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A GENERALIZATION OF QUILLEN'S LEMMA AND ITS APPLICATION TO THE WEYL ALGEBRAS[†]

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ABSTRACT
Quillen's lemma [17] is generalized to modules of arbitrary Krull dimension. This leads to some generalizations of the results of [5] and [12] for the Weyl algebras of index >1.

1. The generalized Quillen lemma

1.1 Let $\mathbf{k}$ be a commutative field, A a $\mathbf{k}$-algebra and $\operatorname{Dim}_{\mathbf{k}} A$ (or simply, $\operatorname{Dim} A$) the Gelfand–Kirillov dimension of A over $\mathbf{k}$ [3, §1.2]. Define the commutative dimension $\operatorname{Cdim} A$ of A over $\mathbf{k}$ through

$$\operatorname{Cdim} A = \sup\{\operatorname{Dim} B : B \subseteq A \text{ commutative subalgebra}\}.$$

By [3, §1.7], we can assume B finitely generated and so $\operatorname{Cdim} A \in \mathbb{N} \cup \infty$. Suppose $\operatorname{Cdim} A = m < \infty$. Then there exists a finitely generated commutative subalgebra B of A with $\operatorname{Dim} B = m$. Write $B = S(V)/I$ and $\sqrt{I} = \cap\{I_i : i = 1, 2, \cdots, n;\ I_i \text{ prime}\}$. By say [3, §3.1 e)], $m = \operatorname{Dim} S(V)/\sqrt{I} = \max\{\operatorname{Dim} S(V)/I_i\}$. Choose j such that $\operatorname{Dim} S(V)/I_j = m$. Since I_j is prime, there exist $b_1, b_2, \cdots b_m \in S(V)$ such that $\mathbf{k}[b_1, b_2, \cdots, b_m] \cap I_j = 0$. Yet $I \subset I_j$, so $\operatorname{Cdim} A$ is just the largest non-negative integer m such that A admits a polynomial subalgebra on m variables.

1.2. Let U be a $\mathbf{k}$-algebra with filtration $U^0 \subset U^1 \subset \cdots$ satisfying $U = \cup\{U^i : i = 0, 1, 2, \cdots\}$. Assume that the associated graded algebra $\operatorname{gr}(U)$ is commutative and finitely generated. Let Kdim denote the Krull dimension of a module [18].

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PROPOSITION. *Let M be a finitely generated U module. Then* $\mathrm{Cdim}\,\mathrm{Hom}_U(M,M) \leqq \mathrm{Kdim}\, M$.

Let $A \subseteq \mathrm{Hom}_U(M,M)$ be a polynomial algebra. Choose a system $m_1, m_2, \cdots, m_s$ of generators for M over U and set

$$M^i = \sum_{j=1}^{s} U^i A m_j.$$

Then M is a filtered module for the filtered ring

$$U \otimes_{\mathbf{k}} A = \bigcup_{i=0}^{\infty} (U^i \otimes_{\mathbf{k}} A)$$

and the associated graded module $\mathrm{gr}(M)$ is finitely generated on $\mathrm{gr}(U \otimes_{\mathbf{k}} A) = \mathrm{gr}(U) \otimes_{\mathbf{k}} A$. Hence by generic flatness (see for example [6, §2.6.3]) there exists $f \in A$ such that

$$\bigotimes_{i=0}^{\infty} (M^i / M^{i-1})_f$$

is a free A_f module. This implies that for each i, the exact sequence $0 \to (M^{i-1})_f \to (M^i)_f \to (M^i)_f/(M^{i-1})_f \to 0$ splits and so there exists for each i an isomorphism of $(M^i)_f$ onto $(M^{i-1})_f \oplus (M^i)_f/(M^{i-1})_f$ which is the identity on $(M^{i-1})_f$. Thus for all $j \in \mathbb{N}^+$,

$$(M^j)_f = \bigoplus_{i=1}^{j} (M^i)_f/(M^{i-1})_f, \quad (A_f \text{ module isomorphism}),$$

and so M_f is isomorphic to $\mathrm{gr}(M)_f$ as an A_f module. Since A is integral, $f^n \neq 0$ for all n and so $M_f \neq 0$. Let $\varphi: m \to m \otimes 1$ be the canonical embedding of M in $M \otimes_A A_f = M_f$. Then $E = A_f \varphi(\varphi^{-1}(E))$, for each A_f submodule E of M_f. Let $I_1 \subsetneqq I_2$ be distinct ideals of A_f. Since M_f is a free A_f module, $I_1 M_f$ is a proper UA_f submodule of $I_2 M_f$ and so $\varphi^{-1}(I_1 M_f) \subsetneqq \varphi^{-1}(I_2 M_f)$. Hence $\mathrm{Kdim}\, M \geqq \mathrm{Kdim}\, A_f = \mathrm{Dim}\, A_f = \mathrm{Dim}\, A$. Combined with 1.1, this proves the proposition.

