THE THEOREMS OF LOEWNER AND PICK

BY
WILLIAM F. DONOGHUE, JR.

ABSTRACT

The study of the exact interpolation of quadratic norms in vector spaces depends in an essential way on the theory of monotone matrix functions developed by Loewner in 1934 [4]. This theory, in its turn, depends on Loewner's solution of a problem of interpolation by rational functions of a certain class. The discussion of this latter problem is necessarily complicated, and Loewner's text does not lend itself to ready reference. It has therefore seemed worthwhile to recast a portion of Loewner's results in a form more suited to the applications we have in view. Our work, however, is not wholly derivative; none of our theorems are explicitly stated by Loewner and our arguments, which are of a more geometric character, are essentially different. The knowledgeable reader will note that our hypotheses are slightly stronger than Loewner's and that our results are therefore also stronger. For the applications which we have in mind, Theorem III is the most important result; the proof of this theorem depends on all of the previously developed theory.

1. **Introduction.** We consider the class P of functions $\phi(\zeta)$ analytic in the upper half-plane with positive imaginary part: $\phi(\zeta) = U(\zeta) + iV(\zeta)$, $V(\zeta) \geqq 0$. A convenient summary of the properties of this well-known class may be found in [1]. In particular, a function is in P if and only if it has a representation of the form

$$(1) \qquad \phi(\zeta) = \alpha\zeta + \beta + \int \left[\frac{1}{\lambda - \zeta} - \frac{\lambda}{\lambda^2 + 1}\right] d\mu$$

where $\alpha \geqq 0$, β is real and μ a positive Borel measure on the real axis for which $\int(\lambda^2 + 1)^{-1} d\mu(\lambda)$ is finite. The representation is unique.

If (a, b) is the open interval $a < x < b$ of the real axis, by $P(a, b)$ we denote the class of those functions in P which are real and regular on the interval (a, b) and which therefore admit an analytic continuation into the lower half-plane which is given by reflection. It is not difficult to show that $\phi(\zeta)$ belongs to $P(a, b)$ if and only if the corresponding measure μ has no mass in the interval $a < \lambda < b$.

It is important to note that the class $P(a, b)$ has a certain compactness property: If $\phi_n(\zeta)$ is a sequence in $P(a, b)$ such that for a pair of distinct points z' and z'' of the interval the sequences $\phi_n(z')$ and $\phi_n(z'')$ are bounded, then there exists a

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subsequence of those functions converging uniformly on closed subintervals of (a, b) to a function $\phi_0(\zeta)$ in $P(a, b)$. We do not give the proof in detail, but remark that the representation (1) for functions in P may equally well be written

$$\phi(\zeta) = \alpha\zeta + \beta + \int \frac{\lambda\zeta + 1}{\lambda - \zeta} \frac{d\mu(\lambda)}{\lambda_2 + 1} \tag{2}$$

and if the coefficient α is thought of as a positive mass at infinity, we obtain a positive measure $d\nu(\lambda)$ on the compactified real axis consisting of that mass at infinity and the distribution $(\lambda^2 + 1)^{-1}d\mu(\lambda)$. This is a Borel measure of finite total mass, and to measures of this kind we may apply Helly's theorem. We will have

$$\frac{\phi_n(z') - \phi_n(z'')}{z' - z''} = \alpha_n + \int \frac{d\mu_n(\lambda)}{(\lambda - z')(\lambda - z'')}$$

and since there exists a positive C such that $C^{-1}(\lambda^2 + 1) \leqq (\lambda - z')(\lambda - z'') \leqq C(\lambda^2 + 1)$ for all λ outside (a, b), the boundedness of the numbers $\phi_n(z')$ and $\phi_n(z'')$ implies the boundedness of the total masses of the corresponding measures $d\nu_n(\lambda)$.

We also note that the subsequence of $\phi_n(\zeta)$ converges to $\phi_0(\zeta)$ at all points ζ which are bounded away from the supports of the measures $d\nu_n(\lambda)$. In the special case when the functions of the sequence are all rational functions of degree at most N, the measures $d\nu_n$ will consist of at most N point masses, one of them possibly at infinity; the limiting measure $d\nu_0$ will have the same property and hence $\phi_0(\zeta)$ will also be rational, of degree at most N.

Suppose that $\phi(\zeta)$ belongs to $P(a, b)$ and that N is an integer $\geqq 1$; let $\xi_1, \xi_2, \xi_3, \cdots, \xi_N$ be any set of N distinct points in (a, b) and let similarly $\eta_1, \eta_2, \eta_3, \cdots, \eta_N$ be another set of N distinct points in that interval. We make no hypothesis requiring that these two sets be disjoint. Form the matrix M of order N defined by

$$\begin{aligned} M_{ij} &= \frac{\phi(\xi_i) - \phi(\eta_j)}{\xi_i - \eta_j} && \text{if } \xi_i \neq \eta_j \\ &= \phi'(\xi_i) && \text{if } \xi_i = \eta_j. \end{aligned}$$

Evidently the matrix elements M_{ij} are non-negative.

THEOREM 1. *If $\phi(\zeta)$ belongs to $P(a, b)$ and M is a corresponding matrix of order N then* (i) det $(M) = 0$ *if and only if $\phi(\zeta)$ is a rational function of degree at most $N - 1$.*

(ii) *If the sequences $\{\xi_i\}$ and $\{\eta_j\}$ are both monotone increasing then* $\det(M) \geqq 0$.

(iii) *If* $\det(M) = 0$, *the function $\phi(\zeta)$ is uniquely determined by the data, i.e.*

the points $\{\xi_i\}$, $\{\eta_j\}$ *and the values of* $\phi(\zeta)$ *at those points, as well as the values of its derivative if such values occur in* M.

Proof. We first prove (i) and (ii) under the special hypothesis that $\alpha = 0$ in the representation (1) for $\phi(\zeta)$. Using that representation, we easily find for all i and j that

$$M_{ij} = \int \frac{d\mu(\lambda)}{(\lambda - \xi_i)(\lambda - \eta_j)}$$

even when $\xi_i = \eta_j$. We then introduce the N functions $f_i(\lambda) = 1/(\lambda - \xi_i)$ which are integrable square relative to the measure μ, and similarly the N functions

$$g_j(\lambda) = \frac{1}{\lambda - \eta_j}$$

which have the same property. Clearly $M_{ij} = (f_i, g_j)$, the inner product being taken in $L^2(\mu)$, and the determinant of M is a kind of Gramm's determinant.

