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PROJECTIONS ONTO HILBERTIAN SUBSPACES OF BANACH SPACES

BY

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ABSTRACT

In this paper we obtain new estimates for the relative projection constants of subspaces of a Banach space Y in terms of geometrical properties of Y. Our method gives that K-convex spaces are locally π-Euclidean. We also get a version of Maurey's extension theorem for spaces of type $p < 2$.

1. Introduction

In this paper we extend some results of [7] and [2].

Let X be a finite dimensional subspace of a Banach space Y. Recall that the relative projection constant, $\lambda(X, Y)$, is the quantity

$$\inf\{\|P\| : P \text{ is a linear projection of } Y \text{ onto } X\}.$$

In Section 3 we construct projections of Y onto subspaces X whose norms admit good estimates in terms of geometrical properties of X and Y. In fact we obtain estimates for the γ_2-norm of the projections, i.e. the norm of a factorization through a Hilbert space. This is done in terms of moduli of convexity and smoothness (Theorem 5.1) and of type and cotype properties (cf. Proposition 6.2).

Our approach yields a variant of Maurey's extension theorem [11] for operators $u : X \to l_2^n$, where X is a subspace of a space of type $p < 2$.

In Section 8 it is proved that, if Y satisfies a natural duality condition, then there exist $c, \alpha > 0$ such that each $X \subset Y$ has a subspace Z with

$$\dim Z \geqq c(\dim X)^\alpha, \qquad \lambda(Z, Y) \leqq 1/c.$$

The duality condition is shown in Section 9 to be equivalent to that of

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K-convexity considered by Maurey and Pisier [12]. Hence Corollary 8.4 generalizes the results of [3] and [7].

The method of this paper depends on the variational result of D. R. Lewis [6] and on the approach developed in [2]. The construction and estimates make use of some special properties of the operator norm ℓ introduced in Section 2 and studied briefly in Section 4. The norm ℓ on $L(E, Z)$ replaces that used by D. R. Lewis (cf. [7]) in the case of order bounded operators. The two norms are equivalent if Z is a superreflexive Banach lattice.

In the final section we compare our results with those obtained recently in [4].

Our terminology is standard (cf. [2]). Letters X, Y, Z denote always normed linear spaces over the real field and E, F, G, H are Euclidean spaces. The results are valid in the complex case as well, only minor changes in proofs are necessary.

We are indebted to D. R. Lewis who has kindly provided us with a preliminary version of [7]. His proofs depended on the lattice structure but they suggested our more general approach.

2. The norms ℓ and ℓ^*

Let E be a finite dimensional Hilbert space. Let γ denote the canonical Gaussian probability measure on E. If Z is a normed space then the L_2-norm of a linear operator $T \in L(E, Z)$ is defined by the formula

$$\ell(T) = \left(\int_E \| Tx \|^2 d\gamma(x) \right)^{1/2}$$

The normed space $(L(E, Z), \ell)$ is isometric to a linear subspace of the space $L_2(E, \gamma, Z)$ consisting of all linear functions. This fact has some useful consequences which we examine in Section 4.

Clearly, if j is a linear isometry of Z into a space Z_1, then for $T \in L(E, Z)$ one has $\ell(jT) = \ell(T)$. In particular, if rank $T = 1$, then

$$\ell(T) = \| T \|.$$

Obviously, if $U \in L(E, E)$ is a unitary map, then $\ell(TU) = \ell(T)$ and hence for any $V \in L(E, E)$

$$\ell(TV) \leqq \ell(T) \| V \|.$$

The latter estimate holds also if $V \in L(F, E)$, F being another Hilbert space (cf. the proof of Theorem 8.1).

Observe that, if $S \in L(Z, Z_1)$, then

$$\ell(ST) \leqq \|S\| \ell(T).$$

Recall that the space $L(Z, E)$ may be identified with the dual of $L(E, Z)$ by the formula

$$\langle T, \phi \rangle = \operatorname{Tr}(\phi \circ T)$$

for $T \in L(E, Z)$, $\phi \in L(Z, E)$. The induced norm on $L(Z, E)$ is given by

$$\ell^*(\phi) = \sup\{|\operatorname{Tr}(\phi \circ T)| : \ell(T) \leqq 1\}.$$

One obtains easily

$$\ell^*(\phi) = \|\phi\|, \qquad \text{if rank } \phi = 1,$$

$$\ell^*(V\phi) \leqq \|V\| \ell^*(\phi), \qquad \phi \in L(Z, E), \quad V \in L(E, E),$$

$$\ell^*(\phi S) \leqq \|S\| \ell^*(\phi), \qquad S \in L(Z_1, Z), \quad \phi \in L(Z, E).$$

The last estimate implies for $\phi \in L(Z, E)$

$$\ell^*(\phi) \leqq (\operatorname{rank} \phi)^{1/2} \|\phi\|.$$

Our final lemma expresses a well known property of the dual of an injective operator ideal.

