^n} = \bigcup_q A_{u(q)}$, where $u(q) = (t^n)^\frown q$, there exists t_{n+1} such that the set $\{o(s^n \mid X)\colon X \in V_{s_n}(X_n) \cap A_{t^{n+1}}\}$ is unbounded.

For $q > s_n$, let $v(q) = (s^n)^\frown q$. Since $o(s^n \mid X) \leqq \sup_q o(v(q) \mid X) + 1$, we can find $q = s_{n+1}$ such that the set $\{o(s^{n+1} \mid X)\colon X \in V_{s_n}(X_n) \cap A_{t^{n+1}}\}$ is unbounded.

Finally, since there are only finitely many sets of the type $V_{s_{n+1}}(X)$, we can find X_{n+1} such that $V_{s_{n+1}}(X_{n+1}) \subset X_{s_n}(X_n)$ and such that the set $\{o(s^{n+1} \mid X)\colon X \in V_{s_{n+1}}(X_{n+1}) \cap A_{t^{n+1}}\}$ is unbounded. The construction is completed.

First, we notice that for each n, we have $s^n \in X_n$, for otherwise, for $X \in V_{s_n}(X_n)$ we would have $(s^n \mid X) = \varnothing$. Moreover, there exists $Y \in T_0$ such that $V_{s_n}(Y) = V_{s_n}(X_n)$ for each n. Hence for each n, $s^n \in Y$.

For each n, let $Y_n \in A_{t^n} \cap V_{s_n}(Y) \cap T$. For each k, $\{Y_1, \cdots, Y_k\} \in \mathscr{A}_0$, so $B = \{Y_n : n \in N\}$ belongs to $\mathscr{A}_1$. It follows that for the topology of $\tilde{T}$ the set B is closed and discrete. Let $t = (t_n)$. Since $f(t)$ is compact, $B \cap f(t)$ is finite. Let $C = B \setminus f(t)$. Since C is closed and disjoint from $f(t)$, and since f is upper semi-continuous, there exists a finite sequence s with $s < t$ such that for $\sigma \in \mathbf{N}^{\mathbf{N}}$ and $s < \sigma$, we have $f(\sigma) \cap C = \varnothing$. Hence, we have $A_s \cap C = \varnothing$. Since $s < t$, s is of the type $(t_1, \cdots, t_n)$. But then all Y_k for $k \geqq n$ belong to $C \setminus A_s$, except finitely many of them, a contradiction.

THEOREM. *There exists a compact space K such that $(C(K)$, weak$)$ is countably determined but is not K-analytic.*

REMARK. In fact, it can be shown using the methods of [4] that $(C(K)$, weak$)$ is the image of a coanalytic set of $\mathbf{N}^{\mathbf{N}}$ under a compact valued upper semi-continuous map.

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