If $\phi(\zeta)$ is rational, of degree at most $N - 1$, the measure μ consists of point masses at the poles of $\phi(\zeta)$, and there are at most $N - 1$ of them. Thus the space $L^2(\mu)$ has dimension at most $N - 1$ and there exists a non-trivial linear dependance between the f_i, whence $\det(M) = 0$.

Conversely, if the determinant vanishes, there must exist a linear combination $f^*(\lambda) = \sum_{i=1}^N c_i f_i(\lambda)$ which is orthogonal to every $g_j(\lambda)$. We may write $f^*(\lambda) = p(\lambda)/Q(\lambda)$ where $Q(\lambda) = \prod_{i=1}^N(\lambda - \xi_i)$ and $p(\lambda)$ is a polynomial of degree at most $N - 1$, since the rational $f^*(\lambda)$ vanishes at infinity. If $R(\lambda) = \prod_{j=1}^N(\lambda - \eta_j)$, the function $g^*(\lambda) = p(\lambda)/R(\lambda)$ is a linear combination of the functions $g_j(\lambda)$, and hence $(f^*, g^*) = 0$. This may be written

$$\int \frac{p(\lambda)^2}{Q(\lambda)R(\lambda)}\, d\mu(\lambda) = 0$$

and we deduce, since the denominator is bounded from below on the support of μ by a positive constant, that the measure μ is concentrated at the zeros of $p(\lambda)$, a set consisting of at most $N - 1$ points. Thus $\phi(\zeta)$ is rational, of degree at most $N - 1$.

If we suppose that $\phi(\zeta)$ is not rational of degree at most $N - 1$ and that the points $\{\xi_i, \eta_j\}$ are given as monotone sequences, we vary the $\{\eta_j\}$ by writing $\eta_j(t) = t\xi_j + (1 - t)\eta_j$. As t varies over the unit interval, $\det M(t)$ is a continuous function of t. For any t there will be N distinct $\eta_j(t)$ as well as ξ_i, hence the determinant will never vanish and therefore keeps a constant sign. For $t = 1$, however, we have $\xi_i = \eta_i$, whence $f_i(\lambda) = g_i(\lambda)$ for all i, and therefore M is a Gramm's determinant. Since it does not vanish, it is positive, whence $\det M(0) > 0$. For $\phi(\zeta)$ rational of degree at most $N - 1$, $\det M(t)$ is of course identically zero.

We next establish (i) and (ii) without the special hypothesis $\alpha = 0$. From the formula (1) it is easy to deduce that the non-negative α is the limit as ζ approaches infinity along the imaginary axis of the ratio $\phi(\zeta)/\zeta$. Accordingly, if $\phi(\zeta)$ in $P(a, b)$ corresponds to a positive α and is positive in that interval, the function $\psi(\zeta) = -1/\phi(\zeta)$ is also in $P(a, b)$ and corresponds to $\alpha = 0$. If M^* is the matrix corresponding to $\psi(\zeta)$, it is easy to see that $\det(M^*) = C^{-1}\det(M)$ where $C = \prod_{i=1}^{N}\phi(\xi_i)\phi(\eta_i) > 0$. Since $\psi(\zeta)$ is rational of degree k if and only if $\phi(\zeta)$ is, we see that (i) and (ii) hold whenever the function $\phi(\zeta)$ is positive on (a, b), or at least on a subinterval containing the points $\{\xi_i\}$ and $\{\eta_j\}$. Since the addition of a constant to $\phi(\zeta)$ does not affect the matrix M at all, and similarly does not affect the rationality or degree of the function, the assertions (i) and (ii) are valid in any case.

To establish (iii) we suppose that $\det(M) = 0$ and therefore that $\phi(\zeta)$ is rational of degree at most $N - 1$. If there were two such functions $\phi(\zeta)$, their difference would be rational of degree at most $2N - 2$, but the total order of its zeros would be $2N$; thus the difference is identically zero. This completes the proof of Theorem I.

REMARK 1. We suppose that $\phi(\zeta)$ is of degree exactly $N - 1$, the determinant, therefore, being zero. Supposing, in addition, that η_1 coincides with none of the $\{\xi_i\}$ we expand the determinant along the first column, displaying the dependence of terms on η_1 and $\phi(\eta_1)$ to obtain

$$\det(M) = \sum_{i=1}^{N} \frac{\phi(\xi_i)m_i}{\xi_i - \eta_1} - \phi(\eta_1) \sum_{i=1}^{N} \frac{m_i}{\xi_i - \eta_1}$$

where the m_i are appropriate (non-zero) minors of the determinant. We introduce the rational functions

$$F(\zeta) = \sum_{i=1}^{N} \frac{\phi(\xi_i)m_i}{\xi_i - \zeta} \quad \text{and} \quad G(\zeta) = \sum_{i=1}^{N} \frac{m_i}{\xi_i - \zeta}$$

and note that since η_1 may be chosen almost arbitrarily in the interval (a, b),

$$0 = \det(M) = F(\zeta) - \phi(\zeta)G(\zeta), \text{ whence } \phi(\zeta) = F(\zeta)/G(\zeta).$$

The determinant is obviously a linear function in the entry $\phi(\eta_1)$ and our purpose here is to emphasize that this linear function is not identically zero. We have $0 = F(\eta_1) - \phi(\eta_1)G(\eta_1)$ and if this function vanished identically in the variable $\phi(\eta_1)$ we would have $0 = F(\eta_1) = G(\eta_1)$; it would follow, since $G(\zeta)$ and $F(\zeta)$ have a common zero at $\zeta = \eta_1$ as well as at infinity that their ratio was a rational function of degree smaller than $N - 1$, contradicting the hypothesis that $\phi(\zeta)$ is of degree exactly $N - 1$. It is not difficult to extend this argument to the case when η_1 happens to coincide with one of the points ξ_i. We shall often make use of this remark in the sequel where we will have a function $\phi(\zeta)$ of degree $N - 1$

in the class $P(a,b)$ and a set of $2N$ points of the interval, as well as another function $g(\zeta)$ defined on those $2N$ points and coinciding with $\phi(\zeta)$ at all but one of them. Then the hypothesis that the determinant of type M computed for $g(\zeta)$ vanishes will imply that $g(\zeta)$ coincides with $\phi(\zeta)$ on all $2N$ points.