LEMMA 2.1. *Let $u \in L(Z, E)$, $\ell^*(u) < \infty$. If Z is a subspace of a space Y then there is $U \in L(Y, E)$ such that $U|_Z = u$ and $\ell^*(U) = \ell^*(u)$.*

PROOF. Apply the Hahn–Banach theorem to the functional $\varphi(T) = \operatorname{Tr}(u \circ T)$ defined on the subspace $L(E, Z)$ of the normed space $(L(E, Y), \ell)$.

3. The basic construction

In the sequel X is a fixed n-dimensional subspace of a normed space Y ($n \geqq 1$). We let $E = l_2^n$ and fix an isomorphism $w \in L(E, X)$ such that

$$\ell(w) = 1, \qquad \ell^*(w^{-1}) = n.$$

(The existence of such a w is a special case of the main result in [6].) We also fix $W \in L(Y, E)$ such that (cf. Lemma 2.1)

$$Ww(e) = e \qquad \text{for } e \in E,$$

$$\ell^*(W) = \ell^*(w^{-1}) = n.$$

Obviously we have

$$d(X, l_2^n) \leqq \| w^{-1} \| \| w \|,$$

$$\lambda(X, Y) \leqq \| wW \| \leqq \| W \| \| w \|.$$

More generally, if $Q \in L(E, E)$ is a projection of rank k and $Z = w(Q(E))$, then

$$d(Z, l_2^k) \leqq \| QW |_Z \| \| w |_{Q(E)} \| \leqq \| QW \| \| wQ \|,$$

$$\lambda(Z, Y) \leqq \| wQW \| \leqq \| wQ \| \| QW \|.$$

The quantities on the right hand side will be estimated by using the following obvious lemma in which

$$\mathcal{P} = \{P \in L(E, E) : P^2 = P = P^* \text{ and rank } P = 1\}.$$

LEMMA 3.1. *Let $T \in L(E, Z)$, $\phi \in L(Z, E)$. If $Q \in L(E, E)$ is a projection, then*

$$\| TQ \| = \sup\{\| Tx \| : x \in Q(E), \| x \| \leqq 1\}$$

$$= \sup\{\ell(TP) : QP = P \in \mathcal{P}\},$$

$$\| Q\phi \| = \sup\{\ell^*(P\phi) : PQ = P \in \mathcal{P}\}.$$

4. Geometric properties of the norm ℓ

We keep the notation of Section 2. Write $L = (L(E, Z), \ell)$ and $L_2(Z) = L_2(E, \gamma, Z)$.

Our first lemma summarizes some results from [1] about moduli of uniform convexity and smoothness. It is used only in the proof of Theorem 5.1.

LEMMA 4.1. *There exists a constant C, $1 < C < 5$, such that one has for $0 < \varepsilon < 2$, $\tau > 0$*

$$\rho_Z(\tau) \leqq \rho_L(\tau) \leqq \rho_{L_2(Z)}(\tau) \leqq \rho_Z(C\tau),$$

$$\delta_Z(\varepsilon) \geqq \delta_L(\varepsilon) \geqq \delta_{L_2(Z)}(\varepsilon) \geqq \delta_Z(\varepsilon/(C+1)).$$

PROOF. Let C be the constant appearing in proposition 17 of [1]. Then we have

$$\rho_{L_2(Z)}(\tau) \leqq C\rho_Z(\tau) \leqq \rho_Z(C\tau),$$

because $\rho_Z(\tau)/\tau$ is non-decreasing.

On the other hand, using proposition 1, the definition of $\tilde{\delta}$, proposition 17 of [1] and the estimate of Lindenstrauss [10], we obtain

$$\begin{aligned}\delta_Z(\varepsilon/(C+1)) &\leqq C\tilde{\delta}_Z(\varepsilon/C)\\ &= \sup_{\tau>0} \tfrac{1}{2}\tau\varepsilon - C\rho_Z(\tau)\\ &\leqq \delta_{L_2(Z)}(\varepsilon).\end{aligned}$$

The remaining assertions of the lemma are trivial, because L is a subspace of $L_2(Z)$ and Z embeds isometrically into L.

Let $r_1, r_2, \cdots$ denote the Rademacher functions on $[0,1]$. Recall that the constant $\alpha_k(Z)$ (resp. $\beta_k(Z)$) is the infimum of those $C>0$ such that for every choice of $z_1, z_2, \cdots, z_k \in Z$ one has

$$\int_0^1 \left\| \sum_{i=1}^k r_i(t) z_i \right\|^2 dt \leqq C^2 \sum_{i=1}^k \| z_i \|^2,$$

$$\left(\text{resp.} \quad \int_0^1 \left\| \sum_{i=1}^k r_i(t) z_i \right\|^2 dt \geqq C^{-2} \sum_{i=1}^k \| z_i \|^2 \right).$$

LEMMA 4.2. *For each* $k = 1, 2, \cdots$ *one has*

$$\beta_k(L) = \beta_k(Z),$$

$$\beta_k(L^*) \leqq \alpha_k(L) = \alpha_k(Z).$$

PROOF. It is obvious that $\beta_k(Z) \leqq \beta_k(L) \leqq \beta_k(L_2(Z))$ and the equality $\beta_k(L_2(Z)) = \beta_k(Z)$ is verified in [12].

