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NONLINEAR PARABOLIC BOUNDARY VALUE PROBLEMS WITH UPPER AND LOWER SOLUTIONS

BY

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ABSTRACT

The existence of at least one periodic solution of a very general second order nonlinear parabolic boundary value problem is proved under the assumption that a lower solution ϕ and an upper solution ψ with $\phi \leqq \psi$ are known.

1. Introduction and statement of the result

In this paper we prove the existence of at least one periodic solution of the parabolic boundary value problem

$$(1) \qquad \begin{cases} \left(\dfrac{\partial u}{\partial t}\right)(x,t)+(\mathscr{A}u)(x,t)+(Pu)(x,t)=f(x,t) & \text{in } Q=\Omega\times(0,T) \\ u(x,t)=0 & \text{on } \Sigma=\partial\Omega\times(0,T) \\ u(x,0)=u(x,T) \text{ in } \Omega \end{cases}$$

provided we know the existence of a lower solution ϕ and an upper solution ψ of (1) with $\phi \leqq \psi$ in Q. Here Ω is a bounded domain in $\mathbf{R}^N$ $(N \geqq 1)$ with smooth boundary $\partial\Omega$, $T>0$ is a given number, and $\mathscr{A}$:

$$(2) \qquad (\mathscr{A}u)(x,t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x,t,u(x,t),(\nabla u)(x,t)), \qquad (x,t)\in Q,$$

is a quasilinear differential operator of second order in divergence form. Further P denotes the Nemitskii operator associated with the function $p: Q\times \mathbf{R}\times \mathbf{R}^N \to \mathbf{R}$, i.e.

$$(Pu)(x,t) = p(x,t,u(x,t),(\nabla u)(x,t))$$

for any function u defined in Q. Moreover, f is a given function defined in Q.

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The following standard conditions of Leray–Lions type are imposed on the coefficient functions A_i $(i = 1, \cdots, N)$ of $\mathcal{A}$ (e.g. Lions [6], p. 322):

(A1) Each $A_i : Q \times \mathbf{R} \times \mathbf{R}^N \to \mathbf{R}$ satisfies the Caratheodory conditions, i.e. $A_i(x, t, s, \xi)$ is measurable in $(x, t) \in Q$ for all fixed $(s, \xi) \in \mathbf{R} \times \mathbf{R}^N$ and continuous in (s, ξ) for a.a. fixed (x, t).

(A2) There exist constants q $(1 < q < \infty)$ and $c_0 \geqq 0$ and a function $k_0 \in L^{q'}(Q)$ $(q' = q/(q-1))$ such that

$$| A_i(x, t, s, \xi) | \leqq k_0(x, t) + c_0(| s |^{q-1} + | \xi |^{q-1}),$$

$i = 1, \cdots, N$, for a.a. $(x, t) \in Q$, $\forall (s, \xi) \in \mathbf{R} \times \mathbf{R}^N$.

(A3) $\sum_{i=1}^N (A_i(x, t, s, \xi) - A_i(x, t, s, \xi'))(\xi_i - \xi'_i) > 0$, for a.a. $(x, t) \in Q$, $\forall s \in \mathbf{R}$, $\forall \xi, \xi' \in \mathbf{R}^N$ with $\xi \neq \xi'$.

(A4) $\sum_{i=1}^N A_i(x, t, s, \xi) \xi_i \geqq \alpha | \xi |^q$ $(\alpha > 0)$, for a.a. $(x, t) \in Q$, $\forall (s, \xi) \in \mathbf{R} \times \mathbf{R}^N$.

We set $\mathcal{V} = L^q(0, T; W^{1,q}(\Omega))$ and $\mathcal{V}_0 = L^q(0, T; W_0^{1,q}(\Omega))$. Further let $\mathcal{V}'_0 = L^{q'}(0, T; W^{-1,q'}(\Omega))$ denote the dual space of $\mathcal{V}_0$. As a consequence of (A1)–(A2) the semilinear form a:

$$a(u, v) = \sum_{i=1}^N \int_Q A_i(x, t, u, \nabla u) \frac{\partial v}{\partial x_i} \, dx dt \tag{2'}$$

is defined on $\mathcal{V} \times \mathcal{V}$.

We further assume that the function p satisfies the Caratheodory conditions, and that $f \in L^{q'}(Q)$. In the following let η denote a sufficiently small positive constant.

Definition 1. A function u is said to be a *weak solution* of problem (1) provided

$$u \in \mathcal{V}_0, \ \frac{\partial u}{\partial t} \in \mathcal{V}'_0 + L^{1+\eta}(Q), \ u(0) = u(T) \text{ in } \Omega, \ Pu \in L^1(Q),$$

and

$$\left(\frac{\partial u}{\partial t}, v\right) + a(u, v) + \int_Q Pu \, v \, dx dt = \int_Q f v \, dx dt, \qquad \forall v \in \mathcal{V}_0 \cap L^\infty(Q).$$

Note that the solution $u : [0, T] \to W^{-1,q'}(\Omega) + L^1(\Omega)$ is continuous. Thus the periodicity condition is meaningful.