REMARK 2. For the applications of Theorem I it is incovenient that the theorem requires that the points $\{\xi_i\}$ and $\{\eta_j\}$ be interior points of the interval (a,b). We therefore introduce the class $P[a,b]$ consisting of those functions in $P(a,b)$ which are continuous on the closed interval $[a,b]$. The proof of Theorem I carries over to functions in this class, where we permit a choice of the points $\{\xi_i\}$ and $\{\eta_j\}$ which may include one or more of the end points. Here, however, we cannot admit that one of the end points be taken both as a ξ and an η, since the derivatives $\phi'(a)$ and $\phi'(b)$ may be infinite for some functions in $P[a,b]$. The proof of this variant of Theorem I is virtually the same, except that if, say, $\xi_1 = a$, the corresponding function $f_1(\lambda) = (\lambda - a)^{-1}$ may no longer be in the space $L^2(\mu)$; however, the integrals which we have written exist in any case.

2. **The cone $P(Z)$).** We suppose given on the real axis a finite set Z of l points:

$$z_1 < z_2 < z_3 < \cdots < z_l$$

and let $C(Z)$ denote the space of all real functions $f(z)$ defined on Z. $C(Z)$ is a real l-dimensional vector space containing a convex cone, $P(Z)$, the restrictions to Z of functions in the class $P[z_1, z_l]$. For any f in $C(Z)$ we introduce its Loewner determinants, which are defined as follows. For any subset S of Z consisting of an even number of points we write the points of S in the following fashion:

$$\xi_1 < \eta_1 < \xi_2 < \eta_2 < \cdots < \xi_N < \eta_N$$

and form the matrix M, where

$$M_{ij} = \frac{f(\xi_i) - f(\eta_j)}{\xi_i - \eta_j} .$$

The determinant of this matrix is the Loewner determinant associated with f and S. Evidently there are as many Loewner determinants as there are non-empty subsets of Z of even cardinal, viz. $2^{l-1} - 1$ of them.

The following assertion is an immediate consequence of Theorem I. If $f(z)$ in $C(Z)$ belongs to $P(Z)$, then

I. *All of the Loewner determinants of f are non-negative* and

II. *If a Loewner determinant of f vanishes, so also do all other Loewner determinants of the same or higher order.*

It is our purpose to show that these conditions are also sufficient for f in $C(Z)$ to belong to $P(Z)$. To establish this, it is necessary to study the convex cone $P(Z)$ in greater detail. We consider it first under the auxiliary hypothesis that l is odd: $l = 2N + 1$, $N \geqq 1$.

LEMMA 1. *If f belongs to $P(Z)$, it is an interior point of that cone if and only if all of its Loewner determinants are positive.*

Proof. Suppose, first, that a Loewner determinant vanishes; from Theorem I it follows that f is the restriction to Z of a uniquely determined function in the class $P[z_1, z_l]$ which is rational of degree at most $N - 1$. Suppose k is the degree of that function and consider a Loewner determinant of order $k + 1$. This determinant vanishes. From Remark 1 we see that the determinant is not identically zero in the variable $f(z_1)$. It follows that if the value of f at z_1 is slightly changed in the appropriate direction, the Loewner determinant becomes negative and the perturbed function is not in the cone $P(Z)$. Thus f is the limit of elements in the complement of $P(Z)$ and hence is a boundary point of the cone. We note also that this part of our argument does not depend on the parity of l.

Conversely, if all of the Loewner determinants are positive f is the restriction to Z of a function $\phi(\zeta)$ in $P[z_1, z_l]$ which is not a rational function of degree smaller than N. Suppose, first, that $\phi(\zeta)$ is rational of degree N, and indeed of the form

$$\phi(\zeta) = \beta + \sum_{i=1}^{N} \frac{m_i}{\lambda_i - \zeta}$$

Here the m_i are positive and the poles λ_i are outside the interval $[z_1, z_l]$. We adjoin $2N + 1 = l$ real variables: $c, a_1, a_2, \cdots, a_N, b_1, b_2, \cdots, b_N$ to form

$$F(\zeta, a_i, b_j, c) = (\beta + c) + \left[\sum_{i=1}^{N} \frac{(1 + a_i)m_i}{\lambda_i + b_i - \zeta} \right].$$

For small values of a_i and b_i this is a function in the class $P[z_1, z_l]$; its restriction to Z then determines a mapping of a neighborhood of 0 in the space of l real variables into a neighborhood of f in $C(Z)$. We have only to show that the mapping is onto, i.e. that its Jacobian does not vanish at the origin. If we compute the partial derivatives of F at the origin we find

$$\frac{\partial F}{\partial a_i} = \frac{m_i}{\lambda_i - \zeta} \qquad \frac{\partial F}{\partial b_j} = \frac{-m_j}{(\lambda_j - \zeta)^2} \qquad \frac{\partial F}{\partial c} = 1.$$

If the Jacobian vanishes, there is a linear combination

$$H(\zeta) = B + \sum_{i=1}^{N} \frac{C_i}{\lambda_i - \zeta} + \sum_{i=1}^{N} \frac{D_i}{(\lambda_i - \zeta)^2}$$

which vanishes on the set Z and hence has $2N + 1$ zeros. Since the total order of the poles is at most $2N$ this is impossible. The same argument goes through if we suppose that the function $\phi(\zeta)$ is rational, of degree N, but with a pole at infinity, viz.

$$\phi(\zeta) = \alpha\zeta + \beta + \sum_{i=1}^{N-1} \frac{m_i}{\lambda_i - \zeta} .$$

The auxiliary functions are

$$F(\zeta, a_i, b_i, c) = \alpha(1 + a_N)\frac{(\zeta + b_N)}{(1 - \zeta b_N)} + (\beta + c) + \sum_{i=1}^{N-1} \frac{(1 + a_i)m_i}{\lambda_i + b_i - \zeta}$$

We remark next that if $\phi(\zeta)$ can be decomposed in any way into a sum $\phi_1(\zeta) + \phi_2(\zeta)$ of functions in P there exists a corresponding decomposition of $f = f_1 + f_2$ in $P(Z)$; if one of the terms is an interior point of $P(Z)$ then f is too. It follows that if the measure μ associated with $\phi(\zeta)$ has N distinct point masses then f is an interior point of $P(Z)$. To complete the proof, therefore, we need only consider the case when the measure μ has no point masses, and we may suppose that μ is concentrated on an interval I lying wholly to one side of the interval $[z_1, z_l]$. On that interval we consider the l functions $h_k(\lambda) = |\lambda - z_k|^{-1}$ which will be linearly independant in $L^2(\mu)$ since otherwise there would exist a linear combination of them which vanished for infinitely many points in I and was a rational function of λ. For a fixed small positive ε we may write $\phi(\zeta)$ in the form
$\phi(\zeta) = \psi(\zeta) + \varepsilon \sum_{i=1}^{l}\phi_i(\zeta)$ where $\psi(\zeta)$ is in $P[z_1, z_l]$ and $\phi_i(\zeta) = \int h_i(\lambda)d\mu(\lambda)/(\lambda - \zeta)$ is in the same class. The functions $\phi_i(\zeta)$, when restricted to Z, form a linearly independant set in $C(Z)$ since the determinant of the matrix $H_{ij} = \phi_i(z_k)$ is essentially the Gramm's determinant (h_i, h_j) and therefore cannot vanish. Thus for small coefficients a_i the functions

$$\psi(\zeta) + \varepsilon \sum_{i=1}^{l} (1 + a_i)\phi_i(\zeta)$$

map onto a neighborhood of f in $C(Z)$.

