ON A CERTAIN BASIS IN c_0

BY

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ABSTRACT

A basis $\{x_n\}_{n=1}^{\infty}$ is constructed in c_0 such that there exists no bounded linear projection of c_0 onto the subspace spanned by a certain subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$.

1. **Introduction.** A. Pełczyński raised the following question ([3], Problem 4): Let $\{x_n\}_{n=1}^{\infty}$ be a basis of a Banach space X. Is each subspace of X spanned by some subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ complemented in X?

In this paper we show that the answer is negative by constructing a suitable example in c_0. Our main tools are the following two propositions:

PROPOSITION 1. (See [1] Theorem 3.) *l_1^{n+1} can be isometrically imbedded into $l_\infty^{2^n}$ and every linear projection P of $l_\infty^{2^n}$ onto l_1^{n+1} has norm*

$$\|P\| \geqq (n+1)2^{-n}\binom{n}{[n/2]}.$$

([$n/2$] *denotes the greatest integer* $\leqq n/2$.)

PROPOSITION 2. (See [2] p. 16, Corollary 3.) *If E is a finite dimensional subspace of a Banach space X for which X^{**} is a P_γ space and there exists a projection with norm c from X onto E, then E is a $P_{\gamma c}$ space. (X is called a P_γ space if for every Banach space Z containing X there is a linear projection P from Z onto X with $\|P\| \leqq \gamma$.)*

If $\{x_i\}_{i\in I}$ is a set of elements of a Banach space X then $[x_i]_{i\in I}$ denotes the closed linear space spanned by $\{x_i\}_{i\in I}$. We denote by $\{e_i^n\}_{i=1}^{2^n}$ the usual basis of $l_\infty^{2^n}$ and by $\{f_i\}_{i=1}^{2^n}$ the corresponding biorthogonal functionals in $l_1^{2^n} = (l_\infty^{2^n})^*$.

2. **Preliminary lemmas.** Denote by A_1 the matrix

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

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and let A_k $(k > 1)$ be the matrix obtained from A_{k-1} by substituting A_1 for $+1$ and $-A_1$ for -1. It is easily proved that $2^{-n}A_n$ is a $2^n \times 2^n$ symmetric orthogonal matrix. Denote by $a_{i,j}^n$ the elements of A_n and 2^n by $g(n)$.

LEMMA 1. *A_k is obtained from A_1 by substituting $+A_{k-1}$ for 1 and $-A_{k-1}$ for -1, $k = 2,3,4,\cdots$.*

The proof follows by induction from the definition of A_n. As a consequence of Lemma 1 we get

LEMMA 2. *For $2 \leqq n$, $1 \leqq i \leqq g(n-1)$ and $1 \leqq j \leqq g(n-1)$*

$$a_{i,j}^n = a_{i,j}^{n-1},\ a_{i,j+g(n-1)}^n = a_{i,j}^{n-1}$$

$$a_{i+g(n-1),j}^n = a_{i,j}^{n-1} \text{ and } a_{i+g(n-1),j+g(n-1)}^n = -a_{i,j}^{n-1}.$$

3. **A basis in $l_\infty^{g(n)}$.** Denote by E_n and F_n the subspaces

$$[e_i^n + e_{i+g(n-1)}^n]_{i=1}^{g(n-1)} \text{ and } [e_i^n - e_{i+g(n-1)}^n]_{i=1}^{g(n-1)}$$

of $l_\infty^{g(n)}$ respectively, and let T_n be the transformation from $l_\infty^{g(n-1)}$ to E_n, defined by

$$T_n\left(\sum_{j=1}^{g(n-1)} c_j e_j^{n-1}\right) = \sum_{j=1}^{g(n-1)} c_j(e_j^n + e_{j+g(n-1)}^n).$$

It is obvious that T_n is a linear isometry onto E_n.
Let us denote $x_i^n = \sum_{j=1}^{g(n)} a_{i,j}^n e_j^n$.

