ISRAEL JOURNAL OF MATHEMATICS, Vol. 32, No. 4, 1979

I-ADIC COMPLETIONS OF NON-COMMUTATIVE RINGS

BY

J. C. McCONNELL†

ABSTRACT

Some sufficient conditions are given for the I-adic completion of a non-commutative ring R to be Noetherian. The case considered is when I is polycentral or has a normalising set of generators. In the polycentral case we use an associated graded ring argument.

1. Introduction

Let R be a ring with 1, I an ideal of R and $\hat{R}$ the I-adic completion of R. If R is a commutative noetherian ring then $\hat{R}$ is also a noetherian ring. However it is possible for $\hat{R}$ to be noetherian even if R itself is not. (For example if I is the augmentation ideal of the group algebra kG, where k is a field of characteristic zero and G is a torsionfree abelian group of finite torsionfree rank. See [2].) If R is commutative then a necessary and sufficient condition for $\hat{R}$ to be noetherian is that R/I be noetherian and I be finitely generated modulo I^2. If R is a non-commutative ring, then the condition that R be (left) noetherian need not imply that $\hat{R}$ is (left) noetherian. (See [7].)

In this paper some sufficient conditions for $\hat{R}$ to be noetherian are given. We show that if R/I is left noetherian and I is polycentral modulo I^2 then $\hat{R}$ is left noetherian. Particular cases of this result are known, [5], [9], [11].

In the example ([7]) of a left and right noetherian ring R for which $\hat{R}$ was not left or right noetherian, I had a normalizing set of generators, $I = (u, v)$, where $uR = Ru$ and R/Ru is a commutative ring. We show that if *either* R/I is left and right artinian and I has a normalizing set of generators modulo I^2 *or* R is left and

† This paper was written while the author was a Fellow of the Institute for Advanced Studies, The Hebrew University, Mount Scopus, Jerusalem, Israel.

Received July 4, 1978

right noetherian and I has a normalizing set of generators $x_1, \cdots, x_n$ such that $x_1 I = I x_1$ and $x_i I = I x_i$ modulo $(x_1, \cdots, x_{i-1})$ for $i > 1$ then $\hat{R}$ is left and right noetherian.

2. Notation and definitions

For the definition and basic properties of the I-adic topology and the corresponding completion of a ring R see [1] chapter 10. Let R be a ring and $\{W_n : n \geqq 1\}$ a chain of ideals of R with $W_n \supset W_{n+1}$ for all n and $W_i W_j \subset W_{i+j}$ for all $i, j \geqq 1$. Then we may form the graded ring

$$R^* = R/W_1 \oplus W_1/W_2 \oplus \cdots \quad \text{(compare [1], p. 111).}$$

The following result is proved just as in the commutative case, [1], 10.23.

2.1. PROPOSITION. *Let $\{W_n\}$ be as above. If $\{W_n : n \geqq 1\}$ and $\{I^n : n \geqq 1\}$ define the same topology on R (i.e. $\forall k, \exists m, n$ s.t. $W_n \subset I^k$ and $I^m \subset W_k$) and R^* is a left noetherian ring then the I-adic completion $\hat{R}$ is left noetherian.* □

2.2. DEFINITION. An ideal K of a ring R is polycentral if there is a finite chain of ideals $K = K_0 \supset \cdots \supset K_{t+1} = 0$ such that, for $0 \leqq j \leqq t$, K_j/K_{j+1} is generated (as an ideal) by finitely many elements of the centre of R/K_{j+1}. A set of generators $\{x_1, \cdots, x_n\}$ of an ideal K is a normalizing set of generators if $x_1 R = R x_1$ and $x_i R = R x_i$ modulo $(x_1, \cdots, x_{i-1})$. ([6] and [10], p. 488.)

