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ON COMBINATORIAL AND AFFINE AUTOMORPHISMS OF POLYTOPES

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ABSTRACT
We disprove the longstanding conjecture that every combinatorial automorphism of the boundary complex of a convex polytope in euclidean space E^d can be realised by an affine transformation of E^d.

1. Introduction

Let P be a (convex) d-dimensional polytope in the euclidean space E^d and $B(P)$ its boundary complex. A combinatorial automorphism of P is a bijective, inclusion-preserving mapping ϕ of the (abstract) complex $B(P)$ onto itself. In [4], p. 289, the following problem is discussed: Given any d-polytope P and a combinatorial automorphism ϕ of P, does there always exist an orthogonal transformation ψ of E^d and a polytope P' combinatorially equivalent to P such that $\psi(P') = P'$ and such that ψ induces the combinatorial automorphism ϕ in $B(P')$. In other words, is every combinatorial automorphism of a polytope "affinely realisable"? Mani [7] has given a positive answer to this question for the case $d = 3$. Perles positively decides the problem for d-polytopes having at most $d + 3$ vertices (compare [3], p. 120). So far, these seem to be the only results in this direction. Our following theorem shows that the question generally has to be answered in the negative.

THEOREM. *There is a 4-polytope P with 10 vertices and a combinatorial automorphism ϕ of P, which cannot be realised by an affine transformation ψ of E^4 and its effect on the boundary complex of a polytope P' combinatorially equivalent to P, where $\psi(P') = P'$.*

The combinatorial scheme $\mathscr{C}$ of the boundary of P and the automorphism ϕ have been found by the third author who also proved that ϕ cannot be realised

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affinely in case $\mathscr{C}$ is isomorphic to the boundary complex of a convex polytope (first part of the proof).

Whether the latter is true remained an open question, being part of the unsolved "Steinitz-problem" of characterising the boundary complexes of convex polytopes intrinsically. Meanwhile the first two authors have independently developed methods of solving special cases of the Steinitz-problem, both suitable to be applied to the example under consideration. The first method proceeds by the calculation of coordinates for the vertices of P after having simplified the problem by means of Plücker–Graßmann-relations. The second method generalizes the concept of stellar subdivisions and presents an existence proof for P which visualizes the Schlegel-diagram of the polytope. We outline both methods.

2. A non-realizable symmetry

We describe $B(P)$ in Table 1 by listing the vertices of its facets, which are all tetrahedra. We describe the automorphism ϕ of $B(P)$ by the images of the vertices under ϕ. The vertices 2, 3, 5 and 6 are left invariant under ϕ. On the set of the remaining vertices ϕ is involutoric, where $\phi(1)=9$, $\phi(7)=8$ and $\phi(4)=10$.

Table 1

1	2	3	7	1	2	4	8	9	2	3	8	9	2	10	7
1	3	4	7	2	3	4	8	9	3	10	8	2	3	10	7
1	4	6	7	1	5	6	8	9	10	6	8	9	5	6	7
4	5	6	7	1	4	5	8	10	5	6	8	9	10	5	7
1	2	6	7	1	2	6	8	9	2	6	8	9	2	6	7
3	4	5	7	3	4	5	8	3	10	5	8	3	10	5	7
1	4	5	6	1	2	3	4	9	10	5	6	9	2	3	10

We assume that there is a polytope P' combinatorially equivalent to P and an affine transformation ψ of E^d with $\psi(P')=P'$ such that ψ induces the combinatorial automorphism ϕ on $B(P')$. We use the same symbols for corresponding vertices of P and P'. The only faces of P' which are fixed under ψ are the edges 23, 35, 56, 26 and their vertices. This can easily be read off from the effect of ϕ on $B(P')$. These faces, which form a circuit in $B(P')$, are even point-wise invariant because their vertices are fixed. Hence, the affine flat $A=\operatorname{aff}(2, 3, 5, 6)$ is point-wise fixed under ψ. Furthermore, A has to be 2-dimensional, for otherwise ψ would leave a 3-dimensional subspace of E^4

invariant, whose intersection with the boundary of P' would be 2-dimensional. Consequently, there would exist an invariant 2-face of P'.

The affine transformation ψ maps the line determined by the vertices 7 and 8 on itself and the line therefore contains a fixed point $p \in A$.

From this and from the planarity of aff $(2, 3, 5, 6) = A$ we may conclude that the affine hull of the vertices 2, 3, 5, 6, 7 and 8 is 3-dimensional and that their convex hull is an octahedron O, the latter following from the fact that the faces formed by these vertices form a complex isomorphic to the boundary-complex of an octahedron.

