ISRAEL JOURNAL OF MATHEMATICS, Vol. 29, No. 1, 1978

ON THE ALMOST-CONVERGENCE OF ITERATES OF A NONEXPANSIVE MAPPING IN HILBERT SPACE AND THE STRUCTURE OF THE WEAK ω-LIMIT SET†

BY

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Dedicated to the memory of my father

ABSTRACT

Let T be a nonexpansive self-mapping of a closed convex subset C of a real Hilbert space. In this paper we deal with the structure of the weak ω-limit set of iterates $\{T^n x\}$, establish conditions under which it is invariant under T, and show that $\{T^n x\}$ converges weakly iff T has a fixed-point and $T^n x - T^{n+1}x \to 0$ weakly.

Introduction

Let H be a real Hilbert space and C a closed convex subset of H. A mapping $T: C \to C$ is called *nonexpansive* provided $\| Tx - Ty \| \leqq \| x - y \| \; \forall x, y$ in C. The *weak* (respectively, *strong*) *ω-limit set* $\omega_w(x)$ (respectively, $\omega_s(x)$) is defined to be the set of weak (respectively, strong) subsequential limits of $\{T^n x\}$.

Under the assumption that $\omega_s(x) \neq \emptyset$ and that T has a fixed-point, Edelstein [10] has shown that for any y in $\omega_s(x)$, the $\lim_n \sum_{k=1}^n T^k y/n = c$ exists, is a fixed-point of T, and is the only fixed-point of T in the closed affine hull of $\omega_s(x)$. (He proved this in any reflexive, strictly convex Banach space.) It is easy to see that also $\lim_n \sum_{k=1}^n T^k x/n = c$ (for the essential idea, see Yen [19, lemma 8]). For extensions to strongly continuous contraction semigroups, see Dafermos and Slemrod [9].

Edelstein's result can be said to be the first nonlinear mean ergodic theorem. But the condition $\omega_s(x) \neq \emptyset$ is quite restrictive; the first ergodic theorem for *general* nonexpansive mappings (in Hilbert space) was established by Baillon

† Supported by NSF Grant MCS 76–08217.
Received January 14, 1977 and in revised form March 15, 1977

[2]: if T has a fixed-point, then the means $\Sigma_{k=1}^{n} T^k x/n$ converge weakly, as $n \to \infty$ to a fixed-point of T. Simpler proofs have since been given by Pazy [17], Brézis and Browder [4], and Bruck [7]. In general, the means do not converge strongly; some additional condition on T, such as oddness, must be imposed (see [1], [3], [4], [6], [7], [17]).

Baillon's theorem suggests that, at least in Hilbert space, the weak ω-limit set might be a more appropriate object of study than the strong ω-limit set. (It has, at least, the virtue that it is non-empty when T has a fixed-point, unlike $\omega_s(x)$.) This is the program which we begin with this paper.

In §1 we recall Lorentz's definition of "almost convergence" and show that the iterates $\{T^n x\}$ are weakly almost convergent to a fixed-point c of T; if T is odd (or isometric, or affine, or if $\omega_s(x) \neq \emptyset$), the almost-convergence is in the strong topology. This amounts to a technical simplification of the results of [4] (in particular, the concept of a "proper" array is eliminated). For *linear* T, the strong almost-convergence is well-known (see Cohen [8]).

Also in §1 we study the properties of $\omega_w(x)$ under the assumption that T has a fixed-point. While it is trivial that $\omega_s(x)$ is T-invariant, we do not know whether the same is true of $\omega_w(x)$ (we conjecture that the answer is negative). $\omega_w(x)$ is certainly *not* minimal, as we show by example. But we do establish desirable properties: clco $\omega_w(x)$ contains exactly one fixed-point of T (namely, c, the almost-limit of $\{T^n x\}$); for any other fixed-point f of T, $f - c$ is orthogonal to $\omega_w(x) - c$ (in particular, the closed affine hulls of $\omega_w(x)$ and the fixed-point set $F(T)$ are orthogonal, intersecting only in c). We also identify c as the asymptotic center (in the sense of Edelstein [11]) of $\{T^n x\}$.

In §2, we introduce the notion of "asymptotically isometric" mappings. These include isometries, affine nonexpansive mappings, nonexpansive mappings for which $\omega_s(x) \neq \emptyset$ and odd nonexpansive mappings. Many of the results of [9] go through for such mappings (with the notable exception of minimality): $\{T^n x\}$ is almost-convergent to c, strongly; $\omega_w(x)$ is T-invariant, and T maps clco $\omega_w(x)$ onto itself affinely and isometrically.

In §1 we establish a necessary and sufficient condition for the iterates $\{T^n x\}$ to converge weakly to a fixed-point of T: that T has a fixed-point and $T^n x - T^{n+1} x \to 0$ weakly. This is reminiscent of the asymptotic regularity condition of Browder and Petryshyn [5]: that $T^n x - T^{n+1} x \to 0$ strongly. Opial [16] showed that if $F(T) \neq \emptyset$ and T is asymptotically regular, then $\{T^n x\}$ converges weakly to a fixed-point of T. The new condition (which might be called "weak asymptotic regularity") is both necessary *and* sufficient.