If K is a polycentral ideal of a ring R then, following Roseblade, [12] or [10], p. 488, we construct a certain graded ring $\mathcal{R}$ containing R as a subring. If the K_i's are as in 2.2, let $\mathcal{R}$ be the subring of $R[X]$ generated by $\{R, K_j X^s : 0 \leqq j \leqq t, 1 \leqq s \leqq 2^j\}$. Then

2.3. $\mathcal{R} = R + L_1 X + L_2 X^2 + \cdots$, where the L's are ideals of R which satisfy $L_1 = K$, $L_1 \supset L_2 \supset \cdots$, $L_n \supset K^n$ and $L_i L_j \subset L_{i+j}$.

2.4. For $m \in N$ and $n > 2^{mt}$, we have that $L_n \subset L_1^m$, since $\mathcal{R}$ is generated by R and $L_1 + L_2 X + \cdots + L_t X^{2^t}$.

2.5. Define $\mathcal{R}_{t+1} = R$ and for $t \geqq i \geqq 0$ define $\mathcal{R}_i$ as the subring of $\mathcal{R}$ generated by $\{\mathcal{R}_{i+1}, K_i X^s : 1 \leqq s \leqq 2^i\}$. Note that $\mathcal{R}_0 = \mathcal{R}$. Then $\mathcal{R}_i$ is generated by $\mathcal{R}_{i+1}$ and finitely many elements $v_1, \cdots, v_l$ such that for $1 \leqq j, k \leqq l$, $[\mathcal{R}_{i+1}, v_j] \subset \mathcal{R}_{i+1}$ and $[v_j, v_k] \in \mathcal{R}_{i+1}$. (See [10] chapter 11, proof of lemma 2.6.)

3. Polycentral ideals

Suppose that I is polycentral modulo I^2, i.e. there is a polycentral ideal K of R with $K \subset I$ and $I = K + I^2$. Let K_i and L_i be as in 2.2 and 2.3. $\{L_n\}$ and $\{I^n\}$ need not define the same topology on R. Define $\{T_n\}$ by $T_n = L_n + I^n$ for $n \geqq 1$.

3.1. PROPOSITION. *$\{T_n\}$ and $\{I^n\}$ define the same topology on R.*

PROOF. From 2.4 we have that for $n > 2^{mt}$, $T_n \subset I^n$. □

The T's need not satisfy $T_iT_j \subset T_{i+j}$ so define $\{W_n\}$ inductively by $W_1 = T_1$ ($= I$) and $W_n = T_n + \Sigma\, W_iW_j$, summed over $i, j \geqq 1$ with $i + j = n$. By induction on n we have

3.2. PROPOSITION. *$W_1 \supset W_2 \supset \cdots$, $W_iW_j \subset W_{i+j}$, $W_n \supset I^n$ and for $n > 2^{mt}$, $W_n \subset I^m$.* □

Thus $\{W_n\}$ and $\{I^n\}$ determine the same topology on R.

3.3. LEMMA. $W_n = L_n$ modulo W_{n+1}.

PROOF. Since $L_1 = K$ and $K = I$ modulo I^2, $L_1^n = I^n$ modulo I^{n+1} and hence $L_n = T_n$ modulo T_{n+1}. Hence $L_1 = W_1$ modulo W_2. Suppose that $L_i = W_i$ modulo W_{i+1} for $1 \leqq i \leqq n$. Then $W_n = T_n + \Sigma\, W_iW_j = L_n + \Sigma\, L_iL_j$ modulo W_{n+1} since $W_iW_j \subset W_{i+j}$. So $W_n = L_n$ modulo W_{n+1} from 2.3. □

Let R^* be the graded ring $R/W_1 \oplus W_1/W_2 \oplus \cdots$ (which is well defined by 3.2).