Definition 2. A function ψ is said to be a *weak upper solution* of problem (1) provided

$$\psi \in \mathcal{V}, \frac{\partial \psi}{\partial t} \in \mathcal{V}'_0 + L^{1+\eta}(Q), \psi |_\Sigma \geqq 0 \text{ a.e., } \psi(0) \geqq \psi(T) \text{ a.e. in } \Omega, P\psi \in L^1(Q),$$

and

$$\left(\frac{\partial\psi}{\partial t}, v\right) + a(\psi, v) + \int_Q P\psi\, v\, dxdt \geqq \int_Q f v\, dxdt,$$

$\forall v \in \mathcal{V}_0 \cap L^\infty(Q)$ with $v \geqq 0$ a.e. in Q.

Similarity a *weak lower solution* ϕ is defined by reversing the inequality signs above.

Our main result is the following

THEOREM. *Suppose ϕ and ψ are given weak lower and upper solutions of problem* (1), *respectively, with $\phi, \psi \in L^\infty(Q)$ and $\phi \leqq \psi$ a.e. in Q. Assume that, with constants $c \geqq 0$, $\delta > 0$ and a suitable function $k \in L^{1+\eta}(Q)$,*

$$|p(x,t,s,\xi)| \leqq k(x,t) + c\,|\xi|^{q-\delta}, \tag{3}$$

for a.a. $(x,t) \in Q$, $\forall \xi \in \mathbf{R}^N$, $\forall s \in \mathbf{R}$ with $\phi(x,t) \leqq s \leqq \psi(x,t)$ in Q. Then there exists a weak solution u of problem (1) *with $\phi \leqq u \leqq \psi$ a.e. in Q.*

REMARKS. 1) A theorem similar to the above — with almost identical proof — holds also for the *initial* boundary value problem.

2) We can always assume $\delta \leqq 1$ and $\eta \leqq \delta/(q-\delta)$. Then in particular $L^{q'}(Q) \subset L^{1+\eta}(Q)$, and the estimate (3) guarantees that $Pv \in L^{1+\eta}(Q)$, $\forall v \in \mathcal{V}$ with $\phi \leqq v \leqq \psi$ a.e. in Q.

3) If $\mathcal{A}$ is a *linear*, uniformly elliptic differential operator with sufficiently smooth coefficient functions (which may depend also on t), it follows from the results of Sobolevskii [9] that $\partial u/\partial t \in L^{1+\eta}(Q)$ and $\mathcal{A}u \in L^{1+\eta}(Q)$.

For *linear* top order part $\mathcal{A}$, this Theorem was proved by Puel [7] (generalizing a result of Sattinger [8]). Our method of proof is however essentially different from Puel's, inasmuch as we operate entirely in the framework of weak solutions and can therefore dispense with any regularity results. It is modelled by the method developed in [4] and [5] for the corresponding stationary problem. Like Puel [7] we associate to problem (1) a parabolic variational inequality; but in contrast to [7] our convex set is stationary. The auxiliary variational inequality is solved by a systematic application of the penalty method.

Also for *linear* $\mathcal{A}$, complete results concerning the existence of *classical* solutions have recently been obtained by Amann [1].

The present Theorem extends some results of the first author's Ph. D. thesis [3] where $\delta = 1$ is assumed. Its proof — though quite elementary — seems rather long. It is therefore split into several parts. The central existence result is stated as a Proposition in Section 2. The proof of the Theorem then follows in Section 3. Some auxiliary results are needed, which are stated as lemmas in the Appendix.

2. Existence theory for a strongly nonlinear parabolic variational inequality

Let $\alpha_1 < 0 < \alpha_2$ be two constants and let the convex set $\mathcal{K}$ be given by

$$\mathcal{K} = \{u \in \mathcal{V}_0 \mid \alpha_1 \leqq u \leqq \alpha_2 \text{ a.e. in } Q\}.$$

Our aim in this section is to prove the following

PROPOSITION. *Let $\mathcal{A}$ be a quasilinear differential operator of the form* (2), *whose coefficient functions satisfy conditions* (A1)–(A4), *and let a denote the associated semilinear form. Further let $\mathcal{F}$ be a mapping having the properties:*