We establish the following conventions, which henceforth always hold: H

denotes a real Hilbert space, C a non-empty closed convex subset of H. (Complex Hilbert spaces can be reduced to this case by the standard trick of ignoring nonreal scalars and taking the real part of the inner product. Orthogonality in that case must be interpreted as real-orthogonality.) Strong convergence is denoted by $\to$ and lim, weak convergence by $\rightharpoonup$ and w-lim. For convenience, sequences are indexed from $n = 0, 1, 2, \cdots$. $F(T)$ denotes the set of fixed-points of the mapping T. co W denotes the convex hull of W, clco W the closed convex hull of W, $H(W)$ the closed affine hull of W.

Our results are stated for discrete semigroups (iterates of a single mapping) for simplicity, but the techniques are also applicable to strongly continuous contraction semigroups. For another approach to this case, see Reich [18].

We wish to thank Simeon Reich for calling our attention to the papers of Cohen and Lorentz, and for reminding us of Edelstein [10]. In the original version of this paper we established the weak convergence of $\{T^n x\}$ (and, in §2, the strong convergence) for the method (E) below (the same method considered by Brézis and Browder [4]). Reich observed that the convergence is actually by method (SR); he has also established the almost-convergence of the iterates by a method independent of ours.

§1. Structure of $\omega_w(x)$ for general T

A key ingredient of our ergodic theorem is Lorentz's definition of almost convergent sequences, which we adapt to locally convex spaces.

Definition 1.1. A sequence $\{x_n\}$ in a locally convex space X is said to be *almost-convergent* to a point $x \in X$ iff

$$\lim_n \sum_{k=1}^{n} x_{k+i}/n = x,$$

uniformly in $i = 0, 1, 2, \cdots$.

It follows from the definition that $\{x_n\}$ is almost-convergent to x iff the shifted sequence $\{x_{n+1}\}$ is almost-convergent to x. Indeed, if $X = \mathbf{R}$, Lorentz [15] shows that $\{x_n\}$ is almost-convergent to x iff $\{x_n\}$ is bounded and $L(\{x_n\}) = x$ for every Banach limit L (motivating the "almost").

We introduce two classes of infinite real matrices $Q = [q_{ij}]$ $(0 \leqq i, j < \infty)$: (SR) (for strongly regular) consists of those Q for which

$$\text{(1.1)} \qquad \sup_n \sum_k |q_{nk}| < \infty, \qquad \lim_n \sum_k q_{nk} = 1;$$

$$\lim_n \sum_k | q_{n,k+1} - q_{n,k} | = 0; \tag{1.2}$$

while (E) (for ergodic) consists of those Q in (SR) satisfying

$$q_{ij} \geqq 0 \ \forall i, j, \qquad \sum_k q_{nk} = 1 \ \forall n.$$

It follows from (1.1) and (1.2) that $\lim_n q_{nk} = 0$, uniformly in k, so each $Q \in$ (SR) defines a regular method of convergence. In particular, for any bounded sequence $\{t_n\}$ of real numbers,

$$\begin{aligned} \liminf t_n &\leqq \liminf_n \sum_k q_{nk} t_k \\ &\leqq \limsup_n \sum_k q_{nk} t_k \leqq \limsup_n t_n. \end{aligned} \tag{1.3}$$

LEMMA 1.1. *If X is sequentially complete, then the following are equivalent:*
(a) $\{x_n\}$ *is almost-convergent to* x;
(b) $\forall Q$ *in* (E), $\lim_n \Sigma_k q_{nk} x_k = x$;
(c) $\forall Q$ *in* (SR), $\lim_n \Sigma_k q_{nk} x_k = x$.

The implication (c) $\Rightarrow$ (b) $\Rightarrow$ (a) is obvious; the reverse implication was proven by Lorentz [15] when $X = \mathbf{R}$, and as no new ideas are needed in the general case, the proof can be safely deleted. Note that sequential completeness is only needed to guarantee the convergence of $\Sigma_k q_{nk} x_k$. (As Lorentz [15, p. 171] notes, the almost-convergence of a sequence implies its boundedness, and this is also true in general X.)

In the sequel we refer to strong or weak almost-convergence in H, meaning almost-convergence in H with the strong (or weak) topology.

Note that on account of (1.2), for any bounded sequence $\{x_n\}$ in X,

$$\lim_n \left(\sum_k q_{nk} x_k - \sum_k q_{nk} x_{k+1} \right) = 0. \tag{1.4}$$

We refer to this as shift-invariance.

Our main result is:

THEOREM 1.1. *Suppose C is a closed convex subset of a real Hilbert space and $T: C \to C$ is nonexpansive and has a fixed-point. Then for each x in C the following hold:*

(i) *$\{T^n x\}$ is weakly almost-convergent to a fixed-point c of T;*

(ii) *c is the asymptotic center of $\{T^n x\}$ in C in the sense of Edelstein* [12];

(iii) $\{c\} = F(T) \cap \operatorname{clco} \omega_w(x)$;

(iv) *$H(F(T))$ is orthogonal to $H(\omega_w(x))$, with $\{c\} = H(F(T)) \cap H(\omega_w(x))$;*

(v) *$\{T^n x\}$ converges weakly to c iff $T^n x - T^{n+1}x \rightharpoonup 0$ as $n \to \infty$.*

Theorem 1.1 follows from two lemmas which we now prove.

