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$M_p(G) \neq M_q(G)$ $(p^{-1}+q^{-1}=1)$

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ABSTRACT
There exists a compact group G having $M_{4/3}(G) \neq M_4(G)$. This answers in the negative (the dual reformulation of) a question of Eymard (Séminaire Bourbaki, 1969/70).

1. Introduction

Let G be a compact group, and let Γ be the dual object of G, the set of equivalence classes of irreducible unitary representations of G. For each $\gamma \in \Gamma$, select a representation $U_\gamma \in \gamma$, let H_γ be the Hilbert space on which U_γ acts, and let d_γ be the dimension of H_γ. Let $\mathscr{B}(H_\gamma)$ denote the space of linear operators on H_γ, and for $f \in L^1(G)$, define $\hat{f}(\gamma)$ to be the operator in $\mathscr{B}(H_\gamma)$ given by

$$\hat{f}(\gamma)=\int_G f(x)U_\gamma(x^{-1})dx.$$

(The measure on G, here denoted dx, is always normalized Haar measure.) To each $f \in L^1(G)$ is associated the Fourier series

$$\sum_{\gamma\in\Gamma} d_\gamma \operatorname{tr}[\hat{f}(\gamma)U_\gamma(x)],$$

where $\operatorname{tr}[\hat{f}(\gamma)U_\gamma(x)]$ stands for the usual trace of the operator $\hat{f}(\gamma)U_\gamma(x)$. Let $\mathscr{C}(\Gamma)$ denote the space $P_{\gamma\in\Gamma}\mathscr{B}(H_\gamma)$.

DEFINITION. Fix $p \in [1,\infty]$. Let m be an element of $\mathscr{C}(\Gamma)$, so that for each $\gamma, m(\gamma) \in \mathscr{B}(H_\gamma)$. The function m is a (left) multiplier of L^p $(=L^p(G))$ if for each $f \in L^p$, the series

$$\sum_{\gamma\in\Gamma} d_\gamma \operatorname{tr}[m(\gamma)\hat{f}(\gamma)U_\gamma(x)]$$

is the Fourier series of some function $L_mf \in L^p$. The collection of all such m is denoted by $M_p(G)$ or simply M_p.

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For each $m \in M_p$, the map $f \to L_m f$ defines a bounded linear operator on L^p, an operator which commutes with left translations by the elements of G. If $p < \infty$, as we shall henceforth assume, each such translation-invariant operator on L^p is an L_m for some $m \in M_p$. We regard M_p as a Banach space under the operator norm.

When G is abelian, an easy argument shows that if $p^{-1} + q^{-1} = 1$, then $M_p = M_q$. If G is not abelian, this same argument shows only that $m \in M_p$ if and only if $m^* \in M_q$, where $m^*(\gamma) \in \mathcal{B}(H_\gamma)$ is the Hilbert space adjoint of $m(\gamma)$. Thus, for nonabelian G, $M_p = M_q$ holds if and only if $m \in M_p$ implies that $m^* \in M_p$. It is the object of the present note to show that this implication sometimes fails.

THEOREM. *There exist a compact group G and an element m of $M_4(G)$ such that $m^* \notin M_4(G)$.*

2. Lemmas and the proof

We shall need a lemma of Bonami concerning product groups. Let G_1 and G_2 be compact groups with dual objects Γ_1 and Γ_2, respectively. Recall that if $G = G_1 \times G_2$, then $\Gamma_1 \times \Gamma_2$ is canonically identifiable with the dual object Γ of G. This identification associates to the pair $(\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2$ the equivalence class of the representation $U_{\gamma_1} \otimes U_{\gamma_2}$, where $U_{\gamma_1} \in \gamma_1$, $U_{\gamma_2} \in \gamma_2$.

LEMMA 1. ([1, Chapitre III, Théorème 2, Lemme 1], see also [2, Lemma 2]). *With notation as above, let m_1 and m_2 be elements of $M_p(G_1)$, $M_p(G_2)$, respectively ($1 \leq p < \infty$). If $m \in \mathscr{C}(\Gamma)$ is defined by $m(\gamma_1, \gamma) = m_1(\gamma_1) \otimes m_2(\gamma_2)$, then $m \in M_p(G)$ and $\| m \|_{M_p(G)} = \| m_1 \|_{M_p(G_1)} \| m_2 \|_{M_p(G_2)}$.*

Our next lemma concerns a particular finite group. The example contained in this lemma was found with the aid of a computer.

LEMMA 2. *There exist a finite group G and a multiplier $m \in M_4(G)$ which has $\| m \|_{M_4} < \| m^* \|_{M_4}$.*

PROOF. Let G be the set of two by two unitary matrices

$$\left\{ \begin{bmatrix} \varepsilon_1 & 0 \\ 0 & \varepsilon_2 \end{bmatrix}, \begin{bmatrix} 0 & \varepsilon_2 \\ \varepsilon_1 & 0 \end{bmatrix} \right\}_{(\varepsilon_1, \varepsilon_2) \in \{\pm 1\}^2}.$$

Then G is a group under matrix multiplication, and is isomorphic to the dihedral group D_4; see [3, 27.61–62]. The self-representation of G on the two-dimensional Hilbert space C^2 is irreducible [3, 29.47]. Let Γ be the dual object of G, and let $\gamma_0 \in \Gamma$ be the equivalence class of this self-representation. Let $m \in \mathscr{C}(\Gamma)$ be defined by

$$m(\gamma_0) = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix}, m(\gamma) = 0 \quad \text{for other} \quad \gamma \in \Gamma.$$