Let H denote the affine hull of O and H^- and H^+ the open halfspaces of E^4 determined by H. Without loss of generality we may assume that the edge 14 of P' lies in H^+ and 9 10 in H^-, for H cannot separate the vertices 1 and 4 or 9 and 10, as otherwise the edge 14 (or 9 10) would have inner points in common with a face of O.

From our construction it follows that H is a supporting hyperplane of the polytope $P^* := \operatorname{conv}(1, 2, \cdots, 8)$ *and* O is the face of P^* which is the intersection of H and P^*. Except for O the facets of P^* are exactly those facets of P' which contain neither 9 nor 10. From Table 1 we deduce that P^* is combinatorially isomorphic to a polytope Q constructed in [6] (in [6] the same numbering is used for the description of the facets). In [6] it is shown that there does not exist a polytope Q' combinatorially equivalent to Q such that the vertices 2, 3, 5 and 6 lie in a plane. In P^*, however, we have seen that $\dim \operatorname{aff}(2, 3, 5, 6) = 2$. Consequently, we have a contradiction to our assumption of the existence of P' and ψ, which completes the proof of the theorem.

We remark that there is even a "central symmetry" of our complex which cannot be realized by an affine mapping. It is involutoric and it can be described by the image of five vertices:

$$\varphi(1) = 10, \quad \varphi(2) = 5, \quad \varphi(3) = 6, \quad \varphi(4) = 9, \quad \varphi(7) = 8.$$

3. Finding the coordinates by using Plücker–Graßmann-relations

The existence of a convex polytope combinatorially equivalent to P can be checked by the coordinates of its vertices presented in Table 2, where ε is a sufficiently small positive number which, e.g., may be chosen to be 10^{-8}.

We briefly describe how we found these coordinates. Identify each vertex $v \in \{1, 2, \cdots, 10\}$ with a vector $(1, x_1^v, x_2^v, x_3^v, x_4^v)$. Let $i, j, k, l \in \{1, 2, \cdots, 10\}$ be the vertices of an arbitrary facet and v another vertex of P.

For this fixed facet and for all choices of v the determinants

$$\left|\begin{array}{cccc} 1 & x_1^i, & \cdots, & x_4^i \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 1 & x_1^l, & \cdots, & x_4^l \\ 1 & x_1^v, & \cdots, & x_4^v \end{array}\right|$$

must have equal signs. On the other hand these conditions for the determinants imply the convexity of P.

We first listed all inequalities of this type. Then we used the Plücker–Graßmann-relations satisfied by our determinants (compare e.g. [5]) to obtain a reduction of the original number of inequalities.

The reduced system finally was clear enough to enable us to find the above solution.

In conclusion, we remark that the contradiction obtained in section 2 from the planarity of aff (2,3,5,6) can also be obtained by use of Plücker–Graßmann-relations.

4. Geometrical construction of P

Let $\mathscr{C}$ be a simplicial complex of dimension d which is realized by a d-diagram (see [2], p. 44) and let $T^1,\cdots,T^{(j)}$, $1\leqq j\leqq d$, be d-cells of $\mathscr{C}$ each having a $(d-1)$-cell in common with a d-cell T. It can be shown that $K:=T\cup T^1\cdots\cup T^{(j)}$ is star-shaped and that the center C of K has points in int T (see [1]). Let $p\in C$, and let $\mathscr{K}$ be the complex such that $K=\text{set}\,\mathscr{K}$.

Table 2

	Coordinates			
Vertex v	x_1^v	x_2^v	x_3^v	x_4^v
1	0	1	1	ε
2	$-2/3$	$+1/3$	$\varepsilon/1000$	1/3
3	0	$-7/18$	1/18	4/3
4	1/3	1/3	1	7/18
5	1	0	0	0
6	0	1	0	0
7	0	0	0	1
8	0	0	1	0
9	0	0	0	0
10	$5\varepsilon/3$	$-\varepsilon$	$\varepsilon/100$	$-\varepsilon/100$

$$\sigma(p;K)\mathscr{C} := (\mathscr{C} \setminus \mathscr{K}) \cup (\{p\} \cdot \partial\mathscr{K})$$

($\{p\} \cdot \partial\mathscr{K}$ the join of the complexes $\{p\}$ and $\partial\mathscr{K}$) is called a hyperstellar subdivision of $\mathscr{C}$, its inverse an inverse hyperstellar subdivision. It can be shown (see [2]) that: If $\mathscr{C}$ is a Schlegel-diagram, then $\sigma(P;K)\mathscr{C}$ is isomorphic to a Schlegel-diagram.

