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General Distribution of Hadron Transverse-Momentum Gaussian Cross-Sections That Are Asymptotic Exponentials.

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Summary. — Reviews of elastic and inelastic hadron scattering data are discussed; they indicate $(\mathrm{d}\sigma/\mathrm{d}t)$ is universally in the form of a superposition of Gaussians (in a transverse-momentum variable x) that is an asymptotic exponential. After discussing two previous models, we use the method of asymptotic expansions to show that a mass m, superposition $(\mathrm{d}\sigma/\mathrm{d}t)_m = F(x)\exp[-f(m)x^2]$ with a distribution function $G(m)\cdot\exp[-g(m)]$ will yield an asymptotic exponential in x if $g(m) = K^2/f(m)$. Experimental and theoretical implications are discussed.

1. – Introduction.

Attempts to phenomenologically parametrize hadron scattering data have yielded a number of striking observations, some of them seemingly contradictory.

In *inelastic* scattering, for example, it has been found that many individual two-body-to-two-body reactions have cross-sections that behave like [1]

$$\frac{\pi}{pp'}\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\mathrm{d}\sigma}{\mathrm{d}t} \propto \exp[\tau A]\,, \tag{1}$$

where $-t$ is the momentum transfer $|(p_\mu - p'_\mu)|$ squared

$$\begin{cases} -t = 2pp'(1-\cos\theta) + 2EE' - m^2 - m'^2 - 2pp' \equiv -t_E - t_{\min} \\ -t_{\text{elastic}} = 2p^2(1-\cos\theta), \end{cases} \tag{2}$$

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[1] AACHEN-BERLIN-CERN COLLABORATION: *Phys. Lett.*, **19**, 608 (1965).

and $-\tau$ is the transverse momentum ($p_\perp$) squared

$$-\tau \equiv p_\perp^2 = p'^2 \sin^2\theta \,, \tag{3}$$

$$\lim_{\theta\to 0}(\tau) = \lim_{\theta\to 0}(t_E)\,. \tag{4}$$

A depends on the particular process involved, but is roughly 10 $(\mathrm{GeV}/c)^{-2}$. However, when the distribution of *long-lived secondaries* is considered, called by COCCONI [2] the « global distribution », it is found that the cross-section varies as [3]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto \exp[-p_\perp/b]\,, \tag{5}$$

i.e., an exponential in $p_\perp$, not a Gaussian. In (5), b is about $(150 \div 200)$ MeV, depending on which mass secondary is considered.

A similar situation has been observed in the case of *elastic* p-p scattering (and indeed for all small-angle hadron scattering). KRISCH [4] has obtained an excellent fit to the p-p elastic data by a sum of Gaussians:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t}\left[\frac{\mathrm{mb}}{(\mathrm{GeV}/c)^2}\right] = \sum_{i=1}^{3} A_i \exp[-a_i\beta^2 p_\perp^2] = \tag{6}$$

$$= 90 \exp[-10.0\beta^2 p_\perp^2] + 0.74 \exp[-3.45\beta^2 p_\perp^2] + 0.0029 \exp[-1.45\beta^2 p_\perp^2]\,. \tag{7}$$

β is the centre-of-mass velocity of the protons. For different regions of $p_\perp$, different Gaussians dominate, with decreasing amplitudes (A_i) for the Gaussians corresponding to smaller inverse widths (a_i). (One should note that this is roughly the case when individual inelastic processes are compared. There, smaller inverse widths correspond to higher-mass secondaries.) Here, in the *elastic* case, the cross-section again goes over to an exponential in $p_\perp$ for high transverse momentum. The form is Orear's law [5]

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\left[\frac{\mathrm{mb}}{\mathrm{sr}}\right] = 34 \exp[-p_\perp/\bar{p}]\,, \qquad \bar{p} \simeq 160\ \mathrm{MeV}/c\,. \tag{8}$$

The above observations raise many questions. Is there a relationship between the widths for the various inelastic and elastic Gaussians? Are there