The proof that $\alpha_k(L) = \alpha_k(Z)$ is similar. Finally, we have $\beta_k(L^*) \leqq \alpha_k(L)$ (cf. e.g. [2]).

We shall only need the following consequence of Lemma 4.2.

COROLLARY 4.3. *Suppose* $P_1, \cdots, P_k \in L(E, E)$ *satisfy*

$$\left\| \sum_{i=1}^k \varepsilon_i P_i \right\| \leqq 1$$

for each choice of signs $\varepsilon_i = \pm 1$. *Then for every* $T \in L(E,Z)$ *and* $\phi \in L(Z,E)$ *one has*

$$\min_{i \leqq k} \ell(TP_i) \leqq k^{-1/2} \beta_k(Z) \ell(T),$$

$$\min_{i \leqq k} \ell^*(P_i \phi) \leqq k^{-1/2} \alpha_k(Z) \ell^*(\phi).$$

PROOF. We have

$$\begin{aligned} k \min_{i \leqq k} \ell(TP_i)^2 &\leqq \sum_{i=1}^{k} \ell(TP_i)^2 \\ &\leqq \beta_k(L)^2 \int_0^1 \ell\Big(\sum_{i=1}^{k} r_i(t) TP_i\Big)^2 dt \\ &\leqq \beta_k(Z)^2 \int_0^1 \Big(\ell(T) \Big\| \sum_{i=1}^{k} r_i(t) P_i \Big\| \Big)^2 dt \\ &\leqq (\beta_k(Z)\ell(T))^2. \end{aligned}$$

The second estimate is proved similarly.

5. Estimates for uniformly convex spaces

THEOREM 5.1. *Let* X *be an* n*-dimensional uniformly convex subspace of a uniformly smooth space* Y. *Let* d, r *satisfy* $\delta_X(d) = 1/n$, $\rho_Y(r) = 1/n$. *Then*

$$d(X, l_2^n) \leqq 30d/r, \qquad \lambda(X, Y) \leqq 30d/r.$$

PROOF. Pick numbers D, R such that

$$\delta_{(L(E,X),\ell)}(D) = 1/n, \qquad \rho_{(L(E,Y),\ell)}(R) = 1/n.$$

Since δ and ρ are non-decreasing functions, Lemma 4.1 yields $1/R \leqq 5/r$, $D \leqq 6d$. Therefore it suffices to prove that the operators w, W from Section 3 satisfy

$$\| w \| \leqq D/2, \qquad \| W \| \leqq 2/R.$$

By Lemma 3.1, this will follow if we show that for each $P \in \mathcal{P}$

$$\ell(wP) \leqq D/2, \qquad \ell^*(PW) \leqq 2/R.$$

Set $x = w$, $y = w(I-2P)$. Then $\ell(x) = \ell(y) = 1$ and

$$\begin{aligned}\ell(x+y) &\geqq n^{-1}\mathrm{Tr}(w^{-1}(x+y))\\ &= n^{-1}\mathrm{Tr}(2(I-P))\\ &= 2-2/n\\ &= 2(1-\delta_{(L(E,X),\ell)}(D))\end{aligned}$$

and hence $\ell(2wP) = \ell(x-y) \leqq D$.

Now let $\rho = \rho_{(L(E,Y),\ell)}$, $\delta = \delta_{(L(Y,E),\ell^*)}$. Put $x = n^{-1}W$, $y = n^{-1}(I-2P)W$. Then $\ell^*(x) = \ell^*(y) = 1$ and

$$\ell^*(x+y) \geqq \mathrm{Tr}((x+y)w) = n^{-1}(2n-2),$$

so that $\delta(\ell^*(x-y)) \leqq 1/n$.

Set $\varepsilon = \ell^*(x-y)$, $\tau = 4/n\varepsilon$. Using the estimate of Lindenstrauss [6] we get

$$\rho(\tau) \geqq \tfrac{1}{2}\tau\varepsilon - \delta(\varepsilon) \geqq 1/n = \rho(R).$$

Since ρ is non-decreasing, this yields

$$\ell^*(PW) = \tfrac{1}{2}n\ell^*(x-y) = 2/\tau \leqq 2/R.$$

This completes the proof of the theorem.

REMARK. Of course, the estimate of $d(X, l_2^n)$ is improved by putting $Y = X$ in the statement of Theorem 5.1.