(i) *$\mathcal{F} : \mathcal{V}_0 \to L^{1+\eta}(Q)$ is bounded and continuous,*

(ii) *$\|\mathcal{F}u\|_{L^1(Q)} \leqq \gamma_1 + \gamma_2 \|u\|_{\mathcal{V}_0}^{q-\delta}$, $\forall u \in \mathcal{V}_0$, with constants $\gamma_1, \gamma_2, \delta > 0$.*

Then there exists

$$(4)\qquad \begin{cases} u \in \mathcal{K} \text{ such that} \\[2mm] \dfrac{\partial u}{\partial t} \in \mathcal{V}_0' + L^{1+\eta}(Q), \qquad u(0) = u(T) \text{ in } \Omega \text{ and} \\[2mm] \left(\dfrac{\partial u}{\partial t}, w - u\right) + a(u, w - u) + \displaystyle\int_Q \mathcal{F}u(w - u)\,dxdt \geqq 0, \qquad \forall w \in \mathcal{K}. \end{cases}$$

REMARK. For *linear* $\mathcal{A}$ with smooth coefficient functions, we have by the results of Sobolevskii [9] that $\partial u/\partial t \in L^{1+\eta}(Q)$ and $\mathcal{A}u \in L^{1+\eta}(Q)$.

Proof.

(i) We may assume $\delta \leqq 1$ and $\eta \leqq \delta/(q - \delta)$. For $n \in \mathbf{N}$ let S_n denote the mapping which truncates functions defined in Q by the constants $\pm n$. We first claim that to each n there exists

$$(5)\qquad \begin{cases} u_n \in \mathcal{K} \text{ such that } \dfrac{\partial u_n}{\partial t} \in \mathcal{V}_0', \ u_n(0) = u_n(T) \text{ in } \Omega \text{ and} \\[2mm] \left(\dfrac{\partial u_n}{\partial t}, w - u_n\right) + a(u_n, w - u_n) + \displaystyle\int_Q (S_n \circ \mathcal{F}u_n)(w - u_n)\,dxdt \geqq 0, \qquad \forall w \in \mathcal{K}. \end{cases}$$

Though this result is not new, we present here a complete proof employing the penalty method. For, many of the intermediate steps will be needed later on.

Let $\beta : L^q(Q) \to L^{q'}(Q)$ be a penalty operator associated to the convex set $\mathcal{K}$, defined as the Nemitskii operator of the function $\beta : \mathbf{R} \to \mathbf{R}$:

$$\beta(s)=\begin{cases}(s-\alpha_2)^{q-1}, & s>\alpha_2+1\\ (s-\alpha_2)^{1/\eta}, & \alpha_2<s\leqq\alpha_2+1\\ 0, & \alpha_1\leqq s\leqq\alpha_2\\ -(-s+\alpha_1)^{1/\eta}, & \alpha_1-1\leqq s<\alpha_1\\ -(-s+\alpha_1)^{q-1}, & s<\alpha_1-1.\end{cases}$$

(Note that this is not the simplest choice of β one can find; in this form it is however suitable also for our further purpose since both the function $s\mapsto|\beta(s)|^{1/(q-1)-1}\beta(s)$ and the function $s\mapsto|\beta(s)|^{\eta-1}\beta(s)$ are Lipschitz continuous on $\mathbf{R}$ as $(q-1)\eta\leqq 1$.)

By a well known result of Lions (e.g. [6], p. 328), to each $n\in\mathbf{N}$ and $\varepsilon>0$ there exists

$$(6)\quad\begin{cases} u_{n\varepsilon}\in\mathcal{V}_0 \text{ such that } \dfrac{\partial u_{n\varepsilon}}{\partial t}\in\mathcal{V}_0',\ u_{n\varepsilon}(0)=u_{n\varepsilon}(T) \text{ in } \Omega \text{ and}\\[2ex] \left(\dfrac{\partial u_{n\varepsilon}}{\partial t},v\right)+a(u_{n\varepsilon},v)+\displaystyle\int_Q(S_n\circ\mathcal{F}u_{n\varepsilon})v\,dxdt+\frac{1}{\varepsilon}\int_Q\beta u_{n\varepsilon}v\,dxdt=0,\\ \hfill\forall v\in\mathcal{V}_0.\end{cases}$$

(ii) We let $n\in\mathbf{N}$ *fixed* and consider the passage to the limit $\varepsilon\to 0^+$. Setting

$$v=v_{n\varepsilon}=\left(\frac{1}{\varepsilon}\right)^{1/(q-1)}|\beta u_{n\varepsilon}|^{1/(q-1)-1}\beta u_{n\varepsilon}$$