(2) G. COCCONI: *Nuovo Cimento*, **57** A, 837 (1968).
(3) G. COCCONI, J. KOESTER and D. H. PERKINS: UCRL-10022 (1961).
(4) A. D. KRISCH: *Phys. Rev. Lett.*, **19**, 1149 (1967).
(5) J. OREAR: *Phys. Rev. Lett.*, **12**, 112 (1964); *Phys. Lett.*, **13**, 190 (1964).

more than three Gaussians for the elastic p-p process? Also, since total cross-sections do not appear to fall off very rapidly, Gaussians would asymptotically violate the Cerulus-Martin bound [6] imposed by analyticity, whereas the exponential form would not. Thus, are we not forced to believe that the Gaussians must asymptotically go over to an exponential; and if we are, by what mechanism do they do it?

Recently, COCCONI [2] and the group of FLEMING, GIOVANNINI, and PREDAZZI [7] (FGP) proposed models which yield an asymptotic exponential cross-section by a superposition of Gaussians. They did this independently and from very different motivations. In Sects. **2** and **3**, we review these models, point out that they are equivalent, and introduce the technique of asymptotic expansions to the problem.

We then proceed to demonstrate (Sect. **4**) that the above models are special cases of a general class of superpositions of Gaussians, which yield an asymptotic exponential. In particular, we show that if individual Gaussians (as a function of a mass or energy variable m) are of the form $F(m)\exp[-f(m)x^2]$, where x is either $p_\perp$ or $\beta p_\perp$, then a distribution function for the Gaussians of the form $G(m)\exp[-g(m)]$ will asymptotically yield an exponential in x if

$$f(m) = K^2/g(m)\,, \tag{9}$$

where K is a constant. (Whether x is actually $\sqrt{-t}$, $p_\perp$, or $\beta p_\perp$, does not affect our results, since asymptotically $\beta \to 1$.)

Thus, the problem of understanding the theory of the increase of Gaussian widths with mass (the form of $f(m)$) is intimately related to finding the distribution function of Gaussians, which will yield an asymptotic exponential in transverse momentum. We close with a few comments on physical models.

2. – Cocconi model.

In seeking a model that yields an asymptotic exponential cross-section, COCCONI [2] observed three points: *a*) the physical results mentioned in Sect. **1**; *b*) recent K^-p and π^+p inelastic scattering data [8] show that Gaussian cross-sections have decreasing A (larger widths) as the secondary mass becomes

(6) F. CERULUS and A. MARTIN: *Phys. Lett.*, **8**, 80 (1964).

(7) H. FLEMING, A. GIOVANNINI and E. PREDAZZI: *Nuovo Cimento*, **56** A, 1131 (1968).

(8) AACHEN-BERLIN-CERN COLLABORATION and AACHEN-BERLIN-CERN-LONDON (I.C.)-VIENNA COLLABORATION: *Phys. Lett.*, **27** B, 336 (1968).

higher; *c*) analysis of scattering data in the Glauber formalism [9] using quark models [10,11] can be interpreted [2] as yielding Gaussian cross-sections with wider widths for multiple quark scattering (*i.e.* higher-mass secondaries).

With this physical motivation, COCCONI proposed that the inelastic cross-section for producing a secondary of mass m is [12]

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right)_m \propto \frac{1}{m}\exp\left[-A\frac{m_0}{m}p_\perp^2\right]. \tag{10}$$

m_0 is the mass of the simplest quark-scattering structure, and the factor m^{-1} is for normalization. The « global » cross-section is then

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\right) \propto \int\limits_{m_{\min}}^{m_{\max}} \mathrm{d}m\,\frac{P(m)}{m}\exp\left[-A\frac{m_0}{m}p_\perp^2\right], \tag{11}$$

where $m_{\max}$ depends on the energy of the system and $m_{\min}$ is approximately the incident-particle mass. $P(m)$ is the probability of producing a secondary of mass m.