LEMMA 2. *Every f in $P(Z)$ is a restriction to Z of a function in $P[z_1, z_l]$ which is rational and of degree at most N. This function is unique.*

Proof. The uniqueness is obvious, since if there were two such functions their difference would be rational of degree at most $2N$ but would have $2N + 1$ zeros. To prove the lemma, we consider the set $R(Z)$, the subset of $P(Z)$ consisting of restrictions to Z of rational functions of degree at most N which are in $P[z_1, z_l]$. We must show that $R(Z)$ coincides with $P(Z)$. Clearly $R(Z)$ contains all of the boundary points of $P(Z)$ which belong to that cone. Moreover, if f belongs to $R(Z)$ and is an interior point of $P(Z)$, the argument of the second part of the proof of the previous lemma shows that f is surrounded by a neighborhood which belongs itself to $R(Z)$. Thus the interior points of $P(Z)$ which belongs to $R(Z)$ are interior points of the latter cone. To complete the proof, we must show that no interior point f of $P(Z)$ can be a boundary point of $R(Z)$.

If such an f exists there is a sequence $f_n(z)$ in $R(Z)$ converging to $f(z)$ uniformly on Z. Using the compactness property mentioned in the Introduction we deduce that there exists a rational function $\phi_0(\zeta)$ of degree at most N which coincides with $f(z)$ at all points z in Z bounded away from the supports of the measures μ_n occuring in the representation (1) for $f_n(\zeta)$. Thus we have $\phi_0(z_k) = f(z_k)$ for all k satisfying $2 \leqq k \leqq l-1$. We cannot have $f(z_1) = \phi_0(z_1)$ and $f(z_l) = \phi_0(z_l)$ simultaneously, since f by hypothesis is not in $R(Z)$. Suppose, say, $f(z_1) \neq \phi_0(z_1)$, but $f(z_l) = \phi_0(z_l)$. Since $f(z_1) = \lim_n f_n(z_1)$ it follows that the point z_1 is not bounded away from the supports of the measures μ_n; in more simple language, the functions f_n have poles λ_n near z_1 and with increasing n those poles converge to z_1. Such poles correspond to terms of the form $m_n/(\lambda_n - z_1)$ and it is easy to infer from the boundedness of the numbers $f_n(z_1)$ that these terms are uniformly bounded in absolute value. Thus, since the denominators tend to zero, the numerators m_n also do, and we infer that the function $\phi_0(\zeta)$ has not so many poles as the $f_n(\zeta)$; i.e. degree $\phi_0(\zeta)$ is at most $N-1$. It follows that the Loewner determinant computed for $\phi_0(\zeta)$ and the set $z_2, z_3, z_3, \cdots, z_l$ vanishes, and since this is also a Loewner determinant for f, that function has a Loewner determinant which vanishes. This contradicts the hypothesis that f was an interior point of $P(Z)$. In the case that $f(z)$ differs from $\phi_0(z)$ at both endpoints, the function $\phi_0(z)$ is of degree at most $N-2$ since at least two poles have disappeared in the limiting process. We then argue as before using a Loewner determinant of lower order.

The uniquely determined rational function associated with f in $P(Z)$ by the previous lemma will be called the canonical representation of f. Note that it exists only when l is odd.

LEMMA 3. *A boundary point f of $P(Z)$ belongs to that cone if and only if II is satisfied.*

Proof. We suppose that f is a boundary point of $P(Z)$ which does not belong to that cone but which does satisfy II and deduce a contradiction. Since f is a limit of a sequence of interior points of $P(Z)$, each representable as a rational function in $P[z_1, z_l]$ of degree at most N, we argue as in the previous lemma to find a rational $\phi_0(\zeta)$ in $P[z_1, z_l]$ of degree $k \leqq N-1$ which coincides with f at all points of Z except perhaps the end points. Since f is not in $P(Z)$, f cannot coincide with $\phi_0(\zeta)$ at both end points.

If $k \leqq N-2$ there exists a subset S of Z consisting of $2k+2$ points and not containing the end points z_1 and z_l; the corresponding Loewner determinant of $\phi_0(\zeta)$, and therefore of f, is zero. Since f satisfies II, all Loewner determinants of that order for f vanish, in particular the one computed for the system of points $z_1, z_2, z_3, \cdots, z_{2k+2}$. Because f coincides with $\phi_0(\zeta)$ at all of th points except perhaps z_1, the Remark following Theorem I guarantees that $f(z_1) = \phi_0(z_1)$ as well. We argue similarly to show $f(z_l) = \phi_0(z_l)$, whence f coincides with ϕ_0, a contradiction.

If $k = N - 1$ the function $\phi_0(\zeta)$ coincides with f for at least one of the end points, because if not, then in the limiting process $f_n(\zeta)$ converging to $\phi_0(\zeta)$ at least two masses would be destroyed, whence degree $\phi_0 < N - 1$. We suppose $f(z_l) = \phi_0(z_l)$ and deduce that the Loewner determinant of f associated with the set of $2N$ points obtained by omitting z_1 vanishes. From II then, it follows that the Loewner determinant of f associated with the first $2N$ points of Z vanishes, and again, by the remark following Theorem I, $f(z_1) = \phi_0(z_1)$, a contradiction.

We pass next to the more complicated case when l is even: $l = 2N, N \geqq 1$. The results of Lemmas 1 and 2 can be brought over to this case by the following device. We select a point $\tilde{z}$ in the interval $z_1 < \tilde{z} < z_2$ and adjoin it to Z to obtain a set $\tilde{Z}$ of $l + 1$ points. Every f in $P(Z)$ is the restriction to Z of an element in $P(\tilde{Z})$. The projection mapping of $C(\tilde{Z})$ onto $C(Z)$ carries $P(\tilde{Z})$ onto $P(Z)$ and maps interior points of $P(\tilde{Z})$ into interior points of $P(Z)$. Thus from property II we obtain most of the following lemma.