REMARKS. This result represents joint work with R. Rentschler and I should like to thank him for his contribution. The special case when M has finite length (i.e. $\mathrm{Kdim}\, M = 0$) is a well-known corollary of Quillen's original result: but there seems to be no way of using the zero Krull dimension result to prove those of higher Krull dimension. Again Quillen's lemma has an elementary proof if $\mathrm{card}\, \mathbf{k} > \dim_{\mathbf{k}} M$ and in which case the above technical constraints concerning the filtration of U may be dropped; yet this does not seem to apply to the higher Krull dimension cases.

2. Primary applications to the Weyl algebras

From now on we assume $\operatorname{char} \mathbf{k} = 0$. The applications given in this section essentially generalize arguments of J. Stein and myself for A_1 [12, Note added in proof]. Entirely new applications are given in Section 3.

2.1. Given $n \in \mathbb{N}$, let A_n denote the Weyl algebra of index n over $\mathbf{k}$ [5]. Recall that A_n has generators q_i, $\partial/\partial q_i$: $i = 1, 2, \cdots, n$, and set $p_i = -\partial/\partial q_i$. For any subalgebra $A \subset A_n$, we have by [8, theor. 1.1] that $\operatorname{Cdim} A \leqq n$, and in particular that $\operatorname{Cdim} A_n = n$.

2.2. Let $A_n^0 \subset A_n^1 \subset \cdots$ denote the standard (canonical) filtration of A_n. Given M a finitely generated A_n module (with generating subspace V) set

$$d(M) = \overline{\lim_{k\to\infty}} \frac{\log \dim (A_n^k V)}{\log k}.$$

Then $d(M)$ is a non-negative integer $\leqq 2n$. Furthermore, after Bernstein [1],

$$\operatorname{Kdim} M \leqq d(M) - n. \tag{2.1}$$

In particular if I is a non-zero left ideal of A_n, then

$$\operatorname{Kdim} (A_n/I) \leqq n - 1. \tag{2.2}$$

2.3. Let $X \subset A_n$ be a subspace which generates the symmetric algebra $S(X)$ over X in A_n. Set $m = \dim X = \operatorname{Dim} S(X)$, then by 2.1, $m \leqq n$. Let $C(X)$ denote the commutant of $S(X)$ in A_n. Let $F(X)$ (resp. $N(X)$, $D(X)$) denote the largest subalgebra of A_n on which each $\operatorname{ad} x$: $x \in X$ is locally finite (resp. nilpotent, semisimple). Clearly $C(X) = N(X) \cap D(X)$. With respect to a basis $\{x_i\}$ for X, set $\Lambda(X) = \{(\lambda_1, \lambda_2, \cdots, \lambda_m)$: $\lambda_i \in \mathbf{k}$: $(\operatorname{ad} x_i)y = \lambda_i y$, $0 \neq y \in A_n\}$. Clearly $\Lambda(X)$ is an additive sub-semigroup of $\mathbf{k}^m$. Given $\lambda = (\lambda_1, \lambda_2, \cdots \lambda_m) \in \Lambda(X)$, set $D_\lambda(X) = \{y \in A_n : (\operatorname{ad} x_i)y = \lambda_i y\}$. Then $D_\lambda(X)$ is a module over $C(X)$ and if $\mathbf{k}$ is algebraically closed $D(X) = \bigoplus\{D_\lambda(X): \lambda \in \Lambda(X)\}$. Given $\dim X = 1$, $0 \neq x \in X$, we write $C(X) = C(x)$, etc. Unless otherwise stated (e.g. 4.1) we shall assume $m = n$.

2.4. LEMMA.
(i) *$C(X)$ is commutative.*
(ii) *Given $y \in C(X)$, then* $\operatorname{ad} y$ *is locally nilpotent on $N(X)$.*
(iii) *$C(X) - \{0\}$ is an Ore set for $N(X)$.*

(i) is due to Makar-Limanov [16]. (ii) follows from (i) exactly as in the special case described in [5, §10.3]. (iii) follows from (ii) and [4, §1.4].

