CODIMENSIONS AND TRACE CODIMENSIONS OF MATRICES ARE ASYMPTOTICALLY EQUAL

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ABSTRACT

The codimensions c_n and the trace codimensions t_n of the $k \times k$ matrices are asymptotically equal: $\lim_{n\to\infty}(t_n/c_n)=1$. Since $t_n \simeq q(n)\cdot k^{2n}$ where $q(x)$ is a known rational function, this asymptotically gives c_n. This has applications to the codimensions of Capelli identities.

Let F be a field of characteristic zero, $0<k$ an integer, and consider $M_k(F)=F_k$, the $k\times k$ matrices. One of the basic questions one can ask about the polynomial identities of a given P.I. algebra, and in particular of F_k, is: How many identities does it have? This question is answered, in a sense, by (calculating) the codimensions $c_n = c_n(F_k)$. Here $Q = I(F_k)$ are the identities of F_k, $V_n = \mathrm{span}_F\{x_{\sigma(1)}\cdots x_{\sigma(n)} \mid \sigma \in S_n\}$ (S_n the symmetric group) and $c_n(F_k)=\dim(V_n/V_n \in Q)$ [3], [4], [7], [9].

Here we give a partial answer to that question by giving the asymptotic value of $c_n(F_k)$ as $n\to\infty$. This has already been done for $c_n(F_2)$ [5], [9]. These computations implied that $c_n(F_2)\simeq t_n(F_2)$ (i.e. $\lim_{n\to\infty}(t_n(F_2)/c_n(F_2)=1)$, where $t_n(F_2)$ are the trace codimensions [5, §6], [9, th. 2.2], and we conjectured that for all k, $c_n(F_k)\simeq t_n(F_k)$ [9, conj. 2.4]. By proving this conjecture here (Theorem 1 below), we capture $c_n(F_k)$ asymptotically since the asymptotic value of $t_n(F_k)$ is known [7, 5.2, 5.4] (correction to [7, 5.4]: $t_n(F_k)\simeq \mathrm{as}(k, n+1)$).

A complete description of the various Poincaré series associated with F_2 was recently given [2], [4], [5]. The asymptotic evaluation of $c_n(F_k)$ may be thought of as a first step towards evaluating these Poincaré — as well as cocharacter — series for the higher matrices.

Our main result is

THEOREM 1. *Let $c_n(F_k)$ be the codimensions of F_k, the $k\times k$ matrices, and $t_n(F_k)$ the trace codimensions, then, as $n\to\infty$,*

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$$c_n(F_k) \simeq t_n(F_k) \simeq \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \left(\frac{1}{2}\right)^{k^2-1/2} \cdot 1! \cdots (k-1)! \cdot k^{(k^2+4)/2} \left(\frac{1}{n}\right)^{(k^2-1)/2} \cdot k^{2n}.$$

The main tool for proving Theorem 1 is [4, th. 16] which indicates that the Poincaré series of the identities and trace identities of F_k are very close. The next few results are also needed for proving Theorem 1.

To save introducing many notations and definitions we assume familiarity with [4].

THEOREM 2 [4, th. 7]. *Let A be any P.I. algebra, $\chi_n(A) = \sum_{\lambda \in \mathrm{Par}(n)} r(\lambda)\chi_\lambda$ its cocharacters. Then $P(A) = \sum_\lambda r(\lambda) s_\lambda$ is the Poincaré series of A.*

THEOREM 3 [1, th. 16]. *Let A, $\{r(\lambda)\}$ be as in Theorem 2. There exists a polynomial $g(x)$ such that for all n and partitions $\lambda \in \mathrm{Par}(n)$, $r(\lambda) \leqq g(n)$.*

We now extend that last theorem to the multiplicities $\bar{r}(\lambda)$ of $P(\bar{R})$. Here R is the ring of $k \times k$ generic matrices and $\bar{R}$ the trace ring of R.