We should mention that the estimate $\lambda(X, Y) \leqq Cn^{1/p-1/2}$ if $d(X, l_2^n) \leqq 2$ and $\rho_Y(\tau) \leqq c\tau^p$ (which follows also from Theorem 5.1 or Corollary 7.3) has been obtained independently by W. B. Johnson, D. R. Lewis and V. D. Milman.

6. Estimates in terms of type and cotype constants

LEMMA 6.1. *For each k, $1 \leqq k \leqq n$, there are subspaces $E_1, E_2 \subset E$ such that* $\dim E_j > n-k$, $j = 1, 2$, *and for $P \in \mathcal{P}$ $P(E) \subset E_1$ implies $\ell(wP) \leqq k^{-1/2}\beta_k(X)$, and $P(E) \subset E_2$ implies*

$$\ell^*(PW) \leqq k^{-1/2}\min\{\alpha_k(Y)n, \beta_k(Y^*)\ell(W^*)\}.$$

PROOF. Let A be a maximal subset of

$$\{P \in \mathcal{P} : \ell(wP) > k^{-1/2}\beta_k(X)\}$$

consisting of mutually orthogonal projections. By Corollary 4.3 A has less than k elements. Thus

$$E_1 = \{e \in E : Pe = 0 \text{ for } P \in A\}$$

has the required properties.

The construction of E_2 is similar. Since $\ell(W^*P) = \| W^*P \| = \| PW \| = \ell^*(PW)$, one can use Corollary 4.3 to estimate $\ell(W^*P)$ or $\ell^*(PW)$, whichever gives better upper bound.

Proposition 6.2. *Let $X \subset Y$ be a subspace of dimension n and let $1 \leqq k \leqq \frac{1}{2}n$. Then there is a subspace $Z \subset X$ with $\dim Z = j \geqq n - 2k + 2$ such that*

$$d(Z, l_2^j) \leqq (n/k)\alpha_k(X)\beta_k(X),$$

$$\lambda(Z, Y) \leqq (n/k)\alpha_k(Y)\beta_k(X).$$

Proof. Given k, let $Z = w(E_1 \cap E_2)$ where E_1, E_2 are constructed in Lemma 6.1. It suffices to prove that if $Q \in L(E, E)$ is the orthogonal projection onto $E_1 \cap E_2$, then

$$\| wQ \| \leqq k^{-1/2}\beta_k(X), \qquad \| QW \| \leqq k^{-1/2} n\alpha_k(Y).$$

These estimates follow directly from Lemmas 3.1 and 6.1.

7. An extension theorem

Lemma 7.1. *Suppose $X \subset Y$ and $u \in L(X, H)$, where H is a Hilbert space, $\dim H < \infty$. Then for each $k = 1, 2, \cdots$ there exist $U \in L(Y, H)$ and an orthogonal projection $Q \in L(H, H)$ such that $U|_X = u$ and*

$$\| QU \| \leqq k^{-1/2}\alpha_k(Y)\ell^*(u),$$

$$\operatorname{rank}(I - Q) < k.$$

Proof. Let U be the extension of u constructed in Lemma 2.1. We repeat the procedure of Lemma 6.1, with A being a subset of

$$\{P \in \mathcal{P} : \ell^*(PU) > k^{-1/2}\alpha_k(Y)\ell^*(U)\}.$$

Using Corollary 4.3 and Lemma 3.1 we obtain easily that the orthogonal projection Q onto the subspace determined by A has the required properties.

THEOREM 7.2. *Let $X \subset Y$ and let $u \in L(X,H)$ have* rank $m \leqq 2^j$. *Then there is an extension $\tilde{u} \in L(Y,H)$ such that*

$$\|\tilde{u}\| \leqq \sqrt{2} \sum_{s=0}^{j-1} \alpha_{2^s}(Y) \|u\|.$$

PROOF. We use induction over j. The case $j = 1$ is well known. Suppose now that $X \subset Y$, $u \in L(X,H)$ has rank n, where $2^j < n \leqq 2^{j+1}$, and that the theorem is true for this j.

Set $k = 2^j$ and let U, Q be those given by Lemma 7.1. Since $\ell^*(u) \leqq \sqrt{n}\|u\|$, we have

$$\|QU\| \leqq k^{-1/2}\alpha_k(Y)\ell^*(u) \leqq \sqrt{2}\alpha_k(Y)\|u\|.$$

Set $X_0 = \ker(I-Q)u$, $X_1 = X/X_0$, $Y_1 = Y/X_0$. Let $u_1 \in L(X_1, H)$ be the operator induced by $(I-Q)u$. Then rank $u_1 \leqq k = 2^j$ and hence, by the inductive assumption, there is an extension $\tilde{u}_1 \in L(Y_1, H)$ such that

$$\|\tilde{u}_1\| \leqq \sqrt{2} \sum_{s=0}^{j-1} \alpha_{2^s}(Y_1) \|u_1\|.$$

Since $\|u_1\| = \|(I-Q)u\| \leqq \|u\|$ and $\alpha_i(Y_1) \leqq \alpha_i(Y)$ for each i, we conclude that the operator

$$u = QU + \tilde{u}_1 q,$$

where $q : Y \to Y/X_0$ is the quotient map, is the required extension of u.