($\in\mathcal{V}_0$) in (6) and observing that

$$\left(\frac{\partial u_{n\varepsilon}}{\partial t},v_{n\varepsilon}\right)\geqq 0\quad\text{(cf. Lemma 1)}$$

and

$$a(u_{n\varepsilon},v_{n\varepsilon})\geqq 0\quad\text{(cf. Lemma 3)},$$

we obtain an estimate of the form

$$\left\|\frac{1}{\varepsilon}\beta u_{n\varepsilon}\right\|^{q'}_{L^{q'}(Q)}\leqq c_1\left\|\left|\frac{1}{\varepsilon}\beta u_{n\varepsilon}\right|^{1/(q-1)}\right\|_{L^q(Q)}=c_1\left\|\frac{1}{\varepsilon}\beta u_{n\varepsilon}\right\|^{q'/q}_{L^{q'}(Q)}.$$

(Remark. The constants $c_1, c_2,\cdots$ may depend on n, but they are independent of $\varepsilon>0$.) It follows that

$$(7)\qquad \left\|\frac{1}{\varepsilon}\beta u_{n\varepsilon}\right\|_{L^{q'}(Q)}\leqq c_2.$$

Hence, by the coercivity of the mapping $\mathcal{A}$,

$$\| u_{n\varepsilon} \|_{\mathcal{V}_0} \leqq c_3 .$$

The uniform boundedness principle now guarantees that

$$\left\| \frac{\partial u_{n\varepsilon}}{\partial t} \right\|_{\mathcal{V}_0'} \leqq c_4 .$$

We may thus extract a sequence, still denoted by $\{u_{n\varepsilon}\}$, such that

$$u_{n\varepsilon} \rightharpoonup u_n \text{ in } \mathcal{V}_0 \text{ and } \frac{\partial u_{n\varepsilon}}{\partial t} \rightharpoonup \frac{\partial u_n}{\partial t} \text{ in } \mathcal{V}_0' \text{ (as } \varepsilon \to 0^+), \tag{8}$$

whence $u_n(0) = u_n(T)$ in Ω.

By Aubin's lemma (e.g. Lions [6], p. 58) we get

$$u_{n\varepsilon} \to u_n \text{ in } L^q(Q) \quad \text{(as } \varepsilon \to 0^+).$$

From (7) we conclude that $\beta u_{n\varepsilon} \to 0$ in $L^{q'}(Q)$ (as $\varepsilon \to 0^+$). Consequently $\beta u_n = 0$; i.e. $u_n \in \mathcal{K}$.

Setting $v = u_{n\varepsilon} - u_n$ in (6) we see readily that

$$\lim_{\varepsilon \to 0^+} a\,(u_{n\varepsilon}, u_{n\varepsilon} - u_n) = 0 .$$

By Lemma 4 of the Appendix it follows that

$$u_{n\varepsilon} \to u_n \text{ in } \mathcal{V}_0 \text{ (as } \varepsilon \to 0^+).$$

Setting now $v = w - u_{n\varepsilon}$ $(w \in \mathcal{K})$ in (6) and passing to the limit $\varepsilon \to 0^+$ we infer immediately that u_n solves (5).

(iii) In order to pass to the limit $n \to \infty$ in (5) we need some a priori estimates on the sequences $\{u_n\}$ and $\{\partial u_n / \partial t\}$.

Setting $w = 0$ in (5) we get

$$a(u_n, u_n) + \int_Q (S_n \circ \mathcal{F} u_n)\, u_n dx dt \leqq 0 .$$

Since $u_n \in \mathcal{K}$, we obtain the estimate

$$\alpha \| u_n \|^q_{\mathcal{V}_0} \leqq d_1 \| S_n \circ \mathcal{F} u_n \|_{L^1(Q)} .$$

(The constants d_1, $d_2, \cdots$ are now independent of both ε *and* n.)

By assumption $\| S_n \circ \mathcal{F} u_n \|_{L^1(Q)} \leqq \gamma_1 + \gamma_2 \| u_n \|^{q-\delta}_{\mathcal{V}_0}$; hence

$$\| u_n \|_{\mathcal{V}_0} \leqq d_2 . \tag{9}$$

Since $\mathcal{F} : \mathcal{V}_0 \to L^{1+\eta}(Q)$ is bounded we have

$$\| S_n \circ \mathcal{F} u_n \|_{L^{1+\eta}(Q)} \leqq d_3. \tag{10}$$

The desired a priori estimates on $\{\partial u_n / \partial t\}$ do not follow directly. We introduce the differential operator $\mathcal{A}_n$:

$$(\mathcal{A}_n v)(x,t) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} A_i(x,t,u_n(x,t),(\nabla v)(x,t)),$$

where u_n is the just obtained solution of (5). To $\mathcal{A}_n$ we associate the semilinear form a_n.