Using physical intuition and the mass-spectrum analysis of ref. [8], COCCONI took

$$P(m\to 0) = P(m\to\infty) \simeq 0\,, \qquad P_{\max}(m) \simeq P(m_0)\,. \tag{12}$$

He therefore proposed that $P(m)$ can be described by [12]

$$P(m) \propto m\exp\left[-m/m_0\right]. \tag{13}$$

By then approximating $m_{\min}=0$ and $m_{\max}=\infty$, he obtained ($p_\perp \equiv x$)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \propto \int\limits_0^\infty \mathrm{d}m\exp\left[-\frac{m}{m_0}-\frac{m_0A}{m}x^2\right] = \tag{14a}$$

$$= (4Am_0^2x^2)^{\frac{1}{2}}K_1\left(\sqrt{4Ax^2}\right) = \tag{14b}$$

$$\underset{x\to\infty}{=} m_0\exp\left[-x\sqrt{4A}\right] = \tag{14c}$$

$$= m_0\exp\left[-x/b\right]. \tag{14d}$$

(9) R. J. GLAUBER: in *Boulder Lectures in Theoretical Physics*, edited by W. E. BRITTIN *et al.*, vol. **1** (New York, 1959); in *High Energy Physics and Nuclear Structure*, edited by G. ALEXANDER (Amsterdam, 1967), p. 311.

(10) T. T. CHOU and C. N. YANG: *Phys. Rev.*, **170**, 1591 (1968).

(11) N. T. DEAN: *Nucl. Phys.*, B **7**, 311 (1968).

(12) COCCONI considered $\mathrm{d}\sigma/\mathrm{d}\Omega$. However, if one wants to discuss $\mathrm{d}\sigma/\mathrm{d}t$, the factor pp' could be incorporated in the normalization or in the probability function $P(m)$. In any event, as we will see in Sect. **3**, such factors would not affect the exponential behaviour of the result.

In (14), $K_1(y)$ is the modified Bessel function ([13]) which goes to $y^{-1}e^{-y}$ as $y \to \infty$. The value of b in (14d) is

$$b = (4A)^{-\frac{1}{2}} = 158 \text{ MeV} , \tag{15}$$

in good agreement with experiment.

For the case of elastic scattering, COCCONI reasoned that, if we let n be the order of scattering in the quark model, then higher-order scattering contributions would yield Gaussian widths with a distribution function of n identical to that for inelastic scattering as a function of m. Thus, the above results would carry over in the same way, with m becoming n. Letting $n_0 = 1$, the value of b in (15) is then in excellent agreement with Orear's law ([5]).

3. – FGP model and asymptotic expansions.

In distinction to that of COCCONI, the phenomenological model of FGP ([7]) is an *ad hoc* four-parameter infinite sum of Gaussians:

$$Y = \frac{(\mathrm{d}\sigma/\mathrm{d}t)}{\Sigma} = \sum_{n=0}^{\infty} \frac{a^{-n}}{(cn+1)^{\nu}} \exp\left[-\frac{Ax^2}{cn+1}\right], \tag{16}$$

where Σ is a normalization constant, and a, A, c, and ν are parameters. FGP took

$$c = 2 , \qquad a = 5 , \qquad \nu = \tfrac{7}{2} , \qquad A = 10 \text{ (GeV}/c)^{-2} , \tag{17}$$

to have the first three terms agree with Krisch's three Gaussians ([4]) and also to have a rough fit to all elastic data.

FGP calculated the asymptotic form of (16) for the restricted case of $c = 2$, $\nu = \frac{7}{2}$. Even so, it suits our purpose to evaluate the general asymptotic form of (16).

To do this, we realize that as $x \to \infty$, the sum in (16) becomes the integral

$$Y = \int_0^{\infty} \frac{\mathrm{d}n}{(cn+1)^{\nu}} \exp\left[-n \ln a - \frac{Ax^2}{cn+1}\right]. \tag{18}$$

Already, by comparing (14a) and (18), the similarity between the Cocconi and FGP models can be seen. It is due to a^{-n} really being an exponential in

([13]) I. S. GRADSTEYN and I. M. RYZHIK: *Table of Integrals, Series and Products* (New York, 1965).

n. However, we shall not proceed in the same way as COCCONI. We will evaluate (18) by using the method of asymptotic expansions [14], since this is the tool we will need to evaluate our general model.