This shows that the theorem is true for $j+1$ too, and hence completes the proof.

COROLLARY 7.3. *If Y is of type $p < 2$, $X \subset Y$ and* $\dim X = n$, *then*

$$\lambda(X,Y) \leqq Cn^{1/p-1/2} d(X, l_2^n),$$

where $C = \sqrt{2}K^{(p)}(Y)/(1-2^{1/2-1/p})$.

PROOF. Let $u \in L(X, l_2^n)$ satisfy $\|u\| = d(X, l_2^n)$, $\|u^{-1}\| = 1$. The projection $P = u^{-1}\tilde{u}$ has the required property because $\alpha_i(Y) \leqq K^{(p)}(Y) i^{1/p-1/2}$ (cf. Section 10).

8. A class of locally π-Euclidean spaces

Let us define $\mathcal{K}_n(Y)$ to be the least C such that for each $u \in L(Y, l_2^n)$ one has

$$\ell(u^*) \leqq C\ell^*(u).$$

THEOREM 8.1. *There is a constant $c > 0$ which has the following property. For each normed space Y, if $X \subset Y$, $\dim X = n$, then there exists $Z \subset X$ such that*

$$\dim Z \geqq c \min\{n/\beta_n(X)^2, n/\beta_n(Y^*)^2\},$$

$$\lambda(Z, Y) \leqq 27\mathcal{K}_n(Y).$$

We shall use a lemma which is implicit in [2]. Its proof is given after that of the theorem.

LEMMA 8.2. *Let λ be the normalized rotation invariant measure on the unit sphere S in a k-dimensional Euclidean space F. If $0 \neq u \in L(F, Z)$, then for each $C \geqq \ell(u)/\|u\|$ one has*

$$\lambda(\{x \in S : \|ux\| \geqq \sqrt{8/k}\,C\|u\|\}) \leqq 4e^{-C^2}.$$

PROOF OF THEOREM 8.1. Set $k = [(n+4)/3]$. Let E_1, E_2 be subspaces of E constructed in Lemma 6.1 for this k. Pick a subspace $F \subset E_1 \cap E_2$ with $\dim F = k$ (one has $\dim E_1 \cap E_2 \geqq n - 2k + 2 \geqq k$). It suffices to prove the estimates

$$(*)\quad \begin{aligned} &\lambda(\{x \in S : \|wx\| \geqq \sqrt{8/k}\}) \leqq 4\exp[-k/\beta_k(X)^2],\\ &\lambda(\{x \in S : \|W^*x\| \geqq \sqrt{8/k}\,\ell(W^*)\}) \leqq 4\exp[-k/\beta_k(Y^*)^2]. \end{aligned}$$

Indeed, (*) enables us to apply the idea used in [2] in order to prove Theorem 2.6. This yields an absolute constant $c' > 0$ for which there exists $G \subset F$ with

$$\dim G > c' \min\{k/\beta_k(X)^2, k/\beta_k(Y^*)^2\}$$

such that for $g \in G$ one has

$$\|wg\| \leqq \sqrt{9/k}\,|g|, \qquad \|W^*g\| \leqq \sqrt{9/k}\,\ell(W^*)\,|g|.$$

It follows that, if P is the orthogonal projection of E onto G, then $\|wP\| \leqq \sqrt{9/k}$ and $\|PW\| = \|W^*P\| \leqq \sqrt{9/k}\,\ell(W^*)$. Thus wPW projects Y down onto $Z = w(G)$ and

$$\| wPW \| \leqq (9/k)\ell(W^*) \leqq 27\mathcal{K}_n(Y).$$

This shows that the theorem holds with $c = \frac{1}{3}c'$.

Let us prove the second estimate (*). Let Q be the orthogonal projection of E onto F. Then

$$\ell(W^*|_F) = \ell(W^*Q) \leqq \ell(W^*),$$

so that we may apply Lemma 8.2 with $u = W^*|_F$ and $C = \ell(W^*)/\| W^*|_F\|$. Moreover, the argument used in the proof of Proposition 6.2 yields that $C > k^{1/2}/\beta_k(Y^*)$. The first estimate is proved similarly.

Proof of Lemma 8.2. Set $f(x) = \| ux \|$ for $x \in S$. Let M be the median of f, i.e. the largest number c such that

$$\lambda(\{x \in S : \| ux \| \geqq c\}) \geqq 1/2.$$

Then $M \leqq \sqrt{2/k}\,\ell(u)$, because

$$\ell(u)^2 = \int_F \| ux \|^2 d\gamma_F(x) = k \int_S \| ux \|^2 d\lambda(x) \geqq \tfrac{1}{2}kM^2.$$

Estimate (2.6) in [2] yields for each $\varepsilon > 0$

$$\lambda(\{x \in S : \| ux \| > M + \varepsilon \| u \|\}) \leqq 4e^{-k\varepsilon^2/2}.$$

Letting $\varepsilon = \sqrt{2/k}\,C$, we have

$$M + \varepsilon \| u \| \leqq \sqrt{8/k}\,C \| u \|$$

and the assertion of the lemma follows immediately.