The solvability of the variational inequality

$$(11) \quad \begin{cases} \tilde{u}_n \in \mathcal{K} \text{ such that } \dfrac{\partial \tilde{u}_n}{\partial t} \in \mathcal{V}_0', \ \tilde{u}_n(0) = \tilde{u}_n(T) \text{ in } \Omega \text{ and} \\ \left(\dfrac{\partial \tilde{u}_n}{\partial t}, w - \tilde{u}_n \right) + a_n(\tilde{u}_n, w - \tilde{u}_n) + \displaystyle\int_Q (S_n \circ \mathcal{F} u_n)(w - \tilde{u}_n) dx dt \geqq 0, \forall w \in \mathcal{K} \end{cases}$$

is proved as above by passing to the limit $\varepsilon \to 0^+$ in the penalized problem

$$(12) \quad \begin{cases} \tilde{u}_{n\varepsilon} \in \mathcal{V}_0 \text{ such that } \dfrac{\partial \tilde{u}_{n\varepsilon}}{\partial t} \in \mathcal{V}_0', \ \tilde{u}_{n\varepsilon}(0) = \tilde{u}_{n\varepsilon}(T) \text{ in } \Omega \text{ and} \\ \left(\dfrac{\partial \tilde{u}_{n\varepsilon}}{\partial t}, v \right) + a_n(\tilde{u}_{n\varepsilon}, v) + \displaystyle\int_Q (S_n \circ \mathcal{F} u_n) v \, dx dt + \frac{1}{\varepsilon} \int_Q \beta \tilde{u}_{n\varepsilon} v \, dx dt = 0, \ \forall v \in \mathcal{V}_0. \end{cases}$$

By strict monotonicity, the variational inequalities (5) and (11) have exactly the same solutions. Thus

$$u_n = \tilde{u}_n$$

and we have

$$\tilde{u}_{n\varepsilon} \to u_n \text{ in } \mathcal{V}_0, \ \frac{\partial \tilde{u}_{n\varepsilon}}{\partial t} \rightharpoonup \frac{\partial u_n}{\partial t} \text{ in } \mathcal{V}_0', \text{ as } \varepsilon \to 0^+. \tag{13}$$

From (12) we can however deduce some a priori estimates on $\{\partial u_n / \partial t\}$. Set

$$v = \tilde{v}_{n\varepsilon} = \left(\frac{1}{\varepsilon} \right)^{\eta} | \beta \tilde{u}_{n\varepsilon} |^{\eta - 1} \beta \tilde{u}_{n\varepsilon}$$

in (12) (note that $\tilde{v}_{n\varepsilon} \in \mathcal{V}_0$ by our choice of the function β). We get

$$\left(\frac{\partial \tilde{u}_{n\varepsilon}}{\partial t}, \tilde{v}_{n\varepsilon} \right) + a_n(\tilde{u}_{n\varepsilon}, \tilde{v}_{n\varepsilon}) + \int_Q (S_n \circ \mathcal{F} u_n) \tilde{v}_{n\varepsilon} dx dt + \left\| \frac{1}{\varepsilon} \beta \tilde{u}_{n\varepsilon} \right\|_{L^{1+\eta}(Q)}^{1+\eta} = 0.$$

Again (by Lemmas 1 and 3)

$$\left(\frac{\partial \tilde{u}_{n\varepsilon}}{\partial t}, \tilde{v}_{n\varepsilon}\right) \geqq 0 \text{ and } a_n(\tilde{u}_{n\varepsilon}, \tilde{v}_{n\varepsilon}) \geqq 0.$$

Knowing that the sequence $\{S_n \circ \mathcal{F}u_n\}$ is bounded in $L^{1+\eta}(Q)$, we infer that

$$\left\|\frac{1}{\varepsilon}\beta\tilde{u}_{n\varepsilon}\right\|^{1+\eta}_{L^{1+\eta}(Q)} \leqq d_4 \left\|\frac{1}{\varepsilon}\beta\tilde{u}_{n\varepsilon}\right\|^{\eta}_{L^{1+\eta}(Q)}.$$

Thus

$$\left\|\frac{1}{\varepsilon}\beta\tilde{u}_{n\varepsilon}\right\|_{L^{1+\eta}(Q)} \leqq d_4, \qquad \forall n \in \mathbf{N}, \qquad \forall \varepsilon > 0. \tag{14}$$

Let $\mathcal{W} = \mathcal{V}_0 \cap L^{(1+\eta)'}(Q)$ and thus $\mathcal{W}' = \mathcal{V}_0' + L^{1+\eta}(Q)$. Since $\{S_n \circ \mathcal{F}u_n + (1/\varepsilon)\beta\tilde{u}_{n\varepsilon}\}$ is bounded in $L^{1+\eta}(Q)$ (by (10) and (14)) and thus in $\mathcal{W}'$, $\forall n \in \mathbf{N}$, $\forall \varepsilon > 0$, we conclude from (12) that

$$\left\|\frac{\partial \tilde{u}_{n\varepsilon}}{\partial t} + \mathcal{A}_n\tilde{u}_{n\varepsilon}\right\|_{\mathcal{W}'} \leqq d_5.$$

