ALGEBRAS SATISFYING A CAPELLI IDENTITY

BY
AMITAI REGEV

ABSTRACT
The sequence of cocharacters (c.c.s.) of a P.I. algebra is studied. We prove that an algebra satisfies a Capelli identity if, and only if, all the Young diagrams associated with its cocharacters are of a bounded height. This result is then applied to study the identities of certain P.I. algebras, including F_k.

Introduction

Call the non-commutative polynomial

$$d_m[x,y] = d_m[x_1,\cdots,x_m;y_1,\cdots,y_{m-1}] = \sum_{\sigma\in S_m}(-1)^\sigma x_{\sigma(1)}y_1x_{\sigma(2)}y_2\cdots y_{m-1}x_{\sigma(m)}$$

the Capelli polynomial of height m. Its degree is $2m-1$, and 2^{m-1} polynomials can be derived from it by replacing some of the y's by 1. Denote them by $\{d_m[x,y]\}$. For example, if $m=3$ then

$$d_3[x_1,x_2,x_3;y_1,y_2] = \sum_{\sigma\in S_3}(-1)^\sigma x_{\sigma(1)}y_1x_{\sigma(2)}y_2x_{\sigma(3)},$$

$$d_3[x_1,x_2,x_3;y_1,1] = \sum_{\sigma\in S_3}(-1)^\sigma x_{\sigma(1)}y_1x_{\sigma(2)}x_{\sigma(3)},$$

$$d_3[x_1,x_2,x_3;1,y_2] = \sum_{\sigma\in S_3}(-1)^\sigma x_{\sigma(1)}x_{\sigma(2)}y_2x_{\sigma(3)},$$

$$d_3[x_1,x_2,x_3;1,1] = s_3[x_1,x_2,x_3]$$

and

$$\{d_3[x,y]\} = \{d_3[x,y], d_3[x_1,x_2,x_3;y_1,1], d_3[x_1,x_2,x_3;1,y_2], s_3[x_1,x_2,x_3]\}.$$

Received July 4, 1978 and in revised form December 20, 1978

For technical reasons we say that an algebra A satisfies the Capelli identity $d_m[x,y]$ if it satisfies all elements of $\{d_m[x,y]\}$. Note that if $1 \in A$ and $d_m[x,y]=0$ is an identity for A, then so are all elements in $\{d_m[x,y]\}$. However, if $1 \not\in A$, this may not be true. A major example of such an algebra is F_k, the $k \times k$ matrices, which satisfies $d_{k^2+1}[x,y]$.

Capelli identities were used by Razmyslov [2], to construct central polynomials for matrix algebras. They were later used by Amitsur, [1], to deduce—among other results—the M. Artin–Procesi theorem on Azumaya algebras from central and Capelli identities.

Assume throughout that F is a field of characteristic zero. In this note we characterize F-algebras satisfying a Capelli identity in terms of their cocharacter's sequence (c.c.s.). The definition of the c.c.s. appears for example in [3]. This result is then applied to study the identities of some specific algebras, including F_k. We wish to thank S. Amitsur for completing this characterization.

§1. A c.c.s. characterization of Capelli identity

Definition 1. Let $\lambda = (a_1, \cdots, a_r) \in \mathrm{Par}(n)$, $a_1 \geqq \cdots \geqq a_r \gneqq 0$. Then $h(\lambda) = r$ is called the height of λ. $h(\lambda)$ is actually the height of the Young diagram associated with λ.

Recall that each $\lambda \in \mathrm{Par}(n)$ defines an irreducible character $[\lambda]$ of S_n, and every character χ_n of S_n can be written as

$$\chi_n = \sum_{\lambda \in \mathrm{Par}(n)} a_\lambda [\lambda].$$

We say that the sequence of S_n characters $\{\chi_n\}_{n=1}^{\infty}$ is of height bounded by l if for all n,

$$\chi_n = \sum_{\substack{\lambda \in \mathrm{Par}(n) \\ h(\lambda) \leqq l}} a_\lambda [\lambda].$$

We thus have the notion of a P.I. algebra whose c.c.s. is of height bounded by l.

Theorem 2. *An algebra satisfies $\{d_m[x,y]\}$ if and only if the height of its c.c.s. is bounded by $m-1$.*

Proof. Let $Q \subseteq F\langle x \rangle$ be the T-ideal of identities of the algebra.