Lemma 4. *A point f of $P(Z)$ is an interior point of that cone if and only if all of its Loewner determinants are positive. Every f in $P(Z)$ is the restriction to that cone of a rational function in $P[z_1, z_l]$ of degree at most N; this rational function is unique if and only if f is a boundary point of $P(Z)$.*

Proof. Since an interior point of $P(Z)$ possesses an extension to $P(\tilde{Z})$ which is an interior point of that cone, it is clear that there exist infinitely many choices for the value of the extension at $\tilde{z}$ each of which corresponds to a different canonical representation of the extended function. Thus the representation cannot be unique for an interior point, however, Theorem I guarantees that it is unique when f is a boundary point, since then a Loewner determinant vanishes.

The assertion of Lemma 3 is also valid when l is even, its demonstration however, is difficult. This is the content of Lemma 6 of the next section; we will assume it now and pass to the proof of our principal theorem.

Theorem II. *A function f in $C(Z)$ belongs to $P(Z)$ if and only if I and II are satisfied.*

Proof. We know that the conditions are necessary for f to belong to $P(Z)$. Our argument is by induction, the assertion for $l = 2$ and $l = 3$ being trivial. Supposing the theorem true for $l - 1$, and that f in $C(Z)$ satisfies I and II, there exists a $g(z)$ in $P(Z)$ so that $f(z_k) = g(z_k)$ for all $k \neq 2$ by the inductive hypothesis. We form $h_t(z) = tg(z) + (1 - t)f(z)$, where $0 \leqq t \leqq 1$.

If $g = h_1$ is an interior point of $P(Z)$ all of its Loewner determinants are positive. Since the Loewner determinants for h_t are linear in t, all positive for $t = 1$ and non-negative for $t = 0$, it follows that no h_t for $t > 0$ can be a boundary point of $P(Z)$; from continuity considerations, then, either $f = h_0$ is in $P(Z)$ itself, or is at least a boundary point of that cone. Because II is satisfied, it follows that f is an element of $P(Z)$.

If $g = h_1$ is only a boundary point of $P(Z)$ it must be the restriction to Z of a uniquely determined function $\phi(\zeta)$ in the class $P[z_1, z_l]$ which is rational of degree k. If we were to suppose that no Loewner determinant of f were zero, we choose any non-rational function $\phi(\zeta)$ in the class $P(z_1 - 1, z_l + 1)$ and form $f + \varepsilon\psi$ for small positive ε. If ε is sufficiently small, $f + \varepsilon\psi$ has all of its Loewner determinants positive, these determinants being continuous functions of their arguments. The function $g + \varepsilon\psi$ belongs to $P(Z)$ and is not the restriction to Z of a rational function in $P[z_1, z_l]$, hence is an interior point of $P(Z)$. Since it coincides with $f + \varepsilon\psi$ at all points of Z except z_2, the argument which we have just given shows that $f + \varepsilon\psi$ is in $P(Z)$; the ε being arbitrary, it follows as before that either f is in the cone $P(Z)$ or it is a boundary point of that cone. Since f satisfies II, it must then be an element of that cone.

Thus, for the balance of the proof, we may suppose that g is rational of degree k and that both f and g have Loewner determinants which vanish. We must have $k \leqq N - 1$ where $l = 2N$ or $l = 2N + 1$. There exists therefore, when l is odd, or when $k \leqq N - 2$ a proper subset of Z consisting of $2k + 2$ points not containing z_2 so that the corresponding Loewner determinant of g (and therefore of f) vanishes. Since both f and g satisfy II, the Loewner determinants computed for

$$z_1, z_2, \cdots, z_{2k+2}$$

vanish, whence $f(z_2) = g(z_2)$ by Remark 1. If $l = 2N$ and $k = N - 1$, only the largest possible Loewner determinant vanishes, but it vanishes for both functions; we obtain, as before, $f(z_2) = g(z_2)$ completing the proof.

3. **Representations for even l.** In this section we suppose that f is an interior point of $P(Z)$ where $l = 2N$ is even. Because f is interior there exist a variety of extensions of f to $P(\tilde{Z})$ where $\tilde{Z}$ is obtained from Z by the adjunction of a point. Thus there exist a variety of functions, rational of degree N belonging to $P[z_1, z_l]$ which coincide with f on Z. Our purpose is to study these representations.

Let $f_0(z)$ and $f_\infty(z)$ be two such functions; we consider their difference $h(z) = f_0(z) - f_\infty(z)$ which is rational of degree at most $2N$. Since this function does not vanish identically and has $2N$ zeros as the points of Z, it follows that it has degree exactly $2N$, and since its poles are simple, both f_0 and f_∞ are of degree exactly N and have N distinct poles. We see that there is no loss of generality if we suppose that the poles of f_0 and f_∞ are all finite.

We write these functions as ratios of relatively prime polynomials with real coefficients:

$$f_0(z) = \frac{\sigma_0(z)}{\tau_0(z)} \qquad f_\infty(z) = \frac{\sigma_\infty(z)}{\tau_\infty(z)}$$

where the denominators are polynomials of degree exactly N with real distinct and simple zeros. For the difference we then have

$$h(z) = \frac{\tau_\infty(z)\sigma_0(z) - \sigma_\infty(z)\tau_0(z)}{\tau_\infty(z)\tau_0(z)} = \frac{\delta(z)}{\tau_\infty(z)\tau(z)}$$

The numerator, $\delta(z)$ cannot vanish identically and is of degree at most $2N$; since it vanishes at the points of Z it has degree exactly $2N$ and has simple zeros at those points.

We consider any two adjacent poles of $h(z)$ in the interval $z < z_1$; since the zeros of h are all in the set Z, there is no zero between these poles, and hence the residues at these poles are of opposite sign. Because h is the difference of two functions in P, and such functions always have negative residues, it follows that one of this pair of poles is a pole of f_0 and the other a pole of f_∞. The same argument holds if we consider a pair of adjacent poles of h in the interval $z_l < z$ or in a projective neighborhood of the point at infinity. We conclude that the zeros of $\tau_0(z)$ and $\tau_\infty(z)$ separate one another on the projective real axis.

From the foregoing it follows that the function $T(z) = -\tau_0(z)/\tau_\infty(z)$ has residues of the same sign at all of its poles; there is no loss of generality in supposing that these residues are all negative, and therefore that $T(z)$ itself is in P.