By (13) and the weak lower semicontinuity of the norm in $\mathcal{W}'$, it follows that

$$\left\|\frac{\partial u_n}{\partial t} + \mathcal{A}_n u_n\right\|_{\mathcal{W}'} \leqq d_5.$$

Since the sequence $\{\mathcal{A}_n u_n\}$ is bounded in $\mathcal{V}_0'$ — and thus in $\mathcal{W}'$ — we get

$$\left\|\frac{\partial u_n}{\partial t}\right\|_{\mathcal{W}'} \leqq d_6. \tag{15}$$

(iv) Because of (9) and (15) it is now easy to go to the limit $n \to \infty$ in (5). We know that (if necessary by passing to subsequences) $u_n \rightharpoonup u$ in $\mathcal{V}_0$, $u \in \mathcal{K}$, and $\partial u_n/\partial t \rightharpoonup \partial u/\partial t$ in $\mathcal{W}'$ (as $n \to \infty$), whence $u(0) = u(T)$ in Ω.

By Aubin's lemma it follows again

$$u_n \to u \text{ in } L^q(Q) \qquad (n \to \infty).$$

Setting $w = u$ in (5) we get

$$\left(\frac{\partial u_n}{\partial t}, u_n - u\right) + a(u_n, u_n - u) + \int_Q (S_n \circ \mathcal{F}u_n)(u_n - u)\,dxdt \leqq 0. \tag{16}$$

Note that $(\partial u_n/\partial t, u_n - u) \to 0$ $(n \to \infty)$.

We show that

$$\int_Q (S_n \circ \mathcal{F}u_n)(u_n - u)\,dxdt \to 0 \qquad (n \to \infty). \tag{17}$$

In fact, $\{S_n \circ \mathcal{F}u_n\}$ is bounded in $L^{1+\eta}(Q)$. Since $u_n \to u$ in $L^q(Q)$ $(n \to \infty)$ and $u_n \in \mathcal{K}$, $\forall n$, it follows by Lebesgue's dominated convergence theorem that $u_n \to u$ in $L^{(1+\eta)'}(Q)$. Thus (17) holds. Therefore we get from (16) that

$$\limsup_{n\to\infty} a(u_n, u_n - u) \leqq 0,$$

and by Lemma 4,

$$u_n \to u \text{ in } \mathcal{V}_0 \qquad (n \to \infty).$$

Hence

$$S_n \circ \mathcal{F}u_n \to \mathcal{F}u \text{ in } L^{1+\eta}(Q) \qquad (n \to \infty).$$

Passing to the limit $n \to \infty$ in (5) we see now that u solves (4). The Proposition is proved.

3. Proof of the Theorem

The idea of the proof consists in first modifying the problem outside the interesting range of functions $v : \phi \leqq v \leqq \psi$ a.e. in Q. To the modified problem there is associated a certain variational inequality, whose solvability follows from the Proposition. Finally it is shown that any solution u of this variational inequality is a weak solution of problem (1) with $\phi \leqq u \leqq \psi$ a.e. in Q.

(i) For $i = 1, \cdots, N$ we set

$$\tilde{A}_i(x, t, s, \xi) = \begin{cases} A_i(x, t, \psi(x, t), \xi), & s > \psi(x, t) \\ A_i(x, t, s, \xi), & \phi(x, t) \leqq s \leqq \psi(x, t) \\ A_i(x, t, \phi(x, t), \xi), & s < \phi(x, t) \end{cases}$$

for a.a. $(x, t) \in Q$, $\forall (s, \xi) \in \mathbf{R} \times \mathbf{R}^N$. The coefficient functions $\tilde{A}_i$ still satisfy the conditions (A1)–(A4). Let $\tilde{\mathcal{A}}$ denote the corresponding differential operator (deduced from (2) by replacing the A_i by $\tilde{A}_i$) and $\tilde{a}$ the associated semilinear form.

(ii) Let S be the mapping truncating functions by ϕ and ψ; i.e. for $u \in \mathcal{V}$

$$(Su)(x, t) = \begin{cases} \psi(x, t), & u(x, t) > \psi(x, t) \\ u(x, t), & \phi(x, t) \leqq u(x, t) \leqq \psi(x, t) \\ \phi(x, t), & u(x, t) < \phi(x, t). \end{cases}$$

It is well known that S is a continuous mapping from $\mathcal{V}$ into itself (e.g. [4]). As a consequence of (3), $P \circ Su \in L^{1+\eta}(Q)$, $\forall u \in \mathcal{V}$.