First, we let $z = cn + 1$, so that

$$Y = \frac{a^{1/c}}{c}\int_1^\infty \frac{\mathrm{d}z}{z^\nu} \exp\left[\left(-\frac{z\ln a}{c} + \frac{Ax^2}{z}\right)\right]. \tag{19}$$

This is of the form

$$X = \int_\varepsilon^\infty \mathrm{d}z\, h(z) \exp[-j(z)]\,, \tag{20}$$

where $h(z)$ and $j(z)$ are polynomially bounded, and $j(z)$ has a positive maximum $z = z_0 \gg 0$, *i.e.*, $j'(z_0) = 0$. If we expand z about z_0, we can say

$$X = \int_\varepsilon^\infty \mathrm{d}z\, h(z) \exp\left[-\left[j(z_0) + O + \frac{(z-z_0)^2}{2} j''(z_0) + \dots\right]\right] \simeq \tag{21}$$

$$\simeq h(z_0)\exp[-j(z_0)] \int_\varepsilon^\infty \mathrm{d}z \exp\left[-\frac{(z-z_0)^2}{2} j''(z_0)\right].$$

The last step is valid, because the exponential dominates the polynomial. For $z_0 \gg \varepsilon$, we can approximate the lower limit by $-\infty$, giving a perfect Gaussian integral with the value $[2\pi/j''(z_0)]^{\frac{1}{2}}$. Thus,

$$X = \left(\frac{2\pi}{j''(z_0)}\right)^{\frac{1}{2}} h(z_0) \exp[-j(z_0)]\,. \tag{22}$$

For the FGP case of (19)

$$z_0 = \left(\frac{cAx^2}{\ln a}\right)^{\frac{1}{2}} \tag{23}$$

and hence

$$Y = a^{1/c}\sqrt{\frac{\pi}{c\ln a}}\left(\frac{\ln a}{cAx^2}\right)^{\nu/2-\frac{1}{4}} \exp\left[-2x\sqrt{A\ln a/c}\right]. \tag{24}$$

In agreement with our asymptotic assumptions, the slope of the exponential does not depend on ν. The decay constants of the FGP and Cocconi models

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[14] A. ERDELYI: *Asymptotic Expansions* (New York, 1956).

differ by the factor

$$(c/\ln a)^{\frac{1}{2}} = \sqrt{2/\ln 5} = 1.1 \,. \tag{25}$$

4. – General model.

For a more general superposition model of Gaussians, we assume mass cross-section and normalized distribution functions of the form

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)_m = F(m)\exp\left[-f(m)x^2\right], \tag{26}$$

$$P(m) = G(m)\exp\left[-g(m)\right]. \tag{27}$$

The only restrictions we place on $F(m)$, $f(m)$, $G(m)$, and $g(m)$ are that they be polynomially bounded functions of m and do not vary rapidly. Then we can use the asymptotic expansion method, as in Sect. **3**, to obtain

$$\lim_{x\to\infty}\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) = \int\limits_{m_{\min}}^{m_{\max}} \mathrm{d}m\, P(m)\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)_m \simeq \\ \simeq G(M)F(M)\left[\frac{2\pi}{g''(M)+x^2 f''(M)}\right]^{\frac{1}{2}} \exp\left[-g(M)-f(M)x^2\right]. \tag{28}$$

M is defined by [15]

$$g'(M)+f'(M)x^2 = 0 \,. \tag{29}$$

The function $g(m)$ that satisfies (29) and allows (28) to be an exponential in x is

$$g(m) = K^2/f(m)\,, \tag{30}$$

where K is a constant. To prove this, we first note that (29) implies

$$g''(m) = \frac{-K^2}{f^2(m)}\left[f''(m)-\frac{2[f'(m)]^2}{f(m)}\right]. \tag{31}$$

Then, putting (30) into (29) gives

$$f(M) = K/x \,. \tag{32}$$

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[15] In the general case, there will be more than one solution to (29). Then there will be a sum of terms in (28), one for each solution of (29). However, usually one term will dominate the others. See ref. [14] for a more detailed discussion.