Recalling that A_n is integral, we let $M(X)$ denote the localization of $N(X)$ at $C(X)-\{0\}$.

2.5. Let Fract A_n denote the left quotient field of A_n. Let V be a finite dimensional subspace of Fract A_n such that $(\operatorname{ad} x)V \subset V$, for all $x \in X$ (notation 2.3).

PROPOSITION. *There exists* $0 \neq z \in S(X)$ *such that* $zV \subset A_n$ *and so* $zV \subset F(X)$.

Set $I = \{a \in A_n : aV \subset A_n\}$. Then I is a left ideal of A_n which is non-zero since V is finite dimensional. Set $M = A_n/I$ and let $N(I)$ denote the normalizer of I in A_n. From $Ix_i V \subset I(\operatorname{ad} x_i)V + IVx_i \subset A_n$ we obtain $Ix_i \subset I$. Hence $S(X) \subset N(I)$. Yet $N(I)/I$ identifies with $\operatorname{Hom}_{A_n}(M, M)$ and so by 1.2 and (2.2), we obtain $\operatorname{Cdim} N(I)/I \leqq \operatorname{Kdim} M \leqq n-1$. Recalling the $\operatorname{Dim} S(X) = n$, by assumption; it follows that $S(X) \cap I \neq 0$, as required.

2.6. COROLLARY.

(i) Fract $C(X)$ *is the commutant of* $S(X)$ *in* Fract A_n.

(ii) $M(X)$ *is the largest subalgebra of* Fract A_n *on which each* $\operatorname{ad} x : x \in X$ *is locally nilpotent.*

(iii) $\Lambda(X)$ *is an additive subgroup of* $\mathbb{Q}^k$: $k \leqq n$.

(i) and (ii) are clear. For (iii) suppose $\lambda \in \Lambda(X)$ and choose $0 \neq y \in D_\lambda(X)$. Then $\mathbf{k}y^{-1}$ is a one-dimensional subspace of Fract A_n which is stable by each $\operatorname{ad} x : x \in X$. Then by 2.5, $-\lambda \in \Lambda(X)$ and so $\Lambda(X)$ is a group. Finally let $\{\lambda_j\}_{j=1}^k$ be a basis for the rational vector space generated by $\Lambda(X)$ and choose $y_j \in D_{\lambda_j}(X)$. Let A be the subalgebra of A_n generated by the y_j over $S(X)$. Then $k + n \leqq \operatorname{Dim} A \leqq \operatorname{Dim} A_n = 2n$. (Familiarity with computations of Dim can be achieved by reading [3].) Hence $k \leqq n$, as required.

2.7. Björk [2] conjectured that equality holds in (2.1). In particular if I is a non-zero principle left ideal of A_n one should have $\operatorname{Kdim}(A_n/I) = n-1$. Suppose in fact that $I = A_n a : 0 \neq a \in A_n$ and let $C(a)$ denote the commutant of $a \in A_n$. Then

LEMMA. $\operatorname{Kdim}(A_n/I) \geqq \operatorname{Cdim} C(a) - 1$.

Set $m = \operatorname{Cdim} C(a)$ and let A be a maximal commutative subalgebra of $C(a)$ with $\operatorname{Dim} A = m$. Then $A \cap I = Aa$. Hence by 1.2, $\operatorname{Kdim}(A_n/I) \geqq \operatorname{Dim} A/(A \cap I) = \operatorname{Dim} A/Aa = m-1$.

In particular taking $m = n$ and $a \in C(X)$, then $\mathrm{Kdim}(A_n/I) = n-1$, so Björk's conjecture is satisfied in this special case. At the other extreme it can happen that $\mathrm{Hom}_{A_n}(A_n/I, A_n/I)$ (which we recall identifies with $N(I)/I$) reduces to scalars. For example, take $n = 2$, $a = p_1p_2 + q_1^3 + q_2^3$ and $I = A_2a$.

3. A power theorem

From now on we assume that $\mathbf{k}$ is algebraically closed.

3.1. We call a non-zero polynomial *separating* if it has no pair of roots $\alpha, \beta \in \mathbf{k}$ with $\alpha - \beta \in \mathbb{N}^+$. Given f a polynomial, let $A_{1,f}$ denote the algebra with identity and generators x, y, z satisfying the relations

$$[x, y] = y, \quad [x, z] = -z, \quad yz = f(x). \tag{3.1}$$

Observe that $zy = f(x+1)$. Given $\deg f \geqq 1$, then $A_{1,f}$ identifies with a subalgebra of the Weyl algebra A_1 and is isomorphic to A_1 iff $\deg f = 1$ (see 3.6). Suppose that $\deg f \geqq 1$.