LEMMA 1.2. *Let $\{x_n\}$ be a sequence in a weakly compact convex subset of a Banach space X and let W denote the set of all weak subsequential limits of $\{x_n\}$. Then*

$$\operatorname{clco} W = \bigcap_{n=0}^{\infty} \operatorname{clco}\{x_k : k \geqq n\}.$$

PROOF. Put $K_n = \operatorname{clco}\{x_k : k \geqq n\}$, $K = \cap K_n$. The inclusion $W \subset K$ (and hence $\operatorname{clco} W \subset K$) being trivial, it suffices to prove $K \subset \operatorname{clco} W$. Suppose not; let $x \in K \setminus \operatorname{clco} W$. Then there exists u in X^* with

$$\begin{aligned} u(x) &> \sup\{u(y) : y \in \operatorname{clco} W\} \\ &= \sup\{u(y) : y \in W\}. \end{aligned} \tag{1.5}$$

On the other hand, since $x \in K \subset K_n$,

$$\begin{aligned} u(x) &\leqq \sup\{u(y) : y \in K_n\} \\ &= \sup\{u(x_k) : k \geqq n\}. \end{aligned}$$

Therefore $u(x) \leqq \limsup_n u(x_n)$. Extracting a subsequence of $\{u(x_n)\}$ which converges to the lim sup, and using the Eberlein–Smulian theorem, we can therefore find a subsequence $\{x_{n(i)}\}_i$ such that $x_{n(i)} \rightharpoonup x'$ and $u(x) \leqq u(x')$. Since $x' \in W$ by definition, this contradicts (1.5). Q.E.D.

LEMMA 1.3. *Suppose $\{x_n\}$ is a bounded sequence in H with*

$$\limsup_m \limsup_n \limsup_i [(x_m, x_{m+i}) - (x_n, x_{n+i})] \leqq 0. \tag{1.6}$$

Let W denote the set of weak subsequential limits of $\{x_n\}$. Then:

(i) *$\{x_n\}$ is weakly almost-convergent to a point c of* clco W;

(ii) $\lim_n (x_n, c) = \|c\|^2$;

(iii) *c is orthogonal to $W - c$;*

(iv) *c is the point of* clco W *of minimum norm*;

(v) *if, in addition, $\{\|x_n\|\}$ converges, then c is the asymptotic center of $\{x_n\}$ in H.*

PROOF. By (1.6) we can find $\varepsilon(m,n,i)>0$ such that

$$\limsup_m \limsup_n \limsup_i \varepsilon(m,n,i)=0, \tag{1.7}$$

$$(x_m,x_{m+i})-(x_n,x_{n+i})\leqq \varepsilon(m,n,i). \tag{1.8}$$

Let $Q\in(E)$ and put $y_n=\Sigma_k q_{nk}x_k$. Let F denote the set of weak subsequential limits of $\{y_n\}$. Since $\{y_n\}$ is clearly bounded, to show that it converges weakly it suffices to show that F is a singleton.

Multiplying (1.8) by q_{ki} and summing over i, we obtain

$$\begin{aligned}(x_m-x_n,y_k)&=\sum_i q_{ki}(x_m,x_i)-\sum_i q_{ki}(x_n,x_i)\\ &\leqq \sum_i q_{ki}(m,n,i)+\sum_i q_{ki}\left[(x_m,x_i)-(x_m,x_{m+i})\right]\\ &\quad-\sum_i q_{ki}\left[(x_n,x_i)-(x_n,x_{n+i})\right].\end{aligned}$$

By the shift-invariance condition (1.4) and (1.3), therefore

$$\limsup_k\ (x_m-x_n,y_k)\leqq \limsup_i \varepsilon(m,n,i).$$

Let $c'\in F$. Then $(x_m-x_n,c')\leqq \limsup_i \varepsilon(m,n,i)$, hence by (1.7),

$$\limsup_m\ (x_m,c')\leqq \liminf_n\ (x_n,c').$$

It follows that $\lim_n (x_n,c'):=a(c')$ exists for each c' in F. Clearly $a(c')=(x',c')$ for any x' in W, and therefore for any x' in clco W. But it is clear from (1.1) and $\lim_n q_{nk}=0$ that $c'\in\bigcap_{n=0}^{\infty}\mathrm{clco}\{x_k : k\geqq n\}$, which is clco W by Lemma 1.2. Thus in particular, $a(c')=(x',c')=(c',c')$ for all x' in clco W. This shows that c' is orthogonal to clco $W-c'$, whence c' is the point of clco W of minimum norm. This means, first, that F is a singleton $\{c\}$, i.e. $y_n\rightharpoonup c$, proving, by virtue of Lemma 1.1, part (i) of Lemma 1.3: and second, parts (ii)–(iv) of the lemma have been established en route. It remains only to prove part (v).