Using this in (31) and (28) gives our result

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right) = G(M)F(M)\left(\frac{\pi K}{x^3 f'(M)}\right)^{\frac{1}{2}} \exp\left[-2Kx\right]. \tag{33}$$

The Cocconi and FGP models are both special cases of (30) with $g(m) \propto m$. In fact, it is an enlightening exercise to prove explicitly that, if $g(m) \propto m^r$ and $f(m) \propto m^s$, the dominant exponential in (28) will be proportional to x only if $r = -s$.

In the asymptotic region, the result (30) could be considered a transformation from the Cocconi-FGP case to a new, more complicated variable. Thus, with hindsight, it might have been expected.

However, the value of the result (30) lies in making clear the general relationship that must hold between the mass cross-section half-widths and the mass probability distribution function.

5. – Discussion.

If the experimental data continue to show Gaussian cross-sections asymptotically going over to exponentials, then our general result (30) must hold, no matter what the exact mass and x variables are. This result would then in principle allow information to be inferred on the form of $g(m)$ if one had experimental information on $f(m)$ and vice versa. Furthermore, it clearly would place a restriction that a dynamical theory would have to meet.

Most previous models, although often quite successful, have been of a statistical nature. The new discussions in terms of quarks, although still quite phenomenological, are opening possibilities for a more fundamental approach to the problem. (Indeed, as mentioned, the quark model results were one of Cocconi's motivations.) In addition, there has been much investigation in terms of diffraction scattering and Reggeized particle exchange. The recent survey by VAN HOVE [16] serves as a good introduction to this literature.

Future work on the Gaussian-to-exponential cross-section problem will hopefully be quite fruitful. The growing volume of experimental data is continually showing that this question is of fundamental importance to all of high-energy hadron scattering. The insight that may be gained here could be an important tool in obtaining a truly dynamical theory.

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Thanks are due K. HANSEN for helpful suggestions in preparing the manuscript.

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[16] L. VAN HOVE: CERN report Ref. TH. 917, lecture delivered at the *International Symposium on Contemporary Physics, Trieste, June,* 1968.

RIASSUNTO (*)

Si discutono le analisi dei dati dello scattering elastico ed inelastico degli adroni; essi indicano che $d\sigma/dt$ è sempre nella forma di una sovrapposizione di gaussiane (nella variabile x della quantità di moto trasversale), e cioè nella forma di un esponenziale asintotico. Dopo aver discusso due modelli precedenti, si usa il metodo dello sviluppo asintotico per dimostrare che una sovrapposizione della massa (m) $(d\sigma/dt)_m = F(x)\exp[-f(m)x^2]$ con una funzione di distribuzione $G(m)\exp[-g(m)]$ fornisce un esponenziale asintotico in x se $g(m) = K^2/f(m)$. Si discutono le implicazioni sperimentali e teoriche.

(*) *Traduzione a cura della Redazione.*

Общее распределение гауссовских дифференциальных сечений по поперечному импульсу адронов, имеющих экспоненциальную асимптотику.

Резюме (*). — Проводится обзор данных по упругому и неупругому рассеянию адронов; данные указывают, что $d\sigma/dt$ в наиболее общем виде представляется как суперпозиция гауссовских членов (по величине поперечного импульса x), асимптотически представляющая экспоненту. После обсуждения двух предыдущих моделей, используется метод асимптотических разложений, чтобы показать, что суперпозиция масс (m) $(d\sigma/dt)_m = F(x)\exp[-f(m)x^2]$ с функцией распределения $G(m)\cdot\exp[-g(m)]$ дает экспоненциальную асимптотику по x, если $g(m) = K^2/f(m)$. Обсуждаются экспериментальные и теоретические следствия.

(*) *Переведено редакцией.*