(a) Assume $\{d_m[x,y]\} \subseteq Q$. It is trivial then that $\{d_h[x,y]\} \subseteq Q$ for $h \geqq m$. We want to show that the c.c.s. of Q is of bounded height $m-1$, i.e., for $\lambda \in \mathrm{Par}(n)$ with $h(\lambda) \geqq m$, $I_\lambda \subseteq Q$, where I_λ is the two sided ideal in FS_n associated with λ. First let us quote from [4, §2] the following results: Let $\lambda \in \mathrm{Par}(n)$, T_λ a table based on the corresponding Young diagram, and e_{T_λ} the corresponding semi-idempotent. Then there exists $\eta \in S_n$ such that $e_{T_\lambda} = \eta^{-1} e_{T_{0,\lambda}} \eta$, and

$$e_{T_{0,\lambda}}(x) = \sum_{p \in P} p s_{h_1}[x_1, \cdots, x_{h_1}] \cdots s_{h_l}[x_{n-h_l+1}, \cdots, x_n].$$

Here $h_1 = h(\lambda)$. Recall that $\sigma f(x_1, \cdots, x_n) = f(x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ and $f(x_1, \cdots, x_n)\sigma$ is obtained from $f(x_1, \cdots, x_n)$ by permuting the places in each monomial of $f(x)$ according to σ. Since the e_{T_λ}'s generate I_λ as a left ideal over FS_n, it is enough to show that

$$s_{h_1}[x_1, \cdots, x_{h_1}] \cdots s_{h_l}[x_{n-h_l+1}, \cdots, x_n]\eta \in Q$$

if $h(\lambda) = h_1 \geqq m$. Write $(x_{h_1+1}, \cdots, x_n) = (y_{h_1+1}, \cdots, y_n)$,

$$s_{h_1}[x_1, \cdots, x_{h_1}] \cdots s_{h_l}[x_{n-h_l+1}, \cdots, x_n] = s_{h_1}[x_1, \cdots, x_{h_1}] g(y),$$

$g(y) = \Sigma_M \alpha_M M(y)$, $M(y)$ monomials, then it is enough to show that for each such monomial M,

$$s_h[x_1, \cdots, x_{h_1}] M(y)\eta \in Q.$$

Write

$$\begin{aligned} & s_{h_1}[x_1, \cdots, x_{h_1}] M(y)\eta \\ &= \Big(\sum_{\sigma \in S_{h_1}} (-1)^\sigma \sigma \Big) (x_1 \cdots x_{h_1} M(y)\eta) \\ &= \Big(\sum_{\sigma} (-1)^\sigma \sigma \Big) a_0(y) x_{i_1} a_1(y) x_{i_2} \cdots a_{h_1-1}(y) x_{i_{h_1}} a_{h_1}(y), \end{aligned}$$

$(i_1, \cdots, i_{h_1})$ a permutation of $(1, \cdots, h_1)$. It follows that

$$\begin{aligned} & s_{h_1}[x_1, \cdots, x_{h_1}] M(y)\eta \\ &= \pm \sum_{\sigma \in S_{h_1}} (-1)^\sigma a_0(y) x_{\sigma(1)} a_1(y) \cdots a_{h_1-1}(y) x_{\sigma(h_1)} a_{h_1}(y) \\ &= \pm a_0(y) d_{h_1}[x_1, \cdots, x_{h_1}; a_1(y), \cdots, a_{h_1-1}(y)] a_{h_1}(y) \in Q, \end{aligned}$$

since $\{d_{h_1}[x,y]\} \subseteq Q$.

(b) This part is due to Amitsur.

Assume $I_\lambda \subseteq Q_n$ for any n and $\lambda \in \mathrm{Par}(n)$, $h(\lambda) \geqq m$. We show that $d_m[x, y] \in Q_{2m-1}$. Let $\lambda = (1^m) \in \mathrm{Par}(m)$, then

$$e_{T_\lambda} = e_m = \sum_{\sigma \in S_m} (-1)^\sigma \sigma.$$

Write $J = e_m(FS_{2m-1})$. By the Branching Theorem,

$$J \subseteq \sum_{\substack{\lambda \in \mathrm{Par}(2m-1) \\ h(\lambda) \geqq m}} \oplus I_\lambda \quad \text{hence } J \subseteq Q_{2m-1}.$$

In particular, $e_m \eta \in Q_{2m-1}$ for any $\eta \in S_{2m-1}$. In $V_{2m-1} \equiv FS_{2m-1}$,

$$e_m(x_1, \cdots, x_{2m-1}) = \sum_{\sigma \in S_m} \mathrm{sgn}(\sigma) x_{\sigma(1)} \cdots x_{\sigma(m)} x_{m+1} \cdots x_{2m-1}.$$

Denote

$$(x_{m+1}, \cdots, x_{2m-1}) = (y_1, \cdots, y_{m-1})$$

and choose $\eta \in S_{2m-1}$ such that

$$x_1 \cdots x_m y_1 \cdots y_{m-1} \eta = x_1 y_1 x_2 y_2 \cdots y_{m-1} x_m,$$

then clearly $e_m \eta = d_m[x, y]$ and $d_m[x, y] \in Q$. A similar proof shows that $\{d_m[x, y]\} \subseteq Q$.

§2. Applications

The $k \times k$ matrices F_k satisfy $d_{k^2+1}[x, y]$ (see [1] for the minimality of $k^2 + 1$), thus

THEOREM 3. *Let $\{\chi_n(F_k)\}_{n=1}^\infty$ be the c.c.s. of F_k, then*

$$\chi_n(F_k) = \sum_{\substack{\lambda \in \mathrm{Par}(n) \\ h(\lambda) \leqq k^2}} a_\lambda [\lambda].$$

Only limited information regarding the multiplicities a_λ is known at the moment.