3.4. THEOREM. *Let R be a ring and I an ideal of R such that I is polycentral* modulo I^2. *If R/I is left noetherian, then the I-adic completion $\hat{R}$ is left noetherian.*

PROOF. From 3.3, the natural R-module map: $L_n \to W_n/W_{n+1}$ is surjective. By combining these maps in each degree there is a surjective R-module homomorphism, which is in fact a ring homomorphism, from

$$\mathcal{R} = R + L_1X + L_2X^2 + \cdots$$

onto

$$R^* = R/W_1 \oplus W_1/W_2 \oplus W_2/W_3 \oplus \cdots.$$

Consider $\operatorname{Im} \mathcal{R}_i$, the image of $\mathcal{R}_i$ in R^* for $t + 1 \geqq i \geqq 0$. $\operatorname{Im} \mathcal{R}_{t+1} = R/W_1 = R/I$ is left noetherian. Hence, by 2.5 and [10] lemma 2.5, each of $\operatorname{Im} \mathcal{R}_t, \cdots, \operatorname{Im} \mathcal{R}_1, \operatorname{Im} \mathcal{R}_0 = R^*$ is left noetherian. So $\hat{R}$ is left noetherian by Proposition 2.1. □

4. Normalising sets of generators

4.1. Let $\bigcap_{n=1}^{\infty} I^n = 0$ so that R is a subring of $\hat{R}$. Let T be an ideal of R contained in I and consider the exact sequence of R-modules

$$0 \to T \to R \to R/T \to 0.$$

Since the topology on R/T induced from the I-topology on R coincides with the natural I-topology on the R-module R/T, on completing these modules we obtain the exact sequence of $\hat{R}$-modules, $\hat{R} \to (R/T)^{\wedge} \to 0$, by [1] corollary 10.3. $(R/T)^{\wedge}$ may be regarded as a ring, viz. the completion of R/T in its I/T-topology and the above homomorphism is a ring homomorphism which we denote by π. (For more details see [5], proposition 7.)

If K is the kernel of π then K is the closure in $\hat{R}$ of T, i.e., $K = \bigcap_{n=1}^{\infty} (T + (\hat{I})^n)$.

4.2. THEOREM. *Let R be a ring and I an ideal of R such that R/I is left and right artinian and I has a normalising set of generators* modulo I^2. *Then $\hat{R}$ is left and right noetherian.*

PROOF. Let the normalising set of generators be $\{u, v, \cdots, w\}$ and set $L = (u, v, \cdots, w)$. Then $I = L + I^2$. We may assume that $\bigcap_{n=1}^{\infty} I^n = 0$. Consider the exact sequence of R-modules, $0 \to Ru \to R \to R/Ru \to 0$ and the corresponding exact sequence of 4.1,

$$(*) \qquad 0 \to K \to \hat{R} \to (R/Ru)^{\wedge} \to 0.$$

Since $\hat{I} = \hat{R}u + \cdots + \hat{R}w$ modulo $\hat{I}^2$ and $\hat{I} \subset J$, the Jacobson radical of $\hat{R}$, $\hat{I} = \hat{R}u + \cdots + \hat{R}w$, by Nakayama's Lemma, i.e., $\hat{I}$ is finitely generated as a left ideal of $\hat{R}$. Since $\hat{R}/\hat{I} \cong R/I$ is left artinian and hence left noetherian (Hopkins), J is a finitely generated left ideal of $\hat{R}$. Now K is the closure in $\hat{R}$ (in the $\hat{I}$-adic or J-adic topologies) of $\hat{R}u$. But by Hinohara's Theorem, [4] lemma 3, $\hat{R}u$ is closed in $\hat{R}$. Thus $K = \hat{R}u$. Similarly, using right modules instead of left modules throughout, we have $K = u\hat{R}$.

By using an inductive hypothesis on the number of elements in the normalising set of generators, we may assume that $S := (R/Ru)^{\wedge}$ is left and right noetherian. So we have an exact sequence

$$(*) \qquad 0 \to K \to \hat{R} \to S \to 0$$

where S is left and right noetherian and $K = \hat{R}u = u\hat{R}$. So the associated graded ring $\operatorname{Gr}\hat{R} := \hat{R}/K \oplus K/K^2 \oplus \cdots$ is generated by $S = \hat{R}/K$ and the element