Suppose $\{\|x_n\|\}$ converges, and put $r(y)=\limsup_n\|x_n-y\|^2$ for y in H. By Edelstein [11], [12], r has a unique minimizer (the point which we have called the asymptotic center of $\{x_n\}$ in H in the sense of Edelstein). Since $\|x_n-c\|^2=\|x_n\|^2-2(x_n,c)+\|c\|^2$, we have by part (ii) $r(c)=\lim_n\|x_n-c\|^2$ (that is, the limit exists). Finally, for any $Q\in(E)$, with $y_n=\Sigma_k q_{nk}x_k$ we have the identity

$$\sum_k q_{nk}\|x_k-y\|^2=\sum_k q_{nk}\|x_k-c\|^2+2(y_n-c,c-y)+\|c-y\|^2.$$

By (1.3) and part (i), therefore

$$\begin{aligned} r(y) &\geqq \lim_n \sum_k q_{nk} \| x_k - y \|^2 \\ &= \lim_n \| x_n - c \|^2 + \| c - y \|^2 \\ &= r(c) + \| c - y \|^2. \end{aligned}$$

Thus c is the minimizer of r, i.e., the asymptotic center of $\{x_n\}$. Q.E.D.

PROOF OF THEOREM 1.1. Let f be a fixed-point of T, and put $x_n = T^n x - f$. Then $\| x_{n+1} \| \leqq \| x_n \|$ by the nonexpansiveness of T; so $\{\| x_n \|\}$ converges. On the other hand, again by nonexpansiveness, we have for any $n \geqq m \geqq 0$, $i \geqq 0$ the inequality $\| x_{n+i} - x_n \| \leqq \| x_{m+i} - x_m \|$. Squaring, expanding norms, and noting that $\{\| x_n \|\}$ converges, we deduce (1.6). We now read off consequences from Lemma 1.3. First, $\{T^n x - f\}$ is weakly almost-convergent, hence $\{T^n x\}$ is weakly almost-convergent to a point c; moreover, $c \in \operatorname{clco} \omega_w(x)$. Second, $c - f$ is the asymptotic center of $\{T^n x - f\}$ in H, hence c is the asymptotic center of $\{T^n x\}$ in H. As Edelstein [12] remarks, however, in Hilbert space this implies c is the asymptotic center of $\{T^n x\}$ in C. Third, $c - f$ is orthogonal to $\omega_w(x) - c$. Thus for any $f_1, f_2 \in F(T)$, $w_1, w_2 \in \omega_w(x)$ there holds $(f_i - c, w_i - c) = 0$ $(i = 1, 2)$, whence $(f_1 - f_2, w_1 - w_2) = 0$. This shows $H(\omega_w(x))$ is orthogonal to $H(F(T))$ as affine subspaces of H. Fourth, $c - f$ is the point of $\operatorname{clco} \omega_w(x) - f$ of minimum norm, i.e. c is the point of $\operatorname{clco} \omega_w(x)$ closest to f. By [12, theor. 1] $c \in F(T)$; thus this proves $\{c\} = F(T) \cap \operatorname{clco} \omega_w(x)$. Since $H(F(T))$ is orthogonal to $H(\omega_w(x))$, we also have $H(F(T)) \cap H(\omega_w(x)) = \{c\}$.

All that remains is part (v) of the theorem. But this follows from the easy Tauberian condition: if $\{u_n\}$ is almost-convergent to u, and $u_n - u_{n+1} \to 0$, then $\{u_n\}$ converges to u. For a proof in **R** (the general case being similar) see Lorentz [15, §4].

COROLLARY 1.1. (Pazy [17]). *Suppose $T: C \to C$ is nonexpansive and has a fixed-point. Then a necessary and sufficient condition for $\{T^n x\}$ to converge weakly to a fixed-point of T is that $\omega_w(x) \subset F(T)$.*

PROOF. Theorem 1.1 says $F(T) \cap \operatorname{clco} \omega_w(x)$ is a singleton. Thus $\omega_w(x)$ is a singleton—i.e., $\{T^n x\}$ converges weakly—iff $\omega_w(x) \subset F(T)$, in which case the weak limit is a fixed-point of T. Q.E.D.

§2. Asymptotically isometric mappings

DEFINITION 2.1. Let $T: C \to C$ be nonexpansive. T is said to be *asymptotically isometric* on a subset S of C provided for all x, y in S, the

$\lim_n \| T^n x - T^{n+i} y \|$ exists uniformly in $i = 0, 1, 2, \cdots$. T is said to be ε-*approximately affine* on a convex subset K of C provided

$$\left\| T\left(\sum t_k u_k\right) - \sum t_k T u_k \right\| \leqq \varepsilon$$

for all choices of n in $\mathbf{Z}^+$, $u_1, \cdots, u_n$ in K, $t_1, \cdots, t_n$ in $\mathbf{R}^+$ with $\Sigma t_k = 1$.

Note that for any nonexpansive mapping T, $\| T^n x - T^{n+i} y \|$ is, for any fixed i, nonincreasing (and therefore convergent) as $n \to \infty$. We require the *uniformity* of the limit in i.