LEMMA 3. *For $2 \leqq n$ and $1 \leqq i \leqq g(n-1)$, $T_n(x_i^{n-1}) = x_i^n$.*

Proof.

$$\begin{aligned} T_n(x_i^{n-1}) &= T_n\left(\sum_{j=1}^{g(n-1)} a_{i,j}^{n-1} e_j^{n-1}\right) \\ &= \sum_{j=1}^{g(n-1)} a_{i,j}^{n-1}(e_j^n + e_{j+g(n-1)}^n) \\ &= \sum_{j=1}^{g(n)} a_{i,j}^n e_j^n. \end{aligned}$$

The last equality follows from Lemma 2.
Let $y_i^1 = x_i^1$ for $i = 1,2$ and define

$$y_i^k = \begin{cases} T_k y_i^{k-1} & 1 \leqq i \leqq g(k-1) \\ e_{i-g(k-1)}^k - e_i^k & g(k-1)+1 \leqq i < g(k) \\ x_i^k & i = g(k) \end{cases}$$

for $k > 1$.

Denote by $I(n)$ the set $\{i: i = g(k), 0 \leqq k \leqq n\}$.

LEMMA 4. *For* $n \geqq 1$ *and* $i \in I(n)$, $x_i^n = y_i^n$.

Proof. The case $n = 1$ is clear. Suppose that $x_i^k = y_i^k$ for $k < n$ and $i \in I(k)$. By the definition $y^n_{g(n)} = x^n_{g(n)}$. Since $I(n) = \{g(n)\} \cup I(n-1)$, if $i \in I(n)$ and $i < g(n)$ then $i \in I(n-1)$; therefore, by the induction hypothesis

$$y_i^n = T_n y_i^{n-1} = T_n x_i^{n-1} = x_i^n.$$

(The last equality follows from Lemma 3.)

LEMMA 5. *For* $n \geqq 1$, $g(n-1) \leqq k \leqq m \leqq g(n)$ *and every sequence of scalars* $\{c_i\}_{i=1}^m$,

$$\left\| \sum_{i=1}^{g(n-1)} c_i(e_i^n + e^n_{i+g(n-1)}) \right\|$$

$$\leqq \left\| \sum_{i=1}^{g(n-1)} c_i(e_i^n + e^n_{i+g(n-1)}) + \sum_{i=g(n-1)+1}^{k} c_i(e^n_{i-g(n-1)} - e_i^n \right\|$$

$$\leqq \left\| \sum_{i=1}^{g(n-1)} c_i(e_i^n + e^n_{i+g(n-1)}) + \sum_{i=g(n-1)+1}^{m} c_i(e^n_{i-g(n-1)} - e_i^n) \right\|$$

We omit the trivial proof.

LEMMA 6. *For* $k \geqq 1$, $1 \leqq n \leqq q \leqq g(k)$ *and every sequence of scalars* $\{c_i\}_{i=1}^{g(k)}$

$$(1) \qquad \left\| \sum_{i=1}^{n} c_i y_i^k \right\| \leqq 2 \left\| \sum_{i=1}^{q} c_i y_i^k \right\|$$

Proof. The case $k = 1$ is obvious. Suppose (1) holds for $k \leqq m$ and let us prove the assertion for $k = m+1$. We discuss separately the following four cases:

(a) $q \leqq g(m)$

In this case (1) follows from the definition of y_i^{m+1}, the fact that T_{m+1} is a linear isometry from $l_\infty^{g(m)}$ onto E_{m+1} and from the induction hypothesis.

(b) $g(m) < n \leqq q < g(m+1)$

By the definitions of T_{m+1}, and y_i^{m+1}, $\bar{y}_i^{m+1} \in E_{m+1}$ for $1 \leqq i \leqq g(m)$; Therefore

$$\sum_{i=1}^{g(m)} c_i y_i^{m+1} = \sum_{i=1}^{g(m)} b_i(e_i^{m+1} + e^{m+1}_{i+g(m)})$$

for some $b_1, b_2, \cdots, b_{g(m)}$. On the other hand, for $g(m) < i < g(m+1)$ $y_i^{m+1} = e^{m+1}_{i-g(m)} - e_i^{m+1}$, hence, by Lemma 5

$$(2)\qquad \left\|\sum_{i=1}^{n} c_i y_i^{m+1}\right\| = \left\|\sum_{i=1}^{g(m)} b_i(e_i^{m+1} + e_{i+g(m)}^{m+1}) + \sum_{i=g(m)+1}^{n} c_i(e_{i-g(m)}^{m+1} - e_i^{m+1})\right\|$$

$$\leqq \left\|\sum_{i=1}^{g(m)} b_i(e_i^{m+1} + e_{i+g(m)}^{m+1}) + \sum_{i=g(m)+1}^{q} c_i(e_{i-g(m)}^{m+1} - e_i^{m+1}\right\|$$