Theorem 2 has already been applied in [3]. To help find the c.c.s. of $T_0(S_3[x])$, we showed that $\{d_3[x, y]\} \subseteq T_0(S_3[x])$. Intrigued by that, we then found that for $1 \leqq i \leqq 7$ there exists some $m = m(i)$ such that $\{d_m[x, y]\} \subseteq T_0(S_i[x])$. The proof for $i \leqq 7$ is in particular long, and it is not clear if it can be generalized to all i.

As an example we now show that

$$\{d_3[x, y]\} \subseteq T_0(s_3[x]).$$

NOTATION. First, write $s_n[x_1, \cdots, x_n] = [x_1, \cdots, x_n]$. Now, for fixed $y_1, \cdots, y_s$, write

$$[x_1, \cdots, x_k \overset{y_1}{\vee} x_{k+1}, \cdots, x_l \overset{y_s}{\vee} x_{l+1}, \cdots, x_n]$$

$$= \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \cdots x_{\sigma(k)} y_1 x_{\sigma(k+1)} \cdots x_{\sigma(l)} y_s x_{\sigma(l+1)} \cdots x_{\sigma(n)}.$$

REMARK 4.

(a)
$$S_4[x_1, x_2, x_3, y] = -y[x_1, x_2, x_3] + [x_1 \overset{y}{\vee} x_2, x_3] - [x_1, x_2 \overset{y}{\vee} x_3] + [x_1, x_2, x_3]y;$$

(b)
$$S_3[x_1, x_2, (x_3 \cdot y)] = x_3 y [x_1, x_2] - [x_1 \overset{x_3 y}{\vee} x_2] + [x_1, x_2] x_3 y.$$

PROOF. A direct computation.

COROLLARY 5. $[x_1 \overset{y}{\vee} x_2, x_3] - [x_1, x_2 \overset{y}{\vee} x_3] \in T_0(S_3)$.

COROLLARY 6. *In* 4(b), *alternate* x_1, x_2, x_3 *and sum, to get*

$$2([x_1 \overset{y}{\vee} x_2, x_3] + [x_1, x_2 \overset{y}{\vee} x_3] + [x_1, x_2, x_3]y) \in T_0(s_3),$$

hence

$$[x_1 \overset{y}{\vee} x_2, x_3] + [x_1, x_2 \overset{y}{\vee} x_3] \in T_0(s_3)$$

LEMMA 7. $[x_1 \overset{y}{\vee} x_2, x_3], [x_1, x_2 \overset{y}{\vee} x_3] \in T_0(s_3)$.

PROOF. Follows from Corollaries 5 and 6.

NOTE. $[x_1 \overset{y}{\vee} x_2, x_3, x_4] = [x_1 \overset{y}{\vee} x_2, x_3]x_4 + \cdots + [x_2 \overset{y}{\vee} x_3, x_4]x_1$, hence

$$[x_1 \overset{y_1}{\vee} x_2, x_3, y_2] \in T_0(s_3).$$

LEMMA 8. $[x_1 \overset{y_1}{\vee} x_2 \overset{y_2}{\vee} x_3] \in T_0(s_3)$

PROOF. $= [x_1 \overset{y_1}{\vee} x_2, x_3, y_2] \in T_0(s_3)$

$$= [x_1 \overset{y_1}{\vee} x_2, x_3]y_2 - [x_1 \overset{y_1}{\vee} x_2 \overset{y_2}{\vee} x_3] + [x_1 \overset{y_1 y_2}{\vee} x_2, x_3] - y_2 y_1 [x_1, x_2, x_3]$$

which clearly implies that $d_3[x, y] \in T_0(s_3)$.

The proof that $\{d_3[x, y]\} \subseteq T_0(s_3)$ is now completed. Theorem 2 has an obvious application for the c.c.s. of $T_0(s_3)$.

REFERENCES

1. S. A. Amitsur, *Identities and linear dependence*, Israel J. Math. **22**(1975), 127–137.

2. Ju. P. Razmyslov, *On the Kaplansky problem*, Izv. Akad. Nauk SSSR Ser. Mat. **37**(1973), 483–501 (Russian).

3. A. Regev, *The T-ideal generated by the standard identity* $s_3[x_1, x_2, x_3]$, Israel J. Math. **26**(1977), 105–125.

4. A. Regev, *The representations of* s_n *and explicit identities for P.I. algebras*, J. Algebra **51** (1978), 25–40.

DEPARTMENT OF PURE MATHEMATICS
THE WEIZMANN INSTITUTE OF SCIENCE
REHOVOT, ISRAEL

AND

UNIVERSITY OF CALIFORNIA
SAN DIEGO
LA JOLLA, CALIF. 92093 USA