LEMMA. *The following three conditions are equivalent:*

(i) *$A_{1,f}$ is simple.*

(ii) *f is separating.*

(iii) *There exists $m \in \mathbb{N}^+$ such that the subalgebra of $A_{1,f}$ generated by y^m, z^m, x is simple.*

Define the polynomial g through $g(m^{-1}x) = f(x)f(x-1)\cdots f(x-m+1)$. Observe that f is separating iff g is separating. Again by (3.1), $y^m z^m = g(m^{-1}x)$ and so $A_{1,g}$ is isomorphic to the subalgebra of $A_{1,f}$ defined in (iii). Hence it suffices to prove (i) $\Leftrightarrow$ (ii).

Set $B = A_{1,f}$. Let I be a non-zero two-sided ideal of B. Set $\mathbf{r} = \mathbf{k}x \oplus \mathbf{k}y$. Since $\mathbf{r}$ is solvable and x, y are locally ad-finite, there exists $0 \neq a \in I$ such that $[y, a] = 0$ and $[x, a] = ka$: $k \in \mathbf{k}$. Then by (3.1) one must have $k \in \mathbb{N}$ and $a = y^k$, up to a scalar and then again by (3.1), there exists $\ell \in \mathbb{N}$, such that $y^\ell, z^\ell \in I$. Then $y^\ell z^\ell \in I \cap \mathbf{k}[x]$, and it follows easily that $\dim B/I < \infty$. That is B admits a finite dimensional module M.

Let $0 \neq v \in M$ be an $\mathbf{r}$ eigenvector. Then $yv = 0$ and $xv = \alpha v$, for some $\alpha \in \mathbf{k}$. This gives $0 = zyv = f(x+1)v = f(\alpha+1)v$, so $f(\alpha+1) = 0$. Since $\dim M < \infty$ and the $z^k v \neq 0$ are linearly independent, there exists $k \in \mathbb{N}$ such that $z^{k+1}v = 0$ and $z^k v \neq 0$. Then $0 = yz^{k+1}v = f(x)z^k v = z^k f(x-k)v = f(\alpha - k)z^k v$, so $f(\alpha - k) = 0$. Hence (ii) $\Rightarrow$ (i). Conversely given $\alpha \in \mathbf{k}$, $k \in \mathbb{N}$ such that $f(\alpha+1) =$

$f(\alpha - k) = 0$, the above construction gives the finite dimensional B module $\bigoplus\{z^{\ell}v\colon \ell = 0, 1, 2, \cdots, k\}$. Since $\dim B = \infty$, it follows that B is not simple and so (i) $\Rightarrow$ (ii).

3.2. Let $f_1, f_2, \cdots, f_n$ be separating polynomials. Set $f = (f_1, f_2, \cdots, f_n)$, $A_{n,f} = A_{1,f_1} \times A_{1,f_2} \times \cdots \times A_{1,f_n}$, taking x_i, y_i, z_i to be the generators of A_{1,f_i}, and assumed non-zero.

THEOREM. *Suppose $A_{n,f}$ is contained as a subalgebra in* $\operatorname{Fract} A_n$. *Given $m \in \mathbb{N}^+$ such that $x_i, y_i^m, z_i^m \in A_n$, for all $i = 1, 2, \cdots n$, then $A_{n,f} \subset A_n$.*

Suppose that $\deg f_i = 0$, for some i. Then $y_i^m z_i^m = (y_i z_i)^m \in \mathbf{k}$. Yet $y_i^m, z_i^m \in A_n$ and so the y_i, z_i are scalar. Substitution in (3.1) gives $y_i = z_i = 0$, contradicting their definition. Hence $\deg f_i \geqq 1$, for all i. Set $g = (g_1, g_2, \cdots, g_n)$: $g_i(x_i/m) = f_i(x_i)f_i(x_i - 1)\cdots f_i(x_i - m + 1)$. Then the g_i are separating polynomials of positive degree, so by 3.1 the subalgebra $A_{1,g}$ of A_n generated by x_i, y_i^m, z_i^m and the identity is simple.