The main result of this section is:

THEOREM 2.1. *Suppose C is a closed convex subset of a real Hilbert space H, $T: C \to C$ is nonexpansive, and T has a fixed-point. If T is asymptotically isometric on a subset S of C, then:*

(i) *for all x in S, $\{T^n x\}$ is strongly almost-convergent*;

(ii) *for all $\varepsilon > 0$ and $x_1, \cdots, x_m$ in S there exists n_0 such that T is ε-approximately affine on* $\operatorname{clco}\{T^k x_i : 1 \leqq i \leqq m,\ k \geqq n_0\}$;

(iii) *if $x \in S$ and $n(i) \to \infty$, $T^{n(i)} x \rightharpoonup y$ as $i \to \infty$, then $T^{1+n(i)} x \rightharpoonup Ty$*;

(iv) *for all x in S, T maps $\omega_w(x)$ onto itself*;

(v) *T maps $K = \operatorname{clco} \cup \{\omega_w(x) : x \in S\}$ onto itself affinely and isometrically.*

REMARK. If T is asymptotically isometric on S then it is easy to see that T is asymptotically isometric on $S \cup F(T)$. Thus in the theorem it may be assumed that $F(T) \subset S$.

The key to part (i) is:

LEMMA 2.1. *If a sequence $\{x_n\}$ in H satisfies*

$$\lim_n\, (x_n, x_{n+k}) \text{ exists uniformly in } k = 0, 1, 2, \cdots. \tag{2.1}$$

then $\{x_n\}$ is strongly almost-convergent to its asymptotic center.

PROOF. Clearly (2.1) implies (1.6). Taking $k = 0$ in (2.1), we see $\{\| x_n \|\}$ converges. By Lemma 1.3, therefore, $\{x_n\}$ is weakly almost-convergent to its asymptotic center.

Put $M = \sup_i \| x_i \|$. Given $\varepsilon > 0$, choose m so large that $i \geqq m$, $k \geqq 0$ imply $|(x_i, x_{i+k}) - (x_m, x_{m+k})| \leqq \varepsilon$ (this is possible by (2.1)). Let $Q \in (E)$ and put $y_n = \Sigma_k q_{nk} x_k$. Clearly

$$\begin{aligned}\|y_n\|^2 &= \sum_{i,j} q_{ni}q_{nj}(x_i, x_j) \\ &\leqq \sum_{i,j\geqq m} q_{ni}q_{nj}(x_m, x_{m+|i-j|}) + 2M^2 \sum_{k<m} q_{nk} \\ &+ \varepsilon \cdot \sum_{i,j\geqq m} q_{ni}q_{nj};\end{aligned}$$

thus

$$\|y_n\|^2 \leqq \sum_{i,j} q_{ni}q_{nj}(x_m, x_{m+|i-j|}) + \varepsilon + 4M^2 \sum_{k<m} q_{nk}. \tag{2.2}$$

Let $P = [p_{ij}]$ be the matrix defined by

$$p_{nk} = \sum_i q_{ni}^2 \qquad (k=0),$$

$$p_{nk} = 2 \cdot \sum_i q_{ni}q_{n,k+i} \qquad (k \neq 0).$$

It is an easy computation (noting that $\lim_n q_{nk} = 0$ uniformly in k) that $P \in (E)$. Now by rearrangement,

$$\sum_{i,j} q_{ni}q_{nj}(x_m, x_{m+|i-j|}) = \left(x_m, \sum_k p_{nk}x_{m+k}\right).$$

Since $\{x_k\}_{k=0}^{\infty}$ is weakly almost-convergent to a point c, by shift-invariance so is $\{x_{m+k}\}_{k=0}^{\infty}$. Thus

$$\lim_n \sum_{i,j} q_{ni}q_{nj}(x_m, x_{m+|i-j|}) = (x_m, c).$$

Exploiting this in (2.2), we obtain

$$\limsup_n \|y_n\|^2 \leqq \|c\|^2 + \varepsilon;$$

or finally, $\limsup_n \|y_n\| \leqq \|c\|$. But $y_n \rightharpoonup c$; in Hilbert space these imply $y_n \to c$ strongly. Q.E.D.

We also need:

LEMMA 2.2. *Suppose $\varepsilon > 0$, $T: C \to C$ is nonexpansive, and for all x, y in a certain subset S of C there holds*

$$\|x-y\|^2 \leqq \|Tx - Ty\|^2 + \varepsilon^2/4. \tag{2.3}$$

Then T is ε-approximately affine on clco *S.*

PROOF. We first show that for finitely many v_k in S, λ_k in $\mathbf{R}^+$ with $\Sigma \lambda_k = 1$, there holds

$$\left\| T\left(\sum \lambda_k v_k\right) - \sum \lambda_k T v_k \right\| \leqq \varepsilon/2. \tag{2.4}$$

Indeed, by Baillon's inequality (stated in [2] and proven in [17]),

$$\left\| T\left(\sum \lambda_k v_k\right) - \sum \lambda_k T v_k \right\|^2 \leqq \sum_{i<j} \lambda_i \lambda_j \left[\| v_i - v_j \|^2 - \| T v_i - T v_j \|^2\right].$$

With (2.3), this immediately implies (2.4).

