ON STEINER SYSTEMS*

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ABSTRACT

Steiner's "combinatorial problems" have so far been solved only for $k=3$ [5, 3] and for $k=4$ [1, 2]. In this paper a complete solution of the problem is given for "closed" Steiner systems, i.e. systems having $n=2^{k-1}-1$ elements. Use is made of methods developed by Zaremba [7] for abelian groups.

1. **Introduction.** Let E be a given set of n elements, and let $R_t(t = 3, 4, \cdots, k)$ be a system of nonempty subsets of E having t elements each, i.e. the elements of R_t are certain t-tuples of E. A t-tuple $(t = 4, 5, \cdots, k)$ will be called free in respect of $R_3, R_4, \cdots, R_{t-1}$ (briefly: *free*) if it does not contain as a subset any j-tuple of R_j, $(j = 3, 4, \cdots, t-1)$. Pairs and triples of elements of E are considered to be free.

DEFINITION 1. A system $S_k = \bigcup_{j=3}^{k} R_j$ will be called a Steiner k-system for E if the following conditions hold:

(A) every element of S_k is free;

(B) every free t-tuple of E $(t = 2, 3, \cdots, k-1)$ which is not an element of R_t is contained as a subset in exactly one element of R_{t+1}.

As early as 1852, Steiner [6] formulated the following problem: given an integer k, $(k = 3, 4, \cdots)$, what conditions should be imposed on n in order to ensure the existence of a k-system S_k?

It may be easily verified (see e.g. [4]) that a necessary condition for the existence of a Steiner k-system S_k is

$$(1) \qquad n \equiv 1 \pmod 2 \text{ and } (t!)^{-1} \prod_{i=0}^{t-2} (n + 1 - 2^i) = \text{integer} \qquad t = 3, 4, \cdots, k.$$

So far, condition (1) has been proved to be also sufficient for $k = 3$ by Reiss [5] and Moore [3] and for $k = 4$ by Hanani [1, 2]. For $k > 4$ the problem is still open.

Received February 26, 1964.

* This research was supported by the United States Air Force under Grant No. AF-EOAR-6360 and monitored by the European Office of Aerospace Research.

2. Closed Steiner systems.

DEFINITION 2. A Steiner k-system S_k for E is closed if every free k-tuple of E is an element of R_k.

The number of free k-tuples which are not elements of S_k is

$$(k!)^{-1} \prod_{i=0}^{k-1} (n+1-2^i). \tag{2}$$

In a closed Steiner k-system, this number is zero and therefore necessarily

$$n = 2^{k-1} - 1. \tag{3}$$

We shall prove that condition (3) is also sufficient, namely that

THEOREM 1. *If E is a set of $n = 2^{k-1} - 1$ elements, then there exists a closed Steiner k-system S_k for E.*

Proof. Let there be given an abelian additive group G, having a base $B = \{b_{(i)}\}_{i=1}^{n}$ consisting of n elements, each of order 2. Denote the elements of E by $e_i (i = 1, 2, \cdots, n)$. Consider the 1–1 correspondence $e_i \leftrightarrow b_{(i)}$. It generates the usual 1–1 correspondence between the subsets of E and the elements of G in which the empty subset of E corresponds to the zero element of G and the t-tuple $\{e_{i_1}, e_{i_2} \cdots e_{i_t}\}$ to the element $\sum_{j=1}^{t} b_{(i_j)}$ of G. Let G_t denote the subset of G composed of the elements g_t corresponding to the t-tuples in E. We shall denote $g_t \supset g_s$ if the inclusion holds for the corresponding subsets of E; we shall also denote g_t as a free element if the corresponding t-tuple is free.