Set $I = \{a \in A_n\colon ay_i^k, az_i^k \in A_n$, for all $i = 1, 2, \cdots, n;\ k = 1, 2, \cdots, m - 1\}$. Then I is a non-zero left ideal of A_n. By (3.1), we obtain $A_{n,g} \subset N(I)$ and so $N(I)/I \supset A_{n,g}/(I \cap A_{n,g})$. By the simplicity of $A_{n,g}$, either $I \cap A_{n,g} = A_{n,g}$ or $I \cap A_{n,g} = 0$. In the first case, $1 \in I$ and so $y_i, z_i \in A_n$, as required. In the second case, we have by 1.2 and (2.2) that $n - 1 \geqq \operatorname{Kdim}(A_n/I) \geqq \operatorname{Cdim} N(I)/I \geqq \operatorname{Cdim} A_{n,g} \geqq n$. This contradiction proves the theorem.

3.3. Let A be integral or prime Noetherian and suppose that $\operatorname{Dim} A < \infty$. Given I a two-sided ideal of A, then $\operatorname{Dim} A/I = \operatorname{Dim} A$ implies $I = 0$ [3, §3.5]. Suppose A has an identity 1 and let B be a subalgebra of A with $1 \in B$ and $\operatorname{Dim} A = \operatorname{Dim} B$.

LEMMA. *B simple implies A simple.*

Let I be a two-sided ideal of A. Then $A/I \supset B/(B \cap I)$. Given B simple, then $B \cap I = B$, or $B \cap I = 0$. In the first case, $1 \in B \subset I$, so $I = A$. In the second case, $\operatorname{Dim} A/I \geqq \operatorname{Dim} B = \operatorname{Dim} A$, so $I = 0$.

3.4. Adopt the hypotheses of 3.2, but *without* assuming the f_i separating.

THEOREM. *Suppose $A_{n,f}$ identifies with a subalgebra of* $\operatorname{Fract} A_n$ *which contains A_n. Assume x_i, y_i^m, $z_i^m \in A_n$, for some $m \in \mathbb{N}^+$ and all i. Then $A_{n,f} = A_n$.*

One has $\operatorname{Dim} A_{n,f} = 2n = \operatorname{Dim} A_n$ and so by 3.3, $A_{n,f}$ is simple. Hence by 3.1, the f_i are separating and then the assertion of the theorem follows from 3.2.

3.5. It is not at all obvious if the conclusion of 3.4 implies that $\deg f_i = 1$, for all i. To analyse this we generalize slightly a result of Solomon and Verma [19]. Let $\mathfrak{g}$ be a semi-simple Lie algebra over $\mathbf{k}$, $U(\mathfrak{g})$ its enveloping algebra, $Z(\mathfrak{g})$ the centre of $U(\mathfrak{g})$ and I a two-sided ideal of $U(\mathfrak{g})$. Set $U = U(\mathfrak{g})/I$ and let Z denote the centre of U.

LEMMA. $[U, U] \oplus Z = U$.

One has $[\mathfrak{g}, U] \oplus Z = U$, through the reductivity of $\mathfrak{g}$, so the assertion holds iff $[\mathfrak{g}, U] = [U, U]$. Let π: $U(\mathfrak{g}) \to U$ be the natural projection. Then since the assertion holds for $I = 0$ [19], we have $[U, U] = \pi[U(\mathfrak{g}), U(\mathfrak{g})] = \pi[\mathfrak{g}, U(\mathfrak{g})] = [\mathfrak{g}, U]$, as required.

3.6. In 3.5, we consider the special case when $\mathfrak{g}$ is a direct sum of n copies of $s\ell(2)$ and I is a two-sided ideal generated by a maximal ideal of $Z(\mathfrak{g})$. Then $Z = \mathbf{k}$.

COROLLARY. *In 3.4 one has $\deg f_i = 1$ for at least one value of i.*

Suppose $\deg f_i > 1$, for all i. Then we can write $f_i(x_i) = (x_i + \alpha_i)(x_i + \beta_i)g_i(x_i)$: $\alpha_i, \beta_i \in \mathbf{k}$, g_i polynomial and so A_{1,f_i} is isomorphic to the subalgebra of A_1 defined through the relations

$$y_i = q_i,\ \ x_i + \alpha_i = -q_i p_i,\ \ z_i = -p_i(-q_i p_i - \alpha_i + \beta_i)g_i(-q_i p_i - \alpha_i).$$

Since the algebra generated by q_i, $q_i p_i + \gamma_i$, $q_i p_i^2 + 2\gamma_i p_i$: $i = 1, 2, \cdots, n$, is isomorphic to a U of the above form for all $\gamma_i \in \mathbf{k}$, it follows that we can take $A_{n,f} \subset U$. Then $\mathbf{k} \subset [A_n, A_n] = [A_{n,f}, A_{n,f}] \subset [U, U]$, which contradicts 3.5.