Next suppose $\{u_k\} \subset \operatorname{co} S$, $\{t_k\} \subset \mathbf{R}^+$, and $\Sigma t_k = 1$. Find $\{v_i\} \subset S$, $\{\lambda_{ki}\} \subset \mathbf{R}^+$ with $\Sigma_i \lambda_{ki} = 1$ for all k, and $u_k = \Sigma_i \lambda_{ki} v_i$. Two applications of (2.4) yield

$$\left\| T\left(\sum_{i,k} t_k \lambda_{ki} v_i\right) - \sum_{i,k} t_k \lambda_{ki} T v_i \right\| \leqq \varepsilon/2,$$

or

$$\left\| T\left(\sum_{k} t_k u_k\right) - \sum_{i,k} t_k \lambda_{ki} T v_i \right\| \leqq \varepsilon/2, \tag{2.5}$$

and

$$\left\| T u_k - \sum_{i} \lambda_{ki} T v_i \right\| \leqq \varepsilon/2. \tag{2.6}$$

Multiplying (2.6) by t_k and summing over k, we obtain

$$\left\| \sum_{k} t_k T u_k - \sum_{i,k} t_k \lambda_{ki} T v_i \right\| \leqq \varepsilon/2,$$

which with (2.5) yields

$$\left\| T\left(\sum_{k} t_k u_k\right) - \sum_{k} t_k T u_k \right\| \leqq \varepsilon.$$

This proves T is ε-approximately affine on $\operatorname{co} S$; by continuity, the same is true on $\operatorname{clco} S$. Q.E.D.

PROOF OF THEOREM 2.1. By Theorem 1.1, $\{T^n x\}$ is weakly almost-convergent to a fixed-point c of T. Put $x_n = T^n x - c$. Then $\{\| x_n \|\}$ converges and $\lim_n \| x_{n+k} - x_n \|^2$ exists uniformly in k since T is asymptotically isometric on $\{x\}$. Therefore $\lim_n (x_n, x_{n+k})$ exists uniformly in k. Now $\{x_n\}$ is weakly almost-convergent to 0, so by Lemma 2.1, it is strongly almost-convergent to 0. Thus $\{T^n x\}$ is strongly almost-convergent to c. This proves part (i).

Part (ii) is an obvious consequence of Lemma 2.2 and Definition 2.1.

To prove (iii), it suffices to show that if $T^{n(i)}x \rightharpoonup y$ and $T^{1+n(i)}x \rightharpoonup w$, then $w = Ty$. Let y_i be a convex combination of the $T^{n(j)}x$ ($j \geqq i$) such that $y_i \to y$, and let w_i be the same convex combination of $T^{1+n(j)}x$ ($j \geqq i$). Then $w_i \rightharpoonup w$. By part (ii), $w_i - Ty_i \to 0$. But T is continuous and $y_i \to y$, so $w_i \to Ty$. Since $w_i \rightharpoonup w$, we have $w = Ty$, finishing part (iii).

It immediately follows from (iii) that T maps $\omega_w(x)$ onto itself. To prove part (v), put $K_0 = \bigcup\{\omega_w(x): x \in S\}$, so that $K = \operatorname{clco} K_0$. We claim T is isometric on K_0.

Let $x, y \in S$ and $f \in F(T)$. Since

$$(T^n x - f, T^m y - f) = \tfrac{1}{2}[\| T^n x - f \|^2 + \| T^m y - f \|^2 - \| T^n x - T^m y \|^2]$$

and $\{\| T^n x - f \|\}$, $\{\| T^m y - f \|\}$ converge while T is asympototically isometric on $\{x, y\}$, there exists a sequence $\varepsilon_n \to 0$ such that

$$(2.7) \qquad |(T^{n+1}x - f,\ T^{m+1}y - f) - (T^n x - f,\ T^m y - f)| \leqq \varepsilon_m \qquad (n \geqq m).$$

Let $u \in \omega_w(x)$, $v \in \omega_w(y)$. Letting $n \to \infty$ through some subsequence of integers such that $T^n x \rightharpoonup u$, and using part (iii), we obtain from (2.7)

$$|(Tu - f,\ T^{m+1}y - f) - (u - f,\ T^m y - f)| \leqq \varepsilon_m.$$

Next letting $m \to \infty$ through an appropriate subsequence, we obtain $(Tu - f, Tv - f) = (u - f, v - f)$ for all $u \in \omega_w(x)$, $v \in \omega_w(y)$, and $f \in F(T)$. It is clear that this identity implies $\| Tu - Tv \| = \| u - v \|$, proving that T is an isometry on K_0. By [13, prop. B], therefore, T maps $\operatorname{co} K_0$ affinely and isometrically into itself. By continuity, T maps K isometrically and affinely into K. But since $T(K_0) = K_0$ we have $T(K) \supset T(\operatorname{co} K_0) = \operatorname{co} K_0$, so that $T(K)$ is dense in K. Finally, $T(K)$ is closed because K is closed and T is isometric; therefore $T(K) = K$. Q.E.D.

COROLLARY 2.1. *Let $T: C \to C$ be nonexpansive and asymptotically isometric on $\{x\}$ for each x in C. Suppose $F(T) \neq \emptyset$ and for each $x \notin F(T)$,*

$$(2.8) \qquad \operatorname{dis}(Tx, F(T)) < \operatorname{dis}(x, F(T)).$$

Then for each x in C, $\{T^n x\}$ converges weakly to a fixed-point of T.