Zaremba [7] proved that there exists a subgroup H of G such that every element $g \in G$ is representable uniquely as a sum of an element $h \in H$ and a base-element $b \in B$ or zero

$$g = h + b \text{ (or } 0). \tag{4}$$

Let $H_t = H \cap G_t$ and let h_t denote a general element of H_t. From (4) follows:

$$H_1 = H_2 = \phi,\ h_{t-2} + b' + b'' \neq h_t \neq h_{t-1} + b,\ (b, b', b'' \in B). \tag{5}$$

Further, we obtain the unique representation:

$$\begin{aligned} &g_1 = b,\ g_2 = h_3 + b,\ g_3 = h_3, \text{ or } h_4 + b, \\ &g_{t-1} = h_{t-2} + b, \text{ or } h_{t-1}, \text{ or } h_t + b, \qquad (b \in B,\ t \geqq 4). \end{aligned} \tag{6}$$

Let us now construct a family of subsets $P_t \subset H_t (t = 3, 4, \cdots, k)$ of G, such that the corresponding systems R_t will form a closed Steiner k-system for E. Put $P_3 = H_3$ and $P_4 = H_4$. By (6), the systems S_3 correponding to P_3 is a Steiner 3-system and S_4, corresponding to $P_3 \cup P_4$, is a Steiner 4-system.

Suppose that S_t corresponding to $\bigcup_{i=3}^{t} P_i$, $(t < k)$ forms a Steiner t-system. We shall construct a set $P_{t+1} \subset H_{t+1}$ such that S_{t+1}, corresponding to $\bigcup_{i=3}^{t+1} P_i$ will form a Steiner $(t+1)$-system.

Denote by p_i a general element of P_i and by q_i a general element of $H_i - P_i$ (if such elements exist). By (2), there exist free elements $g_t \notin P_t$. Let g_t^* be such an element. We prove

$$g_t^* \neq h_t.$$

By definition, $g_t^* \neq p_t$. Suppose $g_t^* = q_t$; there exists $g_{t-1}^* \in G_{t-1}$ satisfying $g_{t-1}^* = g_t^* + b = q_t + b$ with $b \subset g_t^*$. But since $g_{t-1}^* \subset g_t$, g_{t-1}^* is free and according to (B) $g_{t-1}^* = p_t + b'$ in contradiction to the uniqueness of (6). Similarly, we prove

$$g_t^* \neq h_{t-1} + b. \tag{8}$$

g_t^* is free, hence $g_t^* \neq p_{t-1} + b$. Suppose $g_t^* = q_{t-1} + b$. There exists $g_{t-2} \in G_{t-2}$ such that $g_{t-2}^* = g_t^* + b + b' = q_{t-1} + b'$. On the other hand g_{t-2}^* is free and by (B), $g_{t-2}^* = p_{t-1} + b''$, which contradicts the uniqueness of (6).

From (6), (7) and (8) there follows

$$g_t^* = h_{t+1} + b. \tag{9}$$

Denote by P_{t+1} the set of elements h_{t+1} obtainable by (9) from all free elements $g_t \notin P_t$. Then the system S_{t+1} corresponding to $\bigcup_{i=3}^{t+1} P_i$ has the property (B).

In order to show that the system S_{t+1} has property (A) as well, suppose to the contrary that some p_{t+1} satisfies

$$p_{t+1} = p_{t+1-j}^* + g_j^* \tag{10}$$

for some j. From (5) follows $3 \leqq j \leqq t-2$ and by (9) we have

$$g_t^* = p_{t+1} + b = p_{t+1-j}^* + g_j^* + b^* \tag{11}$$

with $p_{t+1-x}^* \in P_{t+1-j}$, $g_j^* \in G_j$ and $b^* \in B$. Clearly $b^* \subset p_{t+1}$ and therefore either

$$p_{t+1-j}^* + b^* = g_{t-j} \tag{12}$$

or

$$g_j^* + b^* = g_{j-1}. \tag{13}$$

Both possibilities lead to a contradiction. Indeed (12) and (11) imply $g_j^* \subset g_t^*$, consequently g_j^* is free and $g_j^* \notin P_x$ and therefore $g_j^* = p_{j+1} + b'$. On the other hand, we know that H is a group and therefore by (10) $g_j^* \in H_j$, which contradicts (5). Further, (13) and (11) imply $g_t^* = p_{t+1-j}^* + g_{j-1}$, contradicting the assumption that g_t^* is free.

This proves the induction from t to $t+1$ for $t < k$. For $t = k$, the system S_k is clearly a closed Steiner k-system.

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