To show that $\deg f_i = 1$, for all i, it suffices to prove that $A_n = A_1 \times B$ implies $A_{n-1} = B$ (algebra isomorphisms). Unfortunately as McConnell points out [15], this sort of "cancellation theorem" is generally quite hard even for very special B.

3.7. let $0 \neq a, b, c \in A_1$ satisfy

$$[a, b] = 2c, \quad [a, c] = a, \quad [b, c] = -b. \tag{3.2}$$

Following [10] we say that (3.2) defines a realization of $s\ell(2)$ in A_1 and that this realization is of type S_n if $\Lambda(c) = 1/n\,\mathbb{Z}$. By [10, theor. 2.6], one can only have $n = 1$ or 2. Theorem 3.2 above gives additional information on S_2. In fact the following result replaces lemma 4.3 and corollary 4.5 in [10] which are incorrect owing to an error in [10, p. 125, line 19]. Set $C = ab + ba - 2c^2$, and recall [10, prop. 2.1], that C is scalar in A_1.

LEMMA. *To each realization of type S_2 there exist $y, z \in \operatorname{Fract} A_1$ and $r \in \mathbf{N}^+$ such that $a = \frac{1}{2}y^2$, $b = \frac{1}{2}z^2$ and*

$$yz = (x-1)\prod_{i=1}^{r-1}\left(\frac{x+2i-1}{x+2i-2}\right), \quad \textit{where } x = -2c - r + \tfrac{3}{2}.$$

Furthermore $y^{2r-1}, z^{2r-1} \in A_1$; $y^{2r-3}, z^{2r-3} \notin A_1$ and $C = -\frac{1}{2}(r-\frac{3}{2})(r+\frac{1}{2})$.

By [12, theor. 1.2], $C(x) = \mathbf{k}[x]$. Since the latter is a principle ideal domain and $\Lambda(x) = \mathbf{Z}$, it follows (cf. 4.4) that $D_n(x)$: $n \in \mathbf{Z}$ is cyclic as a module over $\mathbf{k}[x]$ and we let y_n be a generator. By (3.2), $ay_n = f_n(x)y_{n+2}$: f_n polynomial. As shown in [10, lemma 4.3], inspection of leading terms shows that $f_n \in \mathbf{k}$, for all but finitely many $n \in \mathbf{N}$. Assume f_n scalar for some n odd and set $r = (n+1)/2$. Then (cf. [10, lemma 4.3]) $[a, y_{2r-1}] = 0$ and so $a^{2r-1} = y_{2r-1}^2$, up to a scalar. Set $y = \sqrt{2}a^{-r+1}y_{2r-1}$. Then $y \in \operatorname{Fract} A_1$ and $y^2 = 2a$. Similarly there exists $z \in \operatorname{Fract} A_1$, with $z^2 = 2b$ and $z^{2s-1} \in A_1$, for some $s \in \mathbf{N}^+$. By interchanging y and z if necessary we can assume that $r = s$ and $y^{2r-3} \notin A_1$.

By 2.6 (i) there exists a rational function h such that $yz = h(x)$. Then $y^{2r-1}z^{2r-1} = h(x)h(x-1)\cdots h(x-2r+2) \in \mathbf{k}(x) \cap A_1 = \mathbf{k}[x]$. Again $4ab = y^2z^2 = h(x)h(x-1)$ is a polynomial in x of degree 2 and so $\deg h = 1$. Hence up to translation of x by a scalar we must have

$$h(x) = (x-1)\prod_{i=1}^{t-1}\left(\frac{x+2i-1}{x+2i-2}\right), \quad \text{for some } t \in \mathbf{N}.$$

Furthermore $h(x)h(x-1)\cdots h(x-2u+2)$ is polynomial iff $u \geqq t$ and so $t \leqq r$. One has $y^2z^2 = (x-2)(x+2t-3)$ and so the subalgebra U of A_1 generated by y^2, z^2, x is simple by 3.1. Again by 3.1 the subalgebra of A_1 generated by y^{4t-2}, z^{4t-2}, x is simple and since $y^{2t-1}z^{2t-1}$ is polynomial in x, it follows by 3.2 that $y^{2t-1} \in A_1$ and so $t = r$. Then $8c = 4[a, b] = [y^2, z^2] = (x-2)(x+2r-3) - x(x+2r-1) = -4(x+r-\frac{3}{2})$, so $x = -2c - r + \frac{3}{2}$ and $C = \frac{1}{4}\{y^2z^2 + z^2y^2 - 2(x+r-\frac{3}{2})^2\} = -\frac{1}{2}(r-\frac{3}{2})(r+\frac{1}{2})$, as required. Finally suppose $z^{2r-3} \in A_1$. Then $y^{2r-1}z^{2r-3} \in D_2(x) = y_2\mathbf{k}[x]$. Yet $y^{2r-3}z^{2r-3}$ is not polynomial in x and so y^2 is not a scalar multiple of y_2. Equivalently a is divisible in A_1 by a linear function in x which contradicts [10, lemma 3.2].

REMARKS. Given r as above, we say such realizations are of type S_{2r}. They are inequivalent for different r [10, §5.5]. S_{21} is non-empty and determined up to $\operatorname{End} A_1$. It is not established if S_{2r} is empty for $r > 1$. As pointed out in [10, sect. 4], this is about as difficult as showing that $\operatorname{End} A_1 \neq \operatorname{Aut} A_1$ and in fact unpublished computations of mine show that one must have $\deg x \geqq 147$ for $r > 1$.

3.8. Take $r = 2$ in 3.7. Then $yz = (x-1)(x+1)x^{-1} = x - x^{-1}$, $y^2z^2 = (x-2)(x+1)$, $y^3z^3 = (x-3)(x-1)(x+1)$. Set $z' = \frac{1}{2}[y, z^2]$. Then $[y, z'] = 1$, $yz' = -x$ and $z = z'(1 - x^{-2})$. Hence the algebra generated by x, y, z satisfying the above relations is isomorphic to $(A_1)_S$ where S is the multiplicative subset of $A_1 = \mathbf{k}[y, z']$ generated by the $(x + n)$: $n \in \mathbb{Z}$. (This is an Ore set for A_1 by [4, §1.4].) The subalgebra B generated by x, y^2, z^2, y^3 contains the $s\ell(2)$ subalgebra generated by x, y^2, z^2 and is locally ad-finite as an $s\ell(2)$ module. By 2.6 (i) the commutant $C(y^2)$ of y^2 in B coincides with $\mathbf{k}(y) \cap B$ and so by the above local ad-finiteness we obtain $C(y^2) = \mathbf{k}[y^2, y^3]$. In particular $y \notin B$, and as an ad-$s\ell(2)$ module, B is a direct sum of simple modules of dimensions $3, 4, 5, \cdots$. Again $\operatorname{Dim} B = 2$, so by 3.1 and 3.3, B is simple. Computation gives $1 \in [M, M] \subset [B, B]$, where M is the simple 4 dimensional $s\ell(2)$ submodule of B. Clearly $\operatorname{Fract} B = \operatorname{Fract} A_1$ and B is the largest subalgebra of $\operatorname{Fract} B$ which is locally ad-$s\ell(2)$ finite. Yet B is not isomorphic to A_1 since the commutant of the locally ad-nilpotent element y^2 is not a polynomial algebra (cf. [5, §8.9 (i)]).

3.9. Set $y = p^{-1} - qpq$. Then (cf. [7, sect. 10]) $y^2 = (qpq)^2 - 2q^2 \in A_1$. Set $x = -qp$, $z = p$. Then (3.1) holds with $f(x) = (1 - x)(1 + x)$. This shows that the conclusion of 3.2 fails if f is not separating. Again take x, y as above, but now set $z = q^{-1} - pqp$. Then $y^m, z^m \in A_1$, for all integer $m \geqq 2$. Yet $yz = -x^{-1}(x^2 - 1)^2$, which is not polynomial. Observe that $y^2z^2 = (x+1)^2(x-2)^2(x-1)x$ is not separating, so the algebra generated by x, y^2, z^2 is not simple.

3.10. The following resolves problem 14 in [9].

LEMMA. *Suppose $y, z \in \operatorname{Fract} A_1$ satisfy $yz - zy = -1$. Given $y^m, z^m \in A_1$, for some $m \in \mathbb{N}^+$, then $y, z \in A_1$.*

Set $x = yz$. Then x, y, z satisfy (3.1) with f linear and hence separating. Then by 3.1 and 3.2, it is enough to show that $x \in A_1$ to establish the lemma. By 2.6 (i), there exists a non-zero polynomial g for which $g(z^m)z \in A_1$. Set $k = \deg g$. Application of $\operatorname{ad} y^m$ shows that $(y^{m-1})^{km+1} \in A_1$. Since m and $(km+1)(m-1)$ are coprime, we have $y^{\ell+1} \in A_1$, for all ℓ sufficiently large. Interchanging y and z shows that there exists $n \in \mathbb{N}$, such that $y^k, z^k \in A_1$: $k = n+1, n+2, \cdots$. Set $a_k = y^k z^k$, $b_k = z^k y^k$. We have $a_k = \Gamma(x+1)/\Gamma(x-k+1)$, $b_k = \Gamma(x+k+1)/\Gamma(x+1)$, where Γ is the Gamma function.

Suppose

$$\sum_{k=n+1}^{2n} (\alpha_k a_k + \beta_k b_k) = 0, \text{ for some } \alpha_k, \beta_k \in \mathbf{k}.$$

Take $x = s \in \{1, 2, \cdots, n\}$. Then $\Sigma \beta_k (s+k-1)!/(s-1)! = 0$. Let N_n be the $n \times n$ matrix with entries $(N_n)_{rs} = (r+s+n-1)!$ Then $\det N_n \neq 0$ and so $\beta_k = 0$, for all k. Interchanging x and $-(x+1)$ shows that $\alpha_k = 0$, for all k and so a_k, b_k: $k = n+1, n+2, \cdots, 2n$ are linearly independent. Since they form a system of $2n$ polynomials of degree $\leqq 2n$, it follows that we can write x as a linear combination of them. Hence $x \in A_1$, as required.

4. Strictly semisimple elements

Recall the notation of 2.3 and assume that $D(X) = A_n$. That is each $x \in X$ is locally ad-semisimple on A_1. For $n = 1$, Dixmier [5, §9.2] showed that $S(X) = \mathbf{k}[qp]$, up to $\operatorname{Aut} A_1$. Unfortunately his analysis does not easily extend to A_n since it entails a description of $\operatorname{Aut} A_n$, a problem which is almost certainly intractable for $n > 1$. However by entirely different techniques we are able to go quite far in analysing $S(X)$ and in 5 resolve completely the analogue problem for the Poisson algebra. Though we make use of the generalized Quillen lemma, all the results of this section can be obtained without it and in this connection we remark that in the present special case, 2.6 (i) follows from [13, §6.1]. Nevertheless we believe that a more complete description of $S(X)$ may derive from an extension of the ideas developed in Section 3.

4.1 (Notation 2.3; $m = 1, 2, \cdots, n$). The following result generalizes [5, §9.3].

Lemma. *There exists* $k \in \{1, 2, \cdots n\}$, *such that* $\Lambda(X) = \mathbb{Z}^k$, *up to an isomorphism. Furthermore* $\dim X \leqq k$.

Let V be the rational vector space generated by $\Lambda(X)$ and set $k = \dim_{\mathbb{Q}} V$. Let $\{\lambda_j\}_{j=1}^k$ be a basis for V. Given $\lambda \in \Lambda(X)$, write

$$\lambda = \sum_{j=1}^{k} r_j \lambda_j : \; r_j \in \mathbb{Q},$$

and define linear maps $d_j : D_\lambda(X) \to D_\lambda(X)$, through $d_j y_\lambda = r_j y_\lambda$. The d_j extend to derivations of $D(X) = A_n$ and so by [6, §4.6.8] there exist $x_j' \in A_n$ such that $d_j = \operatorname{ad} x_j'$. Set $X' = \operatorname{lin\,span} x_i'$. Then X' generates a polynomial subalgebra $S(X')$ of A_n and so $k \leqq n$, by 2.1. Clearly $\Lambda(X) = \Lambda(X')$ up to isomorphism and we show that $\Lambda(X') = \mathbb{Z}^k$, for a suitable choice of scalars.

By construction $\Lambda(X')$ is an additive sub-semigroup of $\mathbb{Q}^k$. Since $A_n = D(X)$ is finitely generated we can choose scalars such that $\Lambda(X') \subset \mathbb{Z}^k$. It remains to show that $\Lambda(X')$ is a group and through the x_j' we can assume that $k = 1$ without loss of generality and set $X' = X$.

It is clear that $\Lambda(X)$