Remark. If one uses theorem 5.2 from [2] to obtain a big 2-Euclidean subspace $X_1 \subset X$ and then applies Theorem 8.1 to X_1, one obtains the following corollary.

Corollary 8.3. *If Y is of cotype $q < \infty$ and Y^* is of cotype $q_* < \infty$, then every $X \subset Y$ with $\dim X = n$ has a subspace Z_1 such that*

$$\dim Z_1 = j \geqq c_1 n^{4/qq_*},$$

$$d(Z_1, l_2^j) \leqq 2, \qquad \lambda(Z_1, Y) \leqq 27\mathcal{K}_n(Y),$$

where $c_1 = c_1(Y) > 0$.

COROLLARY 8.4. *If $K(Y) = \sup_n \mathcal{K}_n(Y) < \infty$, then Y is locally π-Euclidean in the sense of* [13].

PROOF OF COROLLARY 8.4. We shall show that $K(Y) < \infty$ implies that Y satisfies the assumption of Corollary 8.3 for some q, $q_* < \infty$. In fact, Y is even B-convex. For, it is easy to check that, for any Z, $\mathcal{K}_n(Z^*) = \mathcal{K}_n(Z)$ and $\mathcal{K}_n(Z) \geqq \mathcal{K}_n(Z_1)$ if $Z_1 \subset Z$. Using Theorem 8.1 and known facts one obtains that if $m \geqq 2^n$, then

$$\mathcal{K}_n(l_1^m) = \mathcal{K}_n(l_\infty^m) \geqq c\sqrt{n},$$

where c is a positive constant. Thus, if $K(Y) < \infty$, then Y cannot contain l_1^n's uniformly.

9. Relation to K-convexity

A normed space Z is said to be K-convex (cf. [12]) if

$$\sup_n \rho_n(Z) < \infty,$$

where $\rho_n(Z)$ is the norm of the operator R_n in $L_2([0,1], dt, Z)$ given by

$$(R_n f)(t) = \sum_{i=1}^{n} r_i(t) \int_0^1 r_i(s) f(s) ds.$$

The main result of this section is the following

PROPOSITION 9.1. *For each normed space Z there is $C < \infty$ such that, for $n = 1, 2, \cdots$,*

$$(\pi/4)\rho_n(Z) \leqq \mathcal{K}_n(Z) \leqq C\rho_n(Z).$$

In particular, Z is K-convex if and only if $K(Z) = \sup_n \mathcal{K}(Z) < \infty$.

We shall compare R_n with the projection Γ_n of $L_2(E, \gamma, Z^*)$, where $E = l_2^n$, onto the space $L(E, Z^*)$ of linear functions given by the formula

$$(\Gamma_n f)(x) = \int_E (x, y) f(y) d\gamma(y).$$

The operator Γ_n may be called the n-th Gaussian projection in $L_2(Z^*)$, because it is equivalent to an operator

$$h \to \sum_{i=1}^{n} \xi_i \int_{\Omega} h\xi_i dP,$$

where $\xi_1, \xi_2, \cdots, \xi_n$ are orthonormal Gaussian random variables.

LEMMA 9.2. *For each $n = 1, 2, \cdots$ one has*

$$\mathcal{K}_n(Z) = \|\Gamma_n\|.$$

PROOF. Recall that $L_2(E, \gamma, Z^*)$ is norming over $L_2(E, \gamma, Z)$ and observe that the annihilator in $L_2(E, \gamma, Z^*)$ of $L = L(E, Z) \subset L_2(E, \gamma, Z)$ is equal to the kernel of Γ_n. Thus the induced mapping

$$L_2(E, \gamma, Z^*)/\mathrm{Ker}\,\Gamma_n \to L_2(E, \gamma, Z^*)$$

can be identified with the mapping of $L^* = (L(Z, E), \ell^*)$ into $L_2(E, \gamma, Z^*)$ given by $u \to u^*$. It follows that the two maps have equal norms, i.e. $\|\Gamma_n\| = \mathcal{K}_n(Z)$.