PROOF. By Theorem 2.1 (v), for any y in $\omega_w(x)$ and f in $F(T)$, $\| Ty - f \| = \| y - f \|$; consequently, $\operatorname{dis}(Ty, F(T)) = \operatorname{dis}(y, F(T))$. By (2.8), therefore, $\omega_w(x) \subset F(T)$. By Corollary 1.1, $\{T^n x\}$ converges weakly to a fixed-point of T.

Q.E.D.

Let (UR) denote the class of matrices satisfying (1.1) and

$$\lim_n \max_k |q_{nk}| = 0,$$

i.e. $\lim_n q_{nk} = 0$ uniformly in k. (UR stands for "uniformly regular"; it is clear that (SR) $\subset$ (UR) and that each (UR) matrix is regular.) We say that a sequence $\{x_n\}$ converges (UR) to x provided $\lim_n \Sigma_k q_{nk}x_k = x$ for all $Q \in$ (UR).

Recently Fong and Sucheston [14] have established (for a *linear* nonexpansive operator T in Hilbert space) that $\{T^n x\}$ converges weakly to y iff $\{T^n x\}$ is strongly (UR)-convergent to y. An example of Baillon [1] can be used to show that such a result is *false* for nnnlinear nonexpansive mappings. But:

THEOREM 2.2. *Suppose $T: C \to C$ is nonexpansive and has a fixed-point. If T is asymptotically isometric on $\{x\}$, then* w-$\lim_n T^n x = y$ *iff $\{T^n x\}$ is strongly (UR)-convergent to y.*

As noted in [14] sufficiency holds in any Banach space. Necessity is by now an obvious consequence of:

LEMMA 2.3. *Suppose $x_n \rightharpoonup x$ and (2.1) holds. Then $\{x_n\}$ is strongly (UR)-convergent to x.*

PROOF. Let $Q \in$ (UR). Using the same notations as in the proof of Lemma 2.1, and slightly refining the estimates, we obtain

$$\| y_n \|^2 \leqq \Big(x_m, \sum_k p_{nk} x_{m+k}\Big) + \varepsilon \cdot \Big(\sum_{k \geqq m} q_{nk}\Big)^2 + 4M^2 \Big(\sum_{k<m} | q_{nk} |\Big)\Big(\sum_k | q_{nk} |\Big). \tag{2.9}$$

It is easy to see (since $Q \in$ (UR)) that P is regular, so that w-$\lim_n \Sigma_k p_{nk} x_{m+k} = x$. The right-hand side of (2.9) therefore tends to $(x_m, x) + \varepsilon$ as $n \to \infty$; whence $\limsup_n \| y_n \|^2 \leqq \| x \|^2$. Since $y_n \rightharpoonup x$ we again obtain $y_n \to x$. Q.E.D.

We conclude this section with conditions sufficient to guarantee that T is asymptotically isometric on a set S.

THEOREM 2.3. *If $T: C \to C$ is nonexpansive, $0 \in C$, $T(0) = 0$, and there exists $c \geqq 0$ such that*

$$\| Tx + Ty \|^2 \leqq \| x + y \|^2 + c\{\| x \|^2 - \| Tx \|^2 + \| y \|^2 - \| Ty \|^2\}, \tag{2.10}$$

then T is asymptotically isometric on C.

The proof of Theorem 2.3 is essentially to be found in the last paragraph of [4]. Note that (2.10) is satisfied (with $c = 0$) if T is odd.

THEOREM 2.4. *Let $T: C \to C$ be nonexpansive. If, for a certain x in C, $\omega_s(x) \neq \emptyset$, then T is asymptotically isometric on $\{x\}$.*

PROOF. Let $y \in \omega_s(x)$; recall that this means there is a subsequence of $\{T^n x\}$ which converges to y. We shall prove

$$\lim_n \| T^{n+i}x - T^n x \| = \| T^i y - y \|$$

uniformly in $i = 0, 1, 2, \cdots$. Let $\varepsilon > 0$ and choose m so that $\| T^m x - y \| \leqq \varepsilon / 2$. For $n \geqq m$ we have

$$\begin{aligned} \| T^{n+i}x - T^n x \| &\leqq \| T^{n-m+i}y - T^{n-m}y \| + \| T^{n+i-m}T^m x - T^{n+i-m}y \| \\ &\quad + \| T^{n-m}T^m x - T^{n-m}y \| \\ &\leqq \| T^i y - y \| + 2\| T^m x - y \| \\ &\leqq \| T^i y - y \| + \varepsilon. \end{aligned}$$

Since $\{\| T^{n+i}x - T^n x \|\}_{n=0}^{\infty}$ is non-increasing, clearly $\| T^i y - y \| \leqq \| T^{n+i}x - T^n x \|$. Since these estimates are independent of i, we are done. Q.E.D.