(iii) Let the function $b : Q \times \mathbf{R} \to \mathbf{R}$ be defined by

$$b(x,t,s)=\begin{cases}(s-\psi(x,t))^{q-1}, & s>\psi(x,t)\\ 0, & \phi(x,t)\leqq s\leqq \psi(x,t)\\ -(-s+\phi(x,t))^{q-1}, & s<\phi(x,t).\end{cases}$$

We note that b satisfies the Caratheodory conditions. The associated Nemitskii operator, which maps $L^q(Q)$ into $L^{q'}(Q)$, is denoted by $\mathscr{B}$.

(iv) Let $\alpha_1<0<\alpha_2$ be constants such that

$$-\infty<\alpha_1+1\leqq\phi(x,t)\leqq\psi(x,t)\leqq\alpha_2-1<+\infty\quad \text{a.e. in } Q.$$

We introduce the convex set

$$\mathscr{K}=\{u\in\mathscr{V}_0 \mid \alpha_1\leqq u\leqq\alpha_2 \text{ a.e. in } Q\}.$$

The mapping $\mathscr{F}:\mathscr{F}u=P\circ Su+\mathscr{B}u-f$ satisfies the assumptions of the Proposition.

It thus follows that there exists

$$(18)\quad \begin{cases} u\in\mathscr{K} \text{ such that } \dfrac{\partial u}{\partial t}\in\mathscr{V}_0'+L^{1+\eta}(Q),\ u(0)=u(T) \text{ in } \Omega \text{ and}\\[2ex] \left(\dfrac{\partial u}{\partial t},w-u\right)+\tilde a(u,w-u)+\displaystyle\int_Q(P\circ Su)(w-u)\,dxdt+\int_Q\mathscr{B}u(w-u)\,dxdt\\ \qquad\qquad \geqq\displaystyle\int_Q f(w-u)\,dxdt,\qquad \forall w\in\mathscr{K}\end{cases}$$

(v) We claim that for any solution u of (18) necessarily $\phi\leqq u\leqq\psi$ a.e. in Q. In fact, setting $w=\min(u,\psi)$ in (18) we obtain

$$(19)\quad \begin{aligned}&\left(\frac{\partial u}{\partial t},(u-\psi)^+\right)+\tilde a(u,(u-\psi)^+)+\int_Q(P\circ Su)(u-\psi)^+dxdt\\ &\qquad+\int_Q\mathscr{B}u(u-\psi)^+dxdt\leqq\int_Q f(u-\psi)^+dxdt\end{aligned}$$

(where v^+ means the positive part of the function v defined in Q.)

Since $(u-\psi)^+\in\mathscr{V}_0\cap L^\infty(Q)$ and ψ is the weak upper solution of problem (1),

$$(20)\quad \begin{aligned}&\left(\frac{\partial\psi}{\partial t},(u-\psi)^+\right)+a(\psi,(u-\psi)^+)+\int_Q P\psi(u-\psi)^+dxdt\\ &\qquad\geqq\int_Q f(u-\psi)^+\,dxdt.\end{aligned}$$

Substracting (20) from (19), we get

$$\left(\frac{\partial}{\partial t}(u-\psi),(u-\psi)^+\right)+\tilde{a}(u,(u-\psi)^+)-a(\psi,(u-\psi)^+)$$

$$+\int_Q (P\circ Su-P\psi)(u-\psi)^+dxdt+\int_Q \mathcal{B}u(u-\psi)^+dxdt \leqq 0.$$

As an immediate consequence of the definition of S,

$$\int_Q (P\circ Su-P\psi)(u-\psi)^+\,dxdt=0.$$

Further, with the notation $Q_+=\{(x,t)\in Q \mid u(x,t)>\psi(x,t)\}$,

$$\begin{aligned}\tilde{a}(u,(u-\psi)^+)&-a(\psi,(u-\psi)^+)\\ &=\int_Q \sum_{i=1}^N(\tilde{A}_i(x,t,u,\nabla u)-A_i(x,t,\psi,\nabla\psi))\frac{\partial}{\partial x_i}(u-\psi)^+dxdt\\ &=\int_{Q_+} \sum_{i=1}^N(A_i(x,t,\psi,\nabla u)-A_i(x,t,\psi,\nabla\psi))\frac{\partial}{\partial x_i}(u-\psi)\,dxdt\\ &\geqq 0, \text{ by hypothesis (A3).}\end{aligned}$$

Moreover, by Lemma 2 of the Appendix,

$$\left(\frac{\partial}{\partial t}(u-\psi),(u-\psi)^+\right)\geqq 0.$$

Hence

$$0\geqq\int_Q \mathcal{B}u(u-\psi)^+dxdt=\|(u-\psi)^+\|^q_{L^q(Q)},$$

i.e. $u\leqq\psi$ a.e. in Q.