LEMMA 9.3. *Suppose the functions $f_i, g_i \in L_2(\Omega, \mu)$, $f'_i, g'_i \in L_2(\Omega', \mu')$, where $i = 1, 2, \cdots, n$, satisfy for any $z_1, \cdots, z_n \in Z$, $z_1^*, \cdots, z_n^* \in Z^*$*

$$\left\| \sum_{i=1}^{n} f'_i z_i^* \right\| \leqq C_1 \left\| \sum_{i=1}^{n} f_i z_i^* \right\|,$$

$$\left\| \sum_{i=1}^{n} g'_i z_i \right\| \leqq C_2 \left\| \sum_{i=1}^{n} g_i z_i \right\|,$$

where $\|\cdot\|$ stands for the L_2 norm. Then the operators A in $L_2(\Omega, \mu, Z)$ and A' in $L_2(\Omega', \mu', Z^*)$ given by

$$Ah = \sum_{i=1}^{n} f_i \int_{\Omega} g_i h d\mu, \qquad A'h = \sum_{i=1}^{n} f'_i \int_{\Omega'} g'_i h d\mu'$$

satisfy $\|A'\| \leqq C_1 C_2 \|A\|$.

PROOF. Fix $\varepsilon > 0$. Given $h \in L_2(\Omega', \mu', Z^*)$, let

$$x = \sum_{i=1}^{n} f_i \int_{\Omega'} g'_i h d\mu' \in L_2(\Omega, \mu, Z^*)$$

and pick $y \in L_2(\Omega, \mu, Z)$ so that $\|y\| < 1 + \varepsilon$ and $\int_\Omega xyd\mu = \|x\|$. Then we have

$$\begin{aligned}\|A'h\| &= \left\| \sum_{i=1}^{n} f_i' \int_{\Omega'} g_i' h \, d\mu' \right\| \\ &\leqq C_1 \|x\| \\ &= C_1 \int_{\Omega} xy \, d\mu \\ &= C_1 \int_{\Omega'} h \left(\sum_{i=1}^{n} g_i' \int_{\Omega} f_i y \, d\mu \right) d\mu' \\ &\leqq C_1 \|h\| \left\| \sum_{i=1}^{n} g_i' \int_{\Omega} f_i y \, d\mu \right\| \\ &\leqq C_1 C_2 \|h\| \|Ay\| \\ &\leqq (1+\varepsilon) C_1 C_2 \|A\| \|h\|.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, the lemma is proved.

PROOF OF PROPOSITION 9.1. Observe first that

$$\rho_n(Z) \leqq \rho_n(Z^{**}) \leqq (4/\pi)\|\Gamma_n\| = (4/\pi)\mathcal{K}_n(Z).$$

The second estimate follows from Lemma 9.3 and the inequality from [14].

Recall that, by corollaire 1.3 in [12], if X does not contain l_∞^n's uniformly, then there is $C < \infty$ such that, for each n and any $x_1, \cdots, x_n \in X$, one has

$$\left\| \sum_{i=1}^{n} \xi_i x_i \right\| \leqq C \left\| \sum_{i=1}^{n} r_i x_i \right\|.$$

Hence, if neither Z nor Z^* contain l_∞^n's uniformly, then the right-hand side estimate follows from Lemma 9.3.

Now observe that $\mathcal{K}_n(Z) \leqq \sqrt{n}$, because for $u \in L(Z, l_2^n)$ one has $\ell(u^*) \leqq \sqrt{n}\|u^*\|$ and $\|u\| \leqq \ell^*(u)$.

Suppose Z contains l_∞^n's uniformly. Then

$$\rho_n(Z) \geqq \rho_n(l_\infty^{2^n}) \geqq 2^{-1/2}\mathcal{K}_n(Z),$$

because $\sqrt{n} \leqq \alpha_n(l_\infty^{2^n}) \leqq \beta_n(l_1^{2^n})\rho_n(l_\infty^{2^n}) \leqq \sqrt{2}\rho_n(l_\infty^{2^n})$.

Finally, if Z^* contains l_∞^n's uniformly, then

$$\mathcal{K}_n(Z) \leqq \sqrt{2}\rho_n(Z^*) \leqq \sqrt{2}\rho_n(Z),$$

where the last estimate follows from Lemma 9.3. This completes the proof of the proposition.

10. Another approach to the extension problem

Using duality relations for operator ideals one can obtain some stronger results (cf. [4]). The comparison will be clearer if we introduce certain new constants.

For $T \in L(X,Z)$, let $b_k(T)$ denote the least $C \geqq 0$ such that $\pi_2(TS) \leqq C\ell(S)$ for $S \in L(l_2^k, X)$. (This is equivalent to $\ell^*(UT) \leqq C\pi_2(U)$ for $U \in L(Z, l_2^k)$.) Analogously, $a_k(T)$ is defined so that $\ell(TS) \leqq a_k(T)\pi_2(S^*)$ for $S \in L(l_2^k, X)$ (or $\pi_2((UT)^*) \leqq a_k(T)\ell^*(U)$ for $U \in L(Z, l_2^k)$). We set $a_k(X) = a_k(I)$, $b_k(X) = b_k(I)$, where $I \in L(X,X)$ is the identity map.

The following lemma is a consequence of the Hahn–Banach theorem and the characterization of Γ_2^* due to Kwapień [5] (the right-hand side is equal to the norm of the functional defined by v on $\{T \in \Gamma_2^*(Z,Y) : T(Z) \subset X\}$; cf. [8] the proof of theorem 3.3).