$$= \left\|\sum_{i=1}^{q} c_i y_i^{m+1}\right\|$$

(c) $n \leqq g(m) < q \leqq g(m+1)$
$y_i^{m+1} \in F_{m+1}$ for $g(m) < i \leqq g(m+1)$ ($y_{g(m+1)}^{m+1} \in F_{m+1}$ by Lemma 2), therefore

$$\sum_{i=g(m)+1}^{q} c_i y_i^{m+1} = \sum_{i=1}^{g(m)} d_i(e_i^{m+1} - e_{i+g(m)}^{m+1})$$

for some $d_1, d_2, \cdots, d_{g(m)}$. If

$$\sum_{i=1}^{g(m)} c_i y_i^{m+1} = \sum_{i=1}^{g(m)} b_i(e_i^{m+1} + e_{i+g(m)}^{m+1}),$$

(as in (b)) then, by Lemma 5,

$$\left\|\sum_{i=1}^{g(m)} c_i y_i^{m+1}\right\| = \left\|\sum_{i=1}^{g(m)} b_i(e_i^{m+1} + e_{i+g(m)}^{m+1})\right\|$$

$$\leqq \left\|\sum_{i=1}^{g(m)} b_i(e_i^{m+1} + e_{i+g(m)}^{m+1}) + \sum_{i=1}^{g(m)} d_i(e_i^{m+1} - e_{i+g(m)}^{m+1})\right\|$$

$$= \left\|\sum_{i=1}^{q} c_i y_i^{m+1}\right\|$$

From case (a) it follows that

$$\left\|\sum_{i=1}^{n} c_i y_i^{m+1}\right\| \leqq 2\left\|\sum_{i=1}^{g(m)} c_i y_i^{m+1}\right\| \leqq 2 \left\|\sum_{i=1}^{q} c_i y_i^{m+1}\right\|.$$

(d) $g(m) < n \leqq q = g(m+1)$
Denote by P_{m+1} the projection of $l_\infty^{g(m+1)}$ onto the one-dimensional subspace $[x_{g(m+1)}^{m+1}]$ defined by

$$P_{m+1}x = \frac{(-1)^m}{2}(f^{m+1}_{g(m)}(x) - f^{m+1}_{g(m+1)}(x))x^{m+1}_{g(m+1)}.$$

(According to §1 $\{f_i^{m+1}\}$ denotes the usual basis of $l_1^{g(m+1)}$.) It is easy to see that $\| P_{m+1} \| = 1$ and that $P_{m+1}y_i^{m+1} = 0$ for $1 \leqq i \leqq g(m+1) - 1$. Hence, $I - P_{m+1}$ is a projection of $l_\infty^{g(m+1)}$ onto $[y_i^{m+1}]_{i=1}^{g(m+1)-1}$ along $[y^{m+1}_{g(m+1)}]$ and $\| I - P_{m+1} \| \leqq 2$. Since $n > g(m)$, it follows from (2) that

$$\left\| \sum_{i=1}^{n} c_i y_i^{m+1} \right\| \leqq \left\| \sum_{i=1}^{g(m+1)-1} c_i y_i^{m+1} \right\|$$

$$= \left\| (I - P_{m+1}) \left(\sum_{i=1}^{q} c_i y_i^{m+1} \right) \right\| \leqq 2 \left\| \sum_{i=1}^{q} c_i y_i^{m+1} \right\|$$

This concludes the proof of Lemma 6.

4. **A non-complemented subspace of c_0.** Denote by $\{e_i\}_{i=1}^{\infty}$ the usual basis n c_0 and let U_n be the natural linear isometry from $l_\infty^{g(n)}$ onto

$$[e_i]_{i=g(n)}^{g(n+1)-1} n = 0, 1, 2, \cdots. \quad \left(U_n \left(\sum_{i=1}^{g(n)} c_i e_i^n \right) = \sum_{i=1}^{g(n)} c_i e_{i+g(n)-1} \right).$$

Put $z_1^0 = e_1$ and $z_i^n = U_n(y_i^n)$ for $n \geqq 1$ and $1 \leqq i \leqq g(n)$.

Lemma 7. *The sequence* $\{z_i^n\}_{i=1}^{g(n)}$ $_{n=0,1,2,\ldots}$ *in its natural order*

$$z_1^0, z_1^1, z_2^1, z_1^2, z_2^2, z_3^2, z_4^2, \cdots$$

forms a basis in c_0.