Next we introduce the family of polynomials $\tau_t(z) = \tau_0(z) + t\tau_\infty(z)$ where t varies over the real axis with the convention that $t = \infty$ corresponds to $\tau_\infty(z)$. All of these polynomials, save one, are of degree N, and the exceptional one will be supposed to have a zero at infinity. With this convention it is immediate that the zeros of $\tau_t(z)$ are exactly the set of points on which $T(z) = t$, and since $T(z)$ belongs to P, it is monotone increasing between its poles and assumes every real value just once in any such interval. Thus for any pair of values, t, s the zeros of $\tau_t(z)$ and $\tau_s(z)$ separate one another. Moreover, all of these zeros are real and simple, and there are exactly N of them.

From this circumstance it follows that the Wronskian

$$W(z) = \tau_0'(z)\tau_\infty(z) - \tau_\infty'(z)\tau_0(z)$$

never vanishes on the real axis since no $\tau_t(z)$ has a multiple zero. We may therefore suppose that this real polynomial is positive on the real axis.

The values taken by the function $T(z)$ on the set Z are called exceptional values of t; since T belongs to $P[z_1, z_l]$ these values are distinct, and there are therefore $2N$ of them. We write $t_k = T(z_k)$ and note that $\tau_{t_k}(z)$ has a zero at $z = z_k$ but does not vanish at any other point of Z.

For values of t which are not exceptional the rational functions

$$f_t(z) = \frac{\sigma_0(z) + t\sigma_\infty(z)}{\tau_0(z) + t\tau_\infty(z)} = \frac{\sigma_t(z)}{\tau_t(z)}$$

coincide with $f(z)$ on the points of Z and are of degree at most N. It will presently become clear that they are of degree exactly N. For exceptional values of t, say $t = t_k$, the denominator $\tau_t(z)$ vanishes at the point z_k, however, the numerator also does, since

$$\sigma_{t_k}(z) = [\tau_\infty(z_k)]^{-1}[\tau_\infty(z_k)\sigma_0(z_k) - \tau_0(z_k)\sigma_\infty(z_k)] = \frac{\delta(z_k)}{\tau_\infty(z_k)} = 0$$

Hence for the corresponding $f_{t_k}(z)$ we must understand the rational function obtained after the common factor $(z - z_k)$ is cancelled from numerator and denominator. This function coincides with f at all points of Z other than z_k; it is of degree $N - 1$ and is the canonical representation for the restriction of f to the set of $2N - 1$ points obtained by omitting z_k from Z. That function therefore belongs to P, however, $f(z_k)$ cannot equal $f_{t_k}(z_k)$ since f is an interior point of $P(Z)$.

If x is a real point not contained in Z, then as t varies over the real axis the quantity $f_t(x)$ is a linear fractional transformation in t which is non-degenerate its determinant, $\delta(x)$ being non-zero. We deduce that if $g(z)$ is any rational function of degree at most N which coincides with f on the set Z, then g is a member of the family f_t; to show this, we remark that g cannot coincide with f_t when t is exceptional; thus there exists a real x not in Z for which $g(x) \neq f_{t_k}(x)$ for all k. We then select t so that $f_t(x) = g(x)$ and note that the difference $g(z) - f_t(z)$ is of degree $2N$ and has $2N + 1$ zeros.

Again, when x is real and not in Z there exists exactly one value of t such that $f_t(z)$ has a pole at x, viz. $t = T(x)$. We compute $r(x)$, the residue of that function at that pole. We choose a small circle C with center at x and radius ε and write

$$r(x) = \frac{1}{2\pi i} \int_C \frac{\sigma_0(z) + T(x)\sigma_\infty(z)}{\tau_0(z) + T(x)\tau_\infty(z)}\, dz\mu$$

If we suppose that $T(x)$ is finite the integrand may be written

$$\frac{\tau_\infty(x)\sigma_0(z) - \tau_0(x)\sigma_\infty(z)}{\tau_\infty(x)\tau_0(z) - \tau_0(x)\tau_\infty(z)}$$

and for small values of ε the numerator approaches

$$\tau_\infty(x)\sigma_0(x) - \tau_0(x)\sigma_\infty(x) = \delta(x)$$

while the denominator is approximated by

$$(z - x)\,[\tau_\infty(x)\tau_0'(x) - \tau_0(x)\tau_\infty'(x)] = (z - x)W(x).$$

Since ε may be arbitrarily small, it follows that $r(x) = \delta(x)/(W(x))$ and this rational function has no poles on the real axis and has exactly $2N$ simple zeros, all of which are at the points of Z. By continuity, the same formula is valid for the points x which appear as poles of $T(z)$.

Since there exists a value of t for which $f_t(z)$ is in $P[z_1, z_l]$ we see that $r(x)$ is negative for x outside the closed interval $[z_1, z_l]$ as well as for x inside intervals of the form $z_{2k} < x < z_{2k+1}$, while $r(x)$ is non-negative elsewhere and zero only at the points of Z. Since Z is contained in an interval between two adjacent zeros of $\tau_\infty(z)$, an interval which we may call (a, b), every f_t has precisely one pole in $[a, b)$. Thus f_t is in $P[z_1, z_l]$ if and only if it has no pole in $[z_1, z_l]$.

We pass to the study of the function for exceptional values of t. We have already remarked that $f_{t_k}(z_k)$ is not equal to $f(z_k)$; it is possible to compute the difference of these numbers. For the sake of simplicity we avoid the special value of t for which $f_t(z)$ has a pole at infinity if that value is exceptional; the functions f_t may then be written in the form

$$f_t(z) = \gamma(t) + \sum_{i=1}^{N} \frac{m_i(t)}{\lambda_i(t) - z}$$

where $m_i(t) = -\delta(\lambda_i(t))/W(\lambda_i(t))$ and the $\lambda_i(t)$ are the roots of $T(z) = t$. The quantity $\gamma(t)$ is the value of $f_t(z)$ at infinity and is a linear fractional function of t. We will suppose that $\lambda_1(t)$ is the root which varies from a to b. As t approaches $t_k = T(z_k)$, $\lambda_1(t)$ approaches z_k and the first term in the sum converges to

$$\lim_{x \Rightarrow z_k} \frac{-\delta(x)}{W(x)(x - z_k)} = \frac{-\delta'(z_k)}{W(z_k)}.$$

The other terms in $f_t(z)$ depend continuously on t and converge to the corresponding terms of $f_{t_k}(z)$; no term corresponding to the first appears in $f_{t_k}(z)$ even though the limit above is non-zero. Thus

$$f_{t_k}(z_k) - f(z_k) = \frac{\delta'(z_k)}{W(z_k)}$$

and the sign of this quantity depends on the parity of k. In particular, one verifies that $f_{t_1}(z_1) > f(z_1)$ as well as $f_{t_l}(z_l) < f(z_l)$.