LEMMA 10.1. *Let $j : X \to Y$ be an isometric embedding and let $v \in L(X,Z)$, rank $v = n$. Then*

$$\min\{\gamma_2(\tilde{v}) : \tilde{v} \in L(Y,Z), \tilde{v}j = v\} =$$
$$\sup\{|\mathrm{Tr}(v_2v_1v)| : v_1 \in L(Z, l_2^n),\ v_2 \in L(l_2^n, X),\ \pi_2(v_1) = \pi_2((jv_2)^*) = 1\}.$$

Since our notation yields the estimate

$$\begin{aligned}|\mathrm{Tr}(v_2v_1v)| &\leqq \ell(v_2)\ell^*(v_1v)\\ &\leqq \ell(jv_2)b_n(v)\pi_2(v_1)\\ &\leqq a_n(Y)\pi_2((jv_2)^*)b_n(v)\\ &= a_n(Y)b_n(v),\end{aligned}$$

we obtain that there exists an extension $\tilde{v}$ with

$$\gamma_2(\tilde{v}) \leqq a_n(Y)b_n(v).$$

It remains to find estimates for a_n and b_n. Clearly, $a_n(Y) \geqq \tilde{\alpha}_n(Y)$, $b_n(v) \geqq \tilde{\beta}_n(v)$, where $\tilde{\alpha}_n$, $\tilde{\beta}_n$ are Gaussian type and cotype constants (cf. [2]). On the other hand, it is easy to check that

$$a_n(Y) \leqq \sup_n \tilde{\alpha}_n(Y) = \tilde{K}^{(2)}(Y), \qquad b_n(v) \leqq \sup_n \tilde{\beta}_n(v) = \tilde{K}_{(2)}(v),$$

and in fact, using the argument of lemma 6.1 in [2], one gets

$$a_n(Y) \leqq \tilde{\alpha}_{n^2}(Y), \quad b_n(v) \leqq \tilde{\beta}_{n^2}(v).$$

This improves slightly the operator case of Maurey's extension theorem [11].

Now, the basic result in [4] asserts that, if rank $V = k$ and $q \geqq 2$, then

$$\pi_2(V) \leqq c(q) k^{1/2-1/q} \pi_{q,2}(V),$$

where $c(q) = 4/(1-2/q)$ if $q > 2$ and $c(2) = 1$.

It follows that, for $T \in L(X, Z)$, $1 < p \leqq 2 \leqq q$ and $k = 1, 2, \cdots$,

$$a_k(T) \leqq c(p/(p-1)) k^{1/p-1/2} \tilde{K}^{(p)}(T),$$

$$b_k(T) \leqq c(q) k^{1/2-1/q} \tilde{K}_{(q)}(T),$$

where $\tilde{K}^{(p)}(T)$, $\tilde{K}_{(q)}(T)$ are the least constants such that

$$\left(\int_\Omega \left\| \sum \xi_i T x_i \right\|^2 dP \right)^{1/2} \leqq \tilde{K}^{(p)}(T) \left(\sum \| x_i \|^p \right)^{1/p},$$

$$\left(\sum \| T x_i \|^q \right)^{1/q} \leqq \tilde{K}_{(q)}(T) \left(\int_\Omega \left\| \sum \xi_i x_i \right\|^2 dP \right)^{1/2},$$

for any finite sequence $x_1, \cdots, x_m \in X$. One needs to check only that, if $U \in L(Z, l_2^k)$, $S \in L(l_2^k, X)$, then

$$\pi_{p/(p-1),2}((UT)^*) \leqq \tilde{K}^{(p)}(T) \ell^*(U), \qquad \pi_{q,2}(T) \leqq \tilde{K}_{(q)}(T) \ell(S).$$

This is similar to the proof of Corollary 4.3 (cf. [4]).

The estimate of the form

$$\gamma_2(\tilde{v}) \leqq c(p,q) \tilde{K}^{(p)}(Y) \tilde{K}_{(q)}(v) n^{1/p-1/q}$$

obtained in [4] is more general than the results of Sections 5, 6, 7, but only Corollary 7.3 is entirely its consequence.

Added in proof. Good estimates for a_n and b_n are a consequence of a recent result of the second-named author (to appear in Ark. Mat.). Using her Theorem 1 it is easy to show that, for any $T \in L(X, Z)$ and $k = 1, 2, \cdots$,

$$a_k(T) \leqq 2\tilde{\alpha}_k(T), \qquad b_k(T) \leqq 2\tilde{\beta}_k(T).$$

The estimate $\gamma_2(\tilde{v}) \leqq 4\tilde{\alpha}_n(Y)\tilde{\beta}_n(v)$ which follows now from Lemma 10.1 is stronger than Proposition 6.2 and Theorem 7.2 It also implies Theorem 5.1.

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