$Y := u + K^2$ which is homogeneous of degree one. Since $u\hat{R} = \hat{R}u$, $YS = SY$ and so $\mathrm{Gr}\,\hat{R}$ is left and right noetherian by [6] theorem 9. Since $\hat{R}$ is complete in its K-topology, $\hat{R}$ is left and right noetherian by Proposition 2.1. □

4.3. THEOREM. *Let R be a ring which is left and right noetherian and I an ideal of R with a normalising set of generators $x_1, \cdots, x_t$ such that $x_1 I = I x_1$ and for $i > 1$, $x_i I = I x_i$* modulo $(x_1, \cdots, x_{i-1})$. *Then $\hat{R}$ is left and right noetherian.*

PROOF. We may assume that $\bigcap_{n=1}^{\infty} I^n = 0$ so that R is a subring of $\hat{R}$. By [8], lemma B, I has the left topological Artin–Rees property, viz. that if M is a finitely generated left R-module and N is a submodule of M, then $\forall k$, $\exists n$ such that $I^n M \cap N \subset I^k N$. A similar statement holds for right modules. So, just as for commutative rings, completion is an exact functor on finitely generated left (or right) R-modules and if M is such a module then $\hat{M} \cong \hat{R} \otimes_R M$. (See e.g. [3] proposition 3 and [11] theorem 1.) Applying $\hat{R} \otimes_R$- to the exact sequence $0 \to Ru \to R \to R/Ru \to 0$ where $u = x_1$ we obtain the exact sequence $0 \to \hat{R}u \to \hat{R} \to (R/Ru)^\wedge \to 0$. Using right modules instead of left modules we have that $0 \to u\hat{R} \to \hat{R} \to (R/Ru)^\wedge \to 0$ is exact. Thus $\hat{R}u = u\hat{R}$. The proof is completed as in Theorem 4.2. □

REMARK. We take this opportunity of pointing out that in the statement of lemma B of [8] the inequality signs are the wrong way round. The statement should read (a) $Ix_1 \subset x_1 I$ and (b) $Ix_i \subset x_i I + x_1 R + \cdots + x_{i-1} R$ for $i = 2, \cdots, n$.

An example of an ideal I satisfying the conditions of Theorem 4.3 is the following. Let g be a completely solvable Lie algebra over a field of characteristic zero, $R = U(g)$ the universal enveloping algebra of g, n a nilpotent ideal of g and I the ideal of R generated by n. (See [8].)

REFERENCES

1. M. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley, Reading, Massachusetts, 1969.
2. J. W. Brewer, D. L. Costa and E. L. Lady, *Prime ideals and localization in commutative group rings*, J. Algebra **34** (1975), 300–308.
3. K. S. Brown and E. Dror, *The Artin–Rees property and homology*, Israel J. Math. **22** (1975), 93–109.
4. Y. Hinohara, *Note on non-commutative semi-local rings*, Nagoya Math. J. **17** (1960), 161–166.
5. T. Levasseur, *Cohomologie des algèbres de Lie nilpotentes et enveloppes injectives*, Bull. Sci. Math. 2e séries, **100** (1976), 377–383.
6. J. C. McConnell, *Localisation in enveloping rings*, J. London Math. Soc. **43** (1968), 421–428.
7. J. C. McConnell, *The noetherian property in complete rings and modules*, J. Algebra **12** (1969), 143–153.

8. J. C. McConnell, *Localisation in enveloping rings — erratum and addendum*, J. London Math. Soc. (2) **3** (1971), 409–410.

9. J. C. McConnell, *On completions of non-commutative noetherian rings*, Comm. Alg. **6**(14) (1978), 1485–1488.

10. D. S. Passman, *The Algebraic Structure of Group Rings*, Wiley, New York, 1977.

11. P. F. Pickel, *Rational cohomology of nilpotent groups and Lie algebras*, Comm. Alg. **6** (1978), 409–419.

12. J. E. Roseblade, *Applications of the Artin–Rees lemma to group rings*, Symposia Mathematica Vol. XVII, Academic Press, New York, 1976.

School of Mathematics
University of Leeds
Leeds LS2 9JT England