(This matrix is viewed in the usual way as an operator on $C^2 = H_{\gamma_0}$). We shall show that

$$\| m \|_{M_4} < \| m^* \|_{M_4}.$$

A. $\| m \|_{M_4} = (353)^{1/4}$.

We start by showing that

$$(1) \qquad \| L_m f \|_{L^4} \leqq (353)^{1/4} \| f \|_{L^4}$$

holds for $f \in L^4(G)$ of the form

$$f(X) = 2\,\mathrm{tr}\,[\hat{f}(\gamma_0)X], \quad \hat{f}(\gamma_0) = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \text{ a real matrix.}$$

For such f, a computation gives

$$\| f \|_{L^4}^4 = 8[c_1^4 + c_2^4 + c_3^4 + c_4^4 + 6(c_1^2 c_4^2 + c_2^2 c_3^2)],$$

$$\| L_m f \|_{L^4}^4 = 8[257(c_3^4 + c_4^4) + 192 c_3^2 c_4^2].$$

Thus it suffices to show that

$$[257(c_3^4 + c_4^4) + 192 c_3^2 c_4^2] \leqq 353(c_3^4 + c_4^4),$$

and it is clearly sufficient to establish this under the additional hypothesis $c_3^4 + c_4^4 = 1$. But then the maximum value of the LHS occurs when $c_3^2 = c_4^2 = \sqrt{2}/2$, and this maximum value is precisely 353.

Next we show that (1) holds for any real-valued $f \in L^4$. Let $\mathscr{X}_0(X) = 2\,\mathrm{tr}\,[X]$ for $X \in G$. Then a computation shows that

$$\| \mathscr{X}_0 \|_{L^1(G)} = 1,$$

so $\mathscr{X}_0$ has norm one as a convolution operator on any space $L^p(G)$. Since

$$\mathscr{X}_0 * f(X) = 2\,\mathrm{tr}\,[\hat{f}(\gamma_0)X],$$

it follows that $L_m f = L_m(\mathscr{X}_0 * f)$. Since $\hat{f}(\gamma_0)$ is a real matrix for any real-valued f, it follows that (1) holds for such f. But a lemma of Bonami [1, Chapitre III, Lemme 2] now implies that (1) holds for any $f \in L^4(G)$. Thus $\| m \|_{M_4} = (353)^{1/4}$ as desired.

B. $\| m^* \|_{M_4} > (353)^{1/4}$.

The function $m^* \in \mathscr{C}(\Gamma)$ is defined by

$$m^*(\gamma_0) = \begin{bmatrix} 0 & 0 \\ 1 & 4 \end{bmatrix}, m^*(\gamma) = 0 \quad \text{for other} \quad \gamma \in \Gamma.$$

If

$$f(X) = 2\,\mathrm{tr}[\hat{f}(\gamma_0)X], \quad \hat{f}(\gamma_0) = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \quad \text{a real matrix},$$

then a computation shows that

$$\| L_{m^*} f \|_{L^4}^4 = 8[(c_1 + 4c_3)^4 + (c_2 + 4c_4)^4].$$

Since

$$\| f \|_{L^4}^4 = 8[c_1^4 + c_2^4 + c_3^4 + c_4^4 + 6(c_1^2 c_4^2 + c_2^2 c_3^2)],$$

it suffices to find real numbers c_1, c_2, c_3, c_4 for which

$$\frac{(c_1 + 4c_3)^4 + (c_2 + 4c_4)^4}{c_1^4 + c_2^4 + c_3^4 + c_4^4 + 6(c_1^2 c_4^2 + c_2^2 c_3^2)} > 353. \tag{2}$$

But when $(c_1, c_2, c_3, c_4) = (\frac{1}{2}, 0, 1, 0)$, the LHS of (2) is 410.0625/1.0625, which is larger than 353. This completes the proof of Lemma 2.

PROOF OF THE THEOREM. It suffices to show that there exist G and $m_1, m_2, \cdots$ in $M_4(G)$ such that

$$\frac{\| m_n^* \|_{M_4}}{\| m_n \|_{M_4}} \to \infty, \tag{3}$$

for if $m^* \in M_4$ for each $m \in M_4$, then the involution $m \to m^*$ would be bounded by the closed graph theorem.

Let G_1 be the group described in Lemma 2, let G_n be the Cartesian product of n copies of G_1, for $n = 2, 3, \cdots$, and let

$$G = \mathbf{P}_{n=1}^{\infty} G_n.$$

Let Γ_n be the dual object of G_n for $n = 1, 2, \cdots$, and let Γ be the dual object of G. Then Γ is canonically identifiable with the weak direct product

$$\mathbf{P}_{n=1}^{\infty}{}^* \Gamma_n,$$

and the canonical *-isomorphism of each $\mathscr{C}(\Gamma_n)$ into $\mathscr{C}(\Gamma)$ preserves the M_4 norm. Thus it suffices to find $m_n \in M_4(G_n)$ such that (3) holds.

Define $m_n \in \mathscr{C}(\Gamma_n) = \mathscr{C}(\mathbf{P}^n_{i-1}\Gamma_m)$ by the formula

$$m_n(\gamma_1, \cdots, \gamma_n) m(\gamma_1) \otimes \cdots \otimes m(\gamma_n) \quad \text{for} \quad (\gamma_1, \cdots, \gamma_n) \in \Gamma_n,$$

where m is as in Lemma 2. Then

$$m_n^*(\gamma_1, \cdots, \gamma_n) = m^*(\gamma_1) \otimes \cdots \otimes m^*(\gamma_n),$$

and so

$$\frac{\| m_n^* \|_{M_4}}{\| m_n \|_{M_4}} = \left(\frac{\| m^* \|_{M_4}}{\| m \|_{M_4}} \right)^n \to \infty,$$

by Lemmas 1 and 2. This concludes the proof of the theorem.

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