This circumstance gives rise to the following remarkable property of the canonical representation in the case when l is odd, $l = 2N + 1$.

LEMMA 5. *If l is odd, f in $P(Z)$ and $\phi(\zeta)$ the canonical representation of f, and if $\psi(\zeta)$ is any function in $P[z_1, z_l]$ which coincides with f at the points of Z, then $\phi(\zeta)$ is regular in any interval containing $[z_1, z_l]$ in which $\psi(\zeta)$ is.*

Proof. Suppose $\psi(\zeta)$ regular in an interval $[z', z_l]$ containing $[z_1, z_l]$; we adjoin z' to Z to obtain a set of $2N + 2 = l + 1$ points Z'. The restriction of $\psi(\zeta)$ to Z' determines a function g in $P(Z')$ which coincides with f at the points of Z. We may suppose that g is an interior point of $P(Z')$, since otherwise the functions $\psi(\zeta)$ and $\phi(\zeta)$ coincide. It follows that $\phi(\zeta)$ is one of the exceptional functions in the family $g_t(\zeta)$ associated with g in $P(Z')$, hence is regular analytic in the interval $[z', z_l]$. We can argue similarly with a point z'' to the right of z_l. In addition, we will have $g(z') = \psi(z') \leqq \phi(z')$ and $g(z'') = \psi(z'') \geqq \phi(z'')$.

In conclusion, we establish a lemma which we used in the proof of Theorem II.

LEMMA 6. *When l is even, a boundary point f of $P(Z)$ belongs to that cone if and only if II is satisfied.*

Proof. We suppose $l = 2N$. As before, we deduce a contradiction from the hypothesis that f is a boundary point of $P(Z)$ which does not belong to that cone but which does satisfy II. Our argument is much the same as before; f is the limit of a sequence f_n in the interior of $P(Z)$, and these functions may be represented by rational functions of degree N in $P[z_1, z_l]$. Since there is a choice for these representatives, we take them in such a fashion that the nearest poles are distributed symmetrically about $[z_1, z_l]$; more precisely, we select each time the representative $f_t^{(n)}$ for f_n so that if λ' is the nearest pole of $f_t^{(n)}$ to the left of z_1 and if λ'' is the nearest pole to the right of z_l, then $z_1 - \lambda' = \lambda'' - z_l$. A subsequence of the sequence of rational functions so determined then converges to a rational $\phi(\zeta)$ which coincides with f at all points of Z except, perhaps, the end points. The representatives have been chosen in such a way that at least two poles are destroyed, i.e. $\phi(\zeta)$ is necessarily of degree at most $N - 2$. Thus the Loewner determinant for $\phi(\zeta)$ associated with the set $z_2, z_3, z_3, \cdots, z_{l-1}$ vanishes and therefore the corresponding Loewner determinant of f does. We now argue exactly as in the proof of Lemma 3 to infer that $f(z_1) = \phi(z_1)$ and $f(z_l) = \phi(z_l)$, hence that f is in $P(Z)$, a contradiction.

4. **The cone $P'(Z)$.** By P' we denote the subclass of P consisting of functions which are regular and positive on the open right half-axis. These functions admit the canonical representation obtained from (1) which follows:

$$\phi(\zeta) = \alpha\zeta + \beta + \int_{-\infty}^{0} \left[\frac{1}{\lambda - \zeta} - \frac{1}{\lambda}\right] d\mu(\lambda) \tag{3}$$

where $\alpha \geqq 0$, $\beta = \phi(0) \geqq 0$ and $\int_{-\infty}^{0}(1 + \lambda^2)^{-1}d\mu(\lambda)$ is finite.

It is easy to see that if $\phi(\zeta)$ belongs to P', so also does $\check{\phi}(\zeta) = [\phi(1/\zeta)]^{-1}$ as well as $\phi^*(\zeta) = \zeta\phi(1/\zeta)$.

In this section we shall suppose that Z is a subset of the open right half-axis and shall seek necessary and sufficient conditions that a function f in $C(Z)$ should belong not just to $P(Z)$ but to $P'(Z)$, the cone consisting of restrictions to Z of functions in P'. It is clear that the cone $P'(Z)$ is closed, for a sequence in P' which converges on the points of Z has a subsequence converging on all points of the positive real axis, those points being bounded away from the supports of the measures; moreover, the limiting function is non-negative on the right half-axis, hence is in P'.

It is also evident that if f belongs to $P'(Z)$ there exists a non-negative value C such that if f is extended to the origin by $f(0) = C$, the extended function is in $P(Z \cup 0)$. Unfortunately, we cannot always take $C = 0$.

We introduce the set Z^* consisting of reciprocals of points in Z

$$1/z_l < 1/z_{l-1} < 1/z_{l-2} < \cdots < 1/z_1$$

which may also be written $z_1^* < z_2^* < z_3^* < \cdots < z_l^*$ and consider the following conditions, concerning f in $C(Z)$.

III. f may be extended to a non-negative function in $P(Z \cup 0)$

IV. The function f^* defined on Z^* by $f^*(z_k^*) = z_k^* f(1/z_k^*)$ may be extended to a non-negative function in $P(Z^* \cup 0)$

V. The function $\check{f}$ defined on Z^* by $\check{f}(z_k^*) = [f(1/z_k^*)]^{-1}$ may be extended to a non-negative function in $P(Z^* \cup 0)$.

We then have

THEOREM III. *A function f in $C(Z)$ belongs to $P'(Z)$ if and only if*

(a) *when l is odd, III and V are valid*

(b) *when l is even, III and IV are valid.*

Proof. The necessity is an immediate consequence of our comment concerning the functions $\check{\phi}(\zeta)$ and $\phi^*(\zeta)$ when $\phi(\zeta)$ is in P'. For the sufficiency we must give different arguments depending on the parity of l. We remark that we possess examples showing that the state of affairs is essentially different when l is odd and when l is even.