THEOREM 4. *Let $P(\bar{R}) = \sum_\lambda \bar{r}(\lambda) s_\lambda$ be the Poincaré series of $\bar{R}$* [4, §5]. *Then there exists a polynomial $f(x)$ such that for all n and $\lambda \in \mathrm{Par}(n)$, $r(\lambda) \leqq f(n)$.*

PROOF. Let $P(R) = \sum_\lambda r(\lambda) s_\lambda$ [4, §5] and let $g(x)$ be a polynomial such that $r(\lambda) \leqq g(n)$ for all n and $\lambda \in \mathrm{Par}(n)$. Let $\Delta^l \in D$ [4, 15(d)] (l instead of k there) and let $\mu = (1^{k^2}) \in \mathrm{Par}(k^2)$. By [4, lemma 13(a)] and [4, th. 16],

$$\bar{r}(\lambda) = \bar{r}(l\mu + \lambda) = r(l\mu + \lambda) \leqq g(n + lk^2) = f(n)$$

where $g(x + lk^2) = f(x)$. Q.E.D.

Note that the polynomial bound $f(x)$ for the $\bar{r}(\lambda)$'s has the same leading term as $g(x)$ for the $r(\lambda)$'s.

Let $d_\lambda = \deg \chi_\lambda$ denote the degree of the S_n character χ_λ, $\lambda \in \mathrm{Par}(n)$; d_λ is given, for example, by the hook formula.

LEMMA 5. *Let $\lambda = (\lambda_1, \cdots, \lambda_u) \in \mathrm{Par}(n)$, $\nu = (1^u) \in \mathrm{Par}(u)$, $0 \leqq a \in \mathbf{Z}$, then for large n,*

$$d_{a\nu+\lambda} \leqq 2 \cdot n^{a(u-1)} \cdot d_\lambda.$$

PROOF. Let h_{ij} denote the hook numbers in the diagram of $a\nu + \lambda$; note that the numbers in the λ part are the hook numbers for λ. Thus

$$d_{a\nu+\lambda} = \frac{(n+au)!}{\prod_{a\nu+\lambda} h_{ij}} = \frac{n!}{\prod_\lambda h_{ij}} \cdot \frac{(n+1)\cdots(n+au)}{(\lambda_i + u - i + j)} \leqq d_\lambda \cdot \frac{(n+1)\cdots(n+au)}{\left(\prod_i (\lambda_i + 1)\right)^a}.$$

Since $\lambda_1 + \cdots + \lambda_u = n$, $\Pi_{i=1}^{u}(\lambda_i + 1) \geqq n$ (induction on u) so

$$d_{a\nu+\lambda} \leqq d_\lambda \cdot \frac{(n+1)\cdots(n+au)}{n^a} \leqq d_\lambda \cdot 2 \cdot n^{a(u-1)} \qquad (n \text{ large}). \quad \text{Q.E.D.}$$

Denote $\Lambda_u(n) = \{(\lambda_1, \lambda_2, \cdots) \in \mathrm{Par}(n) \mid \lambda_i = 0 \text{ if } i \geqq u+1\}$ and recall

THEOREM 6 [8, 4.5].

$$S_u^{(1)}(n) = \sum_{\lambda \in \Lambda_u(n)} d_\lambda \simeq q_u(n) \cdot u^n$$

where $q_u(x)$ is a (known) rational function.

We are now ready to give the

PROOF OF THEOREM 1. By [7, 5.2, 5.3, 5.4],

$$t_n(F_k) \underset{n \to \infty}{\simeq} s(n) \cdot k^{2n}$$

where $s(x)$ is a (known) rational function, and $c_n(F_k) \leqq t_n(F_k)$. Thus the theorem will be proved once we show that for large n,

$$t_n(F_k) - c_n(F_k) \leqq v(n) \cdot (k^2 - 1)^n$$

for some rational function $v(x)$.

Note that

$$t_n(F_k) = \sum_{\lambda \in \Lambda_{k^2}(n)} \bar{r}(\lambda) \cdot d_\lambda$$

[4, §6] while

$$c_n(F_k) = \sum_{\lambda \in \Lambda_{k^2}(n)} r(\lambda) d_\lambda$$

[7, §3], hence, by [4, th. 16] (with l replacing k there),

$$t_n(F_k) - c_n(F_k) = \sum_{\substack{\lambda \in \Lambda'_{k^2}(n) \\ \lambda_{k^2} \not\geqq l}} (\bar{r}(\lambda) - r(\lambda)) d_\lambda \leqq \sum_{j=0}^{l-1} \sum_{\substack{\lambda \in \Lambda_{k^2}(n) \\ \lambda_{k^2} \neq j}} \bar{r}(\lambda) d_\lambda.$$