THEOREM 2.5. *Suppose $T: C \to C$ is affine and nonexpansive and has a fixed-point. Then T is asymptotically isometric on C.*

PROOF. Without loss of generality we may assume $0 \in C$ and $T(0) = 0$. Since C is convex, $H_0 := \text{span } C$ consists of all points of the form $x = \lambda \cdot (x_1 - x_2)$ ($\lambda > 0$, x_1, $x_2 \in C$). Defining $T_0 x = \lambda \cdot (Tx_1 - Tx_2)$, we readily see that T_0 is a well-defined extension of T which is linear. Moreover, T_0 is nonexpansive since

$$\| T_0 x \| = \lambda \| Tx_1 - Tx_2 \| \leqq \lambda \| x_1 - x_2 \| = \| x \|.$$

Extend T_0 to $T_1 : \text{cl } H_0 \to \text{cl } H_0$ by continuity. Then T_1 is linear and nonexpansive, so Theorem 2.5 now follows from Theorem 2.3. Q.E.D.

§3. A counterexample

It is easy to prove that $\omega_s(x)$ is invariant and *minimal* under T, in the sense that for each y in $\omega_s(x)$, $\{T^n x\}$ is dense in $\omega_s(x)$. This is not the case for $\omega_w(x)$, even for unitary operators.

EXAMPLE 3.1. Let $F: [0, 2\pi] \to [0, 1]$ be the Cantor–Lebesgue function with dissection ratio 1/3. Let $\{E_\lambda\}$ be the resolution of the identity on the *complex* space $L^2(0,1)$ defined by $E_\lambda f = f \cdot C_{[0, F(\lambda)]}$, where C_S denotes the characteristic function of the set S. Put $T=\int_0^{2\pi} e^{i\lambda}dE_\lambda$. Then T is unitary (in particular, nonexpansive), and we easily see

$$(g, T^n f) = \int_0^{2\pi} e^{-in\lambda} g(F(\lambda))\overline{f(F(\lambda))}\, dF(\lambda) \tag{3.1}$$

for all f, g in $L^2(0,1)$.

CLAIM 1. $\{T^n 1\}$ does not converge weakly to 0, where 1 denotes the constant function. We have

$$\begin{aligned}(1, T^n 1) &= \int_0^{2\pi} e^{-in\lambda} dF(\lambda)\\ &= (-1)^n \prod_{k=1}^{\infty} \cos(2\pi n 3^{-k})\end{aligned} \tag{3.2}$$

by Zygmund [20, p. 196]. Taking $n=3^m$, we see that

$$(1, T^{3^m}1) = -\prod_{k=1}^{\infty}\cos(2\pi 3^{-k}),$$

a nonzero constant independent of m, thus precluding the weak convergence of $T^n 1$ to 0.

CLAIM 2. $\{T^{n(j)}1\}_{j=0}^{\infty}$ converges weakly to 0, where $n(j)=3^j[3^j/4]$ (here $[\cdot]$ denotes the greatest integer function). Consider any interval of the form $I=[2\pi(k-1)3^{-d}, 2\pi k 3^{-d}]$ (k, d positive integers, $1\leqq k\leqq 3^d$). By (3.1),

$$\begin{aligned}(C_{F(I)}, T^n 1) &= \int_{F^{-1}F(I)} e^{-in\lambda} dF(\lambda)\\ &= \int_I e^{-in\lambda} dF(\lambda)\end{aligned} \tag{3.3}$$

since $F^{-1}F(I)\backslash I$ consists of at most two intervals on which F is constant. Now the form of the Cantor set, and of the Cantor–Lebesgue function, guarantees that either (a) F is a constant on I, or (b) $F(s)=F(s+(k-1)3^{-d})+\text{const}$ for $0\leqq s\leqq 3^{-d}$. Moreover, for such s we have $F(3^d s)=2^d F(s)$. In case (a), $(C_{F(I)}, T^n 1)=0$; in case (b), we find by a change of variable

$$\begin{aligned}\int_I e^{-in\lambda} dF(\lambda) &= \mu \int_0^{3^{-d}} e^{-int} dF(t)\\ &= 2^{-d}\mu \int_0^1 e^{-ins/3^d} dF(s)\end{aligned} \tag{3.4}$$

for some μ with $|\mu| = 1$. For $j \geqq d$, $n(j)3^{-d}$ is an integer, hence by (3.2), (3.3), and (3.4),

$$|(C_{F(I)}, T^{n(j)}1)| = c \cdot \left| \prod_{k=1}^{\infty} \cos(2\pi n(j) 3^{-d-k}) \right|$$

for $c \neq 0$ independent of j. By using only the term $k = 2j - d$ we see that

$$|(C_{F(I)}, T^{n(j)}1)| \leqq c \cdot |\cos(2\pi 3^{-j}[3^j/4])|.$$

But $\lim_j 2\pi 3^{-j}[3^j/4] = \pi/2$, so

$$\lim_j (C_{F(I)}, T^{n(j)}1) = 0. \tag{3.5}$$

Since F maps $[0,2\pi]$ onto $[0,1]$ continuously, the set of $C_{F(I)}$ spans a dense subset of $L^2(0,1)$. Since $\{T^{n(j)}1\}$ is bounded, (3.5) therefore implies $T^{n(j)}1 \to 0$ as $j \to \infty$.

By Claim 2, $0 \in \omega_w(x)$; by Claim 1, $\omega_w(x) \neq \{0\}$. Here we have an example where $\omega_w(x)$ properly contains a fixed-point of T; hence is surely not minimal under T.

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