When $l = 2N + 1$ is odd, we pass from f in $C(Z)$ satisfying III and V to its canonical representation $\phi(\zeta)$, a rational function of degree at most N belonging to $P[z_1, z_l]$. Since there exists a non-negative function in $P[0, z_l]$ which coincides with f on Z, it follows from Lemma 5 that $\phi(\zeta)$ is regular and non-negative in $[0, z_l]$, and we have only to show that this function has no poles to the right of z_l, thereby putting it in P' and therefore putting f in $P'(Z)$. However, we may argue similarly with the function $\check{f}$ defined on Z^* to find that its canonical representation $\psi(\zeta)$ is non-negative and regular in $[0, z_l^*]$. Since both $\phi(\zeta)$ and $\psi(\zeta)$ are rational and of degree at most N and satisfy the equations $\psi(z_k^*) = [\phi(1/z_k^*)]^{-1}$ for all $2N + 1$ values of k, it follows that identically in ζ we have

$$\psi(\zeta) = [\phi(1/\zeta)]^{-1}.$$

The regularity of $\psi(\zeta)$ in $[0, z_l^*]$ therefore implies the regularity of $\phi(\zeta)$ in $z_l < z < +\infty$. Hence $\phi(\zeta)$ is in P'.

When $l = 2N$ is even our argument is somewhat more complicated. Since III and IV surely imply that f is in $P(Z)$ and f^* is in $P(Z^*)$ we will suppose at first that each of these functions is an interior point of the corresponding cone and make use of the representation theory developed in the previous section. Let C be so chosen that when f is extended to 0 by the definition $f(0) = C$ the extended function is in $P(Z \cup 0)$; because of III there exists such a C which is non-negative. In the family $f_t(z)$ associated with f we select the function $f_s(z)$ for which $f_s(0) = C$; this function is rational and of degree at most N and is the canonical representation of the extended function considered on the $l+1$ points of $Z \cup 0$. It follows that $f_s(z)$ is in $P[0, z_l]$. It is not difficult to see that if t is varied so that the poles of f_t move to the right, the number $f_t(0)$ diminishes; it follows that we can pass continuously to that member of the family for which $f_t(0) = 0$ without departing from the class $P[0, z_l]$. We let $t = 0$ correspond to the rational function so determined; $f_0(z)$ is of degree at most N and belongs to $P[0, z_l]$ and satisfies $f_0(0) = 0$. Since $f_0(z)$ is non-negative in $[0, z_l]$ we have only to show that it has no poles to the right of z_l to make sure that it belongs to P'. For this purpose we pass to the ratonal function $g(\zeta) = \zeta f_0(1/\zeta)$ which is also of degree at most N and which coincides with f^* on the points of Z^*. It follows that $g(\zeta)$ is a member of the family $f_t^*$and therefore that the residue of $g(\zeta)$ at any pole to the left of $z_1^* = 1/z_l$ is negative. If, now, $f_0(\zeta)$ had a pole to the right of z_l, $g(\zeta)$ would have one in the interval $0 < z < z_1^*$ and the residue there would be negative. However, if we make the explicit computation we will have

$$f_0(\zeta) = h(\zeta) + \frac{m}{\lambda - \zeta} \text{ where } m > 0 \text{ and } \lambda < z_l \text{ with } h(\zeta)$$

regular near λ, and

$$g(\zeta) = \zeta f(1/\zeta) + \frac{\zeta m}{\lambda - 1/\zeta}$$

which has a positive residue at the pole $1/\lambda$. Thus $f_0(\zeta)$ had no pole to the right of z_l and was therefore in P'.

Finally, if the functions f and f^* are not both interior points of their respective cones, we have only to pass to $f + \varepsilon\sqrt{z}$ which corresponds to $f^* + \varepsilon\sqrt{z}$ for small positive ε. The perturbed functions are interior points of those cones and also satisfy III and IV. From the fact that $P'(Z)$ is closed we infer that f is in $P'(Z)$, completing the proof of Theorem III.

We do not give the easy proof that an element f in $P'(Z)$ may be extended to the origin by $f(0) = 0$ if f is an interior point of $P(Z)$.

5. **The theorems of Pick and Carathéodory.** The problem considered in Section 2 can equally well be studied under the hypothesis that the finite set Z is a subset of the open upper half-plane; we would then seek conditions for a function in $C(Z)$ to belong to the cone $P(Z)$, the restrictions to Z of functions in P. The solution has been given by Pick. [5]

THEOREM. *A function $f(z)$ in $C(Z)$ which is not a real constant is the restriction to Z of a function $\phi(\zeta)$ in P if and only if the imaginary part of $f(z)$ is positive and the matrix of order l*

$$P_{ij} = \frac{f(z_i) - \overline{f(z_j)}}{z_i - \bar{z}_j}$$

is a positive matrix. This matrix has the eigenvalue 0 with multiplicity $k > 0$ if and only if $\phi(\zeta)$ is a rational function of degree $l - k$ and in this case $\phi(\zeta)$ is determined uniquely by the data.

We do not give a proof of this theorem which can be established by the same arguments which we have used to prove Theorems I and II, the proof, however, is substantially easier since in the present case the cone $P(Z)$ is closed and we may also always argue with positive matrices rather than with determinants.

A completely analogous theorem is due to Carathéodory who considered the convex cone of functions $u(z)$ harmonic and positive in the unit circle; such functions admit a Fourier expansion

$$u(r\, e^{i\theta}) = \sum c_k r^{|k|} e^{ik\theta}$$

the summation being taken over all integers. The following theorem is due to Carathéodory.

THEOREM. *A system of numbers $\{c_i\}$ $-N \leqq i \leqq N$ form the Fourier coefficients of order $\leqq N$ of a positive harmonic function $u(z)$ if and only if the matrix of order $N + 1$ defined by*

$$C_{ij} = c_{i-j}$$

is a positive matrix; 0 is an eigenvalue of C_{ij} with multiplicity $k > 0$ if and only if $u(z)$ is the real part of a rational function of degree $N + 1 - k$, and in this case $u(z)$ is uniquely determined by the coefficients.

Since the class of positive harmonic functions in the circle is in a one-to-one correspondence with the positive harmonic functions in the half-plane and since the latter are the imaginary parts of the functions in P, the similarity between the foregoing theorems is to be expected. It is important to note a geometric fact: in both cases one studies the projections on a linear space of dimension l of a certain cone in an infinite dimensional space, and the projections are cones which inherit a certain property of the original cone, viz. the set of (suitably normalized) extreme points forms a skew curve. The cones thus have an extraordinary multiplicity of faces of lower dimension. This geometric situation is more conveniently studied if a suitable normalization condition reduces the study to one of a convex body; we then obtain a convex polytope and the study reduces to the study of "neighborliness" introduced in recent years. [3]

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Michigan State University,
East Lansing, Michigan