Let $\mu = (1^{k^2})$, $\lambda \in \Lambda_{k^2}(n)$, $\lambda_{k^2} = j$, then $\lambda = j \cdot \mu + \lambda'$ where $\lambda' \in \Lambda_{k^2-1}(n - j \cdot k^2)$, so ($n$ large)

$$d_\lambda \leqq 2 \cdot (n - jk^2)^{j(k^2-1)} \cdot d_{\lambda'}.$$

Also, let $\bar{r}(\lambda) \leqq f(n)$ as in Theorem 4. We have:

$$t_n(F_k) - c_n(F_k) \leqq f(n) \cdot \sum_{j=0}^{l-1} 2 \cdot (n - jk^2)^{j(k^2-1)} \sum_{\lambda' \in \Lambda_{k^2-1}(n-jk^2)} d_{\lambda'}$$

$$\underset{n \to \infty}{\simeq} v(n) \cdot (k^2 - 1)^n$$

where $v(x)$ is the rational function

$$v(x) = f(x) \cdot q_{k^2}(x) \cdot \sum_{j=0}^{l-1} \frac{2 \cdot (x - jk^2)^{j(k^2-1)}}{(k^2-1)^{jk^2}},$$

$q_{k^2}(x)$ being given in Theorem 6. Q.E.D.

REMARK 7 [5, §6]. For F_2, $t_n(F_2) - c_n(F_2) = 2^n + \binom{n}{3} - 1$.

As an application we now show that Theorem 1 captures the exponential behaviour of the codimensions associated with the Capelli polynomials $d_m[x; y]$ for $m = k^2 + 1$ [6].

Let $T(d_m) \subset F\langle x \rangle$ be the T ideal generated in $F\langle x \rangle$ by d_m, $U(d_m) = F\langle x \rangle / T(d_m)$ and $c_n(d_m) = c_n(U(d_m))$ the codimensions.

PROPOSITION 8. *Let*

$$p_k(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^{k-1} \cdot \left(\frac{1}{2}\right)^{(k^2-1)/2} \cdot 1! \cdots (k-1)! \cdot k^{(k^2+4)/2} \cdot \left(\frac{1}{x}\right)^{(k^2-1)/2}$$

(*see Theorem* 1). *There exists a second rational function $p'_k(x)$ such that for large n,*

$$p_k(n) \cdot k^{2n} \lesssim c_n(d_{k^2+1}) \lesssim p'_k(n) \cdot k^{2n}.$$

PROOF. The first inequality follows directly from Theorem 1 since F_k satisfies d_{k^2+1}.

Apply "degree" to [6, th. 2] (with $a_\lambda = r(\lambda)$) to get:

$$c_n(d_{k^2+1}) = \sum_{\lambda \in \Lambda_{k^2}(n)} r(\lambda) d_\lambda.$$

By Theorem 3, $r(\lambda) \leqq g(n)$ for some polynomial $g(x)$, so

$$c_n(d_{k^2+1}) \leqq g(n) \cdot \sum_{\lambda \in \Lambda_{k^2}(n)} d_\lambda,$$

and the proof follows from Theorem 6. Q.E.D.

It would be interesting to prove an analogue of Proposition 8 for $c_n(d_m)$ with any $m \geqq 5$.

REFERENCES

1. A. Berele and A. Regev, *Applications of hook Young diagrams to P.I. algebras*, J. Algebra **82** (1983), 559–567.

2. V. S. Drensky, *Codimensions of T-ideals and Hilbert series of relatively free algebras*, C.R. Acad. Bulg. Sci. **34** (1981), 1201–1204.

3. E. Formanek, *The polynomial identities of matrices*, in *Algebraists' Homage: Papers in Ring Theory and Related Topics*, AMS Contemp. Math. **13** (1982), 41–81.

4. E. Formanek, *Invariants and the ring of generic matrices*, J. Algebra, to appear.

5. C. Procesi, *Computing with* 2×2 *matrices*, preprint.

6. A. Regev, *Algebras satisfying a Capelli identity*, Isr. J. Math. **33** (1979), 149–154.

7. A. Regev, *The polynomial identities of matrices in characteristic zero*, Comm. Algebra **8** (15) (1980), 1417–1467.

8. A. Regev, *Asymptotic values for degrees associated with strips of Young diagrams*, Adv. Math. **41** (2) (1981), 115–136.

9. A. Regev, *Combinatorial sums, identities and trace identities of the* 2×2 *matrices*, Adv. Math. **46** (1982), 230–240.

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