Let a 3-simplex D_0 be the outer facet of a 3-diagram, and let its vertices be denoted by a, b, c, d. Consider aff D_0 to be extended to real 4-space $\mathbf{R}^4$. We shall construct a 4-polytope whose boundary complex contains D_0 and is projected onto a Schlegel-diagram by vertical projection onto aff D_0. We use the same letters for vertices in $\mathbf{R}^4$ and their projections into aff D_0. There are 6 steps to be achieved.

Step 1. Consider a 4-simplex conv $(D_0 \cup \{e\})$ such that e is projected into int D_0, and stack a simplex *abdef* onto *abde* obtaining a double simplex P_1.

Step 2. Let, in the Schlegel-diagram $\mathscr{C}_1$ of P_1, $T := abef$, $T' := adef$, $T'' : abce$, $K := T \cup T' \cup T''$. By a hyperstellar subdivision $\sigma(g,K)\mathscr{C}_1$ we obtain a Schlegel-diagram of a polytope P_2 (Fig. 1). Here g must be chosen close enough to ae. In P_2 the vertex g is found by choosing ε_1, μ appropriately in

$$g = \varepsilon_1[\tfrac{1}{2}(b+f)-\tfrac{1}{2}(a+e)]+\tfrac{1}{2}(a+e)+\mu(0,0,0,1).$$

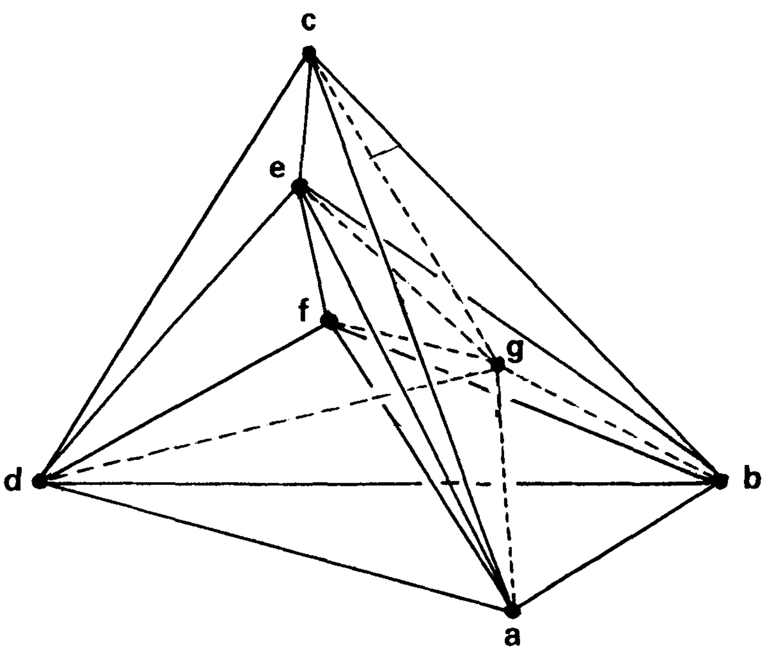

Fig. 1.

Step 3. We choose a point h' close to eg such that

(1) $F := bcegh'$ is a convex double-simplex where ceg is the inner triangle of F.

(2) The simplices $fdeh'$, $fdgh'$, $daeh'$, $dagh'$, $aceh'$, $acgh'$ are new faces replacing $fdeg$, $daeg$, and $aceg$.

(3) a, b, c, d, e, f, g, h' are vertices of a polytope P_3. h' can be found by choosing ε_2, ε_3 appropriately in

$$h' := \tfrac{1}{2}(e+g) + \varepsilon_2[\tfrac{1}{2}(e+g) - b] + \varepsilon_3[(0,0,0,1) - \tfrac{1}{2}(e+g)].$$

Step 4. Now we move h' slightly towards aff D_0 and leave all other vertices unchanged: $h := h' - \varepsilon_4(0,0,0,1)$. The double simplex F breaks down into 2 simplices. We obtain a simplicial polytope P_4 (Fig. 2).

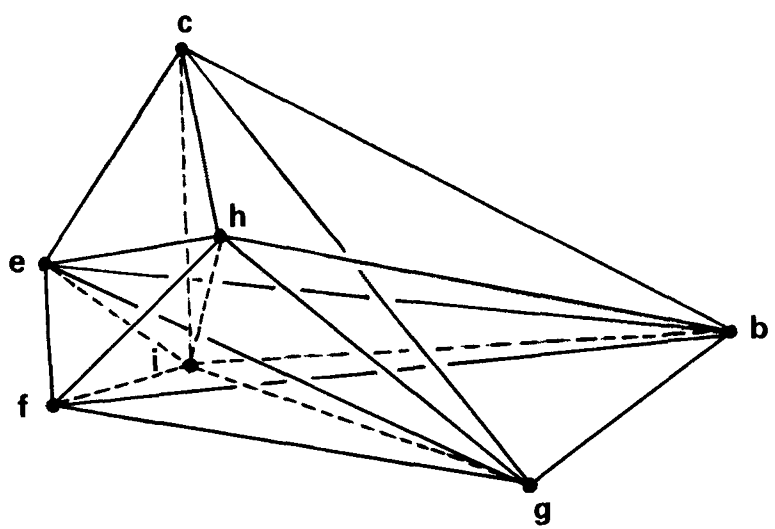

Fig. 2.