Similarly one shows that $\phi\leqq u$ a.e. in Q.

(vi) We now assert that u is a weak solution of problem (1). Indeed, note that $Su=u$, and that (18) is thus reduced to

$$\left(\frac{\partial u}{\partial t},w-u\right)+a(u,w-u)+\int_Q Pu(w-u)\,dxdt\geqq\int_Q f(w-u)\,dxdt, \tag{21}$$
$$\forall w\in\mathcal{K}.$$

Let $v\in\mathcal{V}_0\cap L^\infty(Q)$ be arbitrarily given. Introducing $w=u\pm\theta v$ in (21), with $0<\theta\leqq\|v\|^{-1}_{L^\infty(Q)}$, we obtain

$$\left(\frac{\partial u}{\partial t}, v\right) + a(u, v) + \int_Q Pu\, v\, dxdt = \int_Q f v\, dxdt.$$

Hence the Theorem is proved.

Appendix

LEMMA 1. *Let $w \in \mathcal{V}_0$ be a function with $\partial w/\partial t \in \mathcal{V}_0'$ and $w(0) = w(T)$ a.e. in Ω. Let further $g : \mathbf{R} \to \mathbf{R}$ be a monotone increasing, Lipschitz continuous function with $g(0) = 0$. Then $(\partial w/\partial t, g(w)) = 0$.*

PROOF. With the aid of a partition of unity and double regularization we find a sequence $\{w_j\}$ of functions having the following properties:

$$\begin{cases} w_j \in C^\infty([0, T];\ C^\infty(\bar{\Omega})),\ w_j\big|_\Sigma = 0, \\ w_j \to w \ \text{ in } \ \mathcal{V}_0, \\ \dfrac{\partial w_j}{\partial t} \to \dfrac{\partial w}{\partial t} \ \text{ in } \ \mathcal{V}_0' \ (j \to \infty). \end{cases}$$

Thus $w_j(t) \to w(t)$ in $L^2(\Omega)$, $\forall t \in [0, T]$, because of the continuous embedding of the space $\mathcal{Y} = \{v \in \mathcal{V}_o \mid \partial v/\partial t \in \mathcal{V}_0'\}$ into $C^0([0, T];\ L^2(\Omega))$. We get

$$\left(\frac{\partial w}{\partial t}, g(w)\right) = \lim_{j\to\infty} \int_Q \frac{\partial w_j}{\partial t} g(w_j)\, dxdt.$$

Let further the function G be a primitive of g. We have

$$\frac{\partial}{\partial t} G(w_j) = g(w_j) \frac{\partial w_j}{\partial t}.$$

Hence

$$\left(\frac{\partial w}{\partial t}, g(w)\right) = \lim_{j\to\infty} \int_\Omega [G(w_j(T)) - G(w_j(0))]\, dx = 0,$$

since $w(0) = w(T)$.

By a similar method one proves the following

LEMMA 2. *Let $w \in \mathcal{V} \cap L^\infty(Q)$ be such that $\partial w/\partial t \in \mathcal{V}_0' + L^{1+\eta}(Q)$, $w(0) \leqq w(T)$ a.e. in Ω, and $w\big|_\Sigma \leqq 0$. Then $(\partial w/\partial t, w^+) \geqq 0$.*

LEMMA 3. *Let $g : \mathbf{R} \to \mathbf{R}$ be a monotone increasing, Lipschitz continuous function with $g(0) = 0$, which is differentiable except at a finite number of points. Let a be a semilinear form as defined by (2') with coefficient functions satisfying (A1)–(A4). For $u \in \mathcal{V}_0$ we then have $a(u, g(u)) \geqq 0$.*

PROOF.

$$a(u, g(u)) = \int_Q \sum_{i=1}^{N} A_i(x, t, u, \nabla u) \frac{\partial}{\partial x_i}(g(u))\, dxdt$$

$$\geqq 0,$$

by (A4) and the monotonicity of g.

LEMMA 4. *Let the semilinear form a be as in Lemma* 3, *and let* $\{u_n\}$ *be a sequence such that*

$$u_n \rightharpoonup u \text{ in } \mathcal{V}_0 \text{ and } \frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \text{ in } \mathcal{V}_0' + L^{1+\eta}(Q) \qquad (n \to \infty).$$

Suppose further that $\limsup_{n\to\infty} a(u_n, u_n - u) \leqq 0$. *Then* $u_n \to u$ *in* $\mathcal{V}_0$ $(n \to \infty)$.

PROOF. The proof is similar to that for the elliptic case (Browder [2], p. 25), replacing Sobolev's embedding theorem by Aubin's lemma.

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