Proof. Obviously $[z_i^n]_{i=1}^{g(n)}$ $_{n=0,1,2,\ldots} = c_0$.
If $q \leqq r \leqq g(m+1)$ then by Lemma 6

$$\text{(3)} \qquad \left\| \sum_{k=0}^{m} \left(\sum_{i=1}^{g(k)} c_i^k z_i^k \right) + \sum_{i=1}^{q} c_i^{m+1} z_i^{m+1} \right\|$$

$$= \max \left\{ \max_{k \leqq m} \left\{ \left\| \sum_{i=1}^{g(k)} c_i^k z_i^k \right\| \right\}, \left\| \sum_{i=1}^{q} c_i^{m+1} z_i^{m+1} \right\| \right\}$$

$$\leqq \max \left\{ \max_{k \leqq m} \left\{ \left\| \sum_{i=1}^{g(k)} c_i^k z_i^k \right\| \right\}, 2 \left\| \sum_{i=1}^{r} c_i^{m+1} z_i^{m+1} \right\| \right\}$$

$$\leqq 2 \left\| \sum_{k=0}^{m} \left(\sum_{i=1}^{g(k)} c_i^k z_i^k \right) + \sum_{i=1}^{r} c_i^{m+1} z_i^{m+1} \right\|.$$

Similarly, for $m > n$, $q \leqq g(n+1)$ and $r \leqq g(m+1)$ it follows from (3) that

$$(4)\qquad \left\| \sum_{k=0}^{n} \left(\sum_{i=1}^{g(k)} c_i^k z_i^k \right) + \sum_{i=1}^{q} c_i^{n+1} z_i^{n+1} \right\|$$

$$\leqq 2 \left\| \sum_{k=0}^{n} \left(\sum_{i=1}^{g(k)} c_i^k z_i^k \right) + \sum_{i=1}^{g(n+1)} c_i^{n+1} z_i^{n+1} \right\|$$

$$= 2 \max_{k \leqq n+1} \left\{ \left\| \sum_{i=1}^{g(k)} c_i^k z_i^k \right\| \right\}$$

$$\leqq 2 \max \left\{ \max_{k \leqq m} \left\{ \left\| \sum_{i=1}^{g(k)} c_i^k z^k_i \right\| \right\}, \left\| \sum_{i=1}^{r} c_i^{m+1} z^{m+1} \right\| \right\}$$

$$= 2 \left\| \sum_{k=0}^{m} \left(\sum_{i=1}^{g(k)} c_i^k z_i^k \right) + \sum_{i=1}^{r} c_i^{m+1} z_i^{m+1} \right\|.$$

The last inequalities show that the sequence $\{z_i^n\}$ in its natural order forms a basis in c_0 and Lemma 7 is proved.

By [1], p. 459, $[x_i^n]_{i \in I(n)}$ is isometrically isomorphic to l^{n+1} and since U_n is a linear isometry, we get by Lemma 4 that $[z_i^n]_{i \in I(n)}$ is also isometrically isommorphic to l_1^{n+1}. Suppose that P is a bounded linear projection from c_0 onto the subspace Y spanned by the sequence $\{z_i\}_{i \in I(n)}$ $n = 0, 1, 2, \cdots$. It is obvious that the sequence $\{z_i^n\}_{i \in I(n)}$ $n = 1, 2, \cdots$ forms a basis in Y. From the proof of Lemma 7 it follows that there exists a sequence of projections $\{Q_n\}$ from Y onto $[z_i^n]_{i \in I(n)}$ with $\| Q_n \| \leqq 2$. Now, $Q_n P$ is a projection from c_0 onto $[z_i^n]_{i \in I(n)}$ and $\| Q_n P \| \leqq 2 \| P \|$; $c_0^{**} = m$ is a P_1 space; it follows from Proposition 2 that l_1^{n+1} is a P_γ space, $n = 1, 2, 3, \cdots$, where $\gamma = 2 \| P \|$. This contradicts Proposition 1; therefore, there exists no bounded linear projection from c_0 onto Y. Since Y is spanned by a subsequence of the basis $\{z_i^n\}_{i=1}^{g(n}$ $n = 0, 1, 2, \cdots$ of c_0 we have constructed the desired example.

As J. Lindenstrauss has remarked, a similar example can be constructed in the reflexive space $\sum_{n=1}^{\infty} \oplus_p l_\infty^{g(n)}$. The proof will be almost the same.

References

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