Step 5. We choose a new vertex i close to eg such that

(1) $\tilde{F} := gdebi$ is a convex double simplex where bci is the inner triangle of $\tilde{F}$.

(2) The simplices $fghi$, $efhi$, $cehi$, $cghi$, $bfgi$, $befi$ are new faces, replacing $fheg$, $hceg$, $bfeg$ (see Fig. 2 where a, d and the 1-cells emanating from a, d are left out).

The vertex i can be found by choosing ε_5 sufficiently small >0 in

$$i := \tfrac{1}{2}(e+g) + \varepsilon_5[\tfrac{1}{2}(e+g) - \tfrac{1}{2}(b+c)].$$

We obtain a polytope P_5 (Fig. 3).

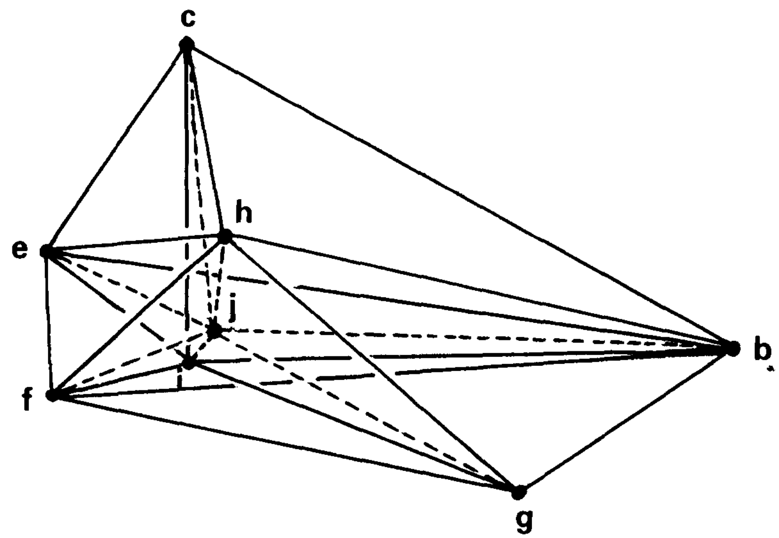

Fig. 3.

Step 6. In the diagram of P_5 consider $\tilde{F} = bcegi$ dissected at the triangle bci, obtaining a complex $\mathscr{C}'$. Let

$$T := cghi, \quad T' := bcgi, \quad T'' := fghi, \quad K := T \cup T' \cup T''.$$

We apply a hyperstellar subdivision $\sigma(j, K)$ to $\mathscr{C}'$ where j is chosen in T such that the line segment $[j, e]$ intersects the triangle bci in its interior. Now "lift" j sufficiently high so that a polytope P is obtained whose boundary complex is isomorphic to $\mathscr{C}$. We can find j by choosing ε_6, ε_7 appropriately in

$$j = \tfrac{1}{2}(g + i) + \varepsilon_6[(\tfrac{1}{2}(g + i) - \tfrac{1}{2}(b + f)] + \varepsilon_7(0,0,0,1).$$

J. Bokowski and B. Neidt have carried out the search for appropriate $\varepsilon_1, \cdots, \varepsilon_8, \mu$; they obtained a polytope with coordinate vectors:

$$a = (1,0,0,0), \quad b = (0,1,0,0), \quad c = (0,0,1,0), \quad d = (-1,-1,-2,0),$$

$$e = (0,0,0,0.5), \quad f = (0,0,-0.5,3.25), \quad g = (0.45,0.05,-0.025,0.3),$$

$$h = (0.227025, 0.015225, -0.023225, 0.4),$$

$$i = (0.225225, 0.024525, -0.0130125, 0.4004),$$

$$j = (0.33795011, 0.36799763, -0.018775256, 0.35175).$$

We obtain an isomorphism of this polytope and that presented in Table 2 if we map the vertices as follows:

$$a \to 1, \quad b \to 7, \quad c \to 6, \quad d \to 4, \quad e \to 5,$$

$$f \to 3, \quad g \to 2, \quad h \to 8, \quad i \to 10, \quad j \to 9.$$

Note added in proof. In the meantime, A. Altshuler has found yet another proof for the polytopality of our complex $\mathscr{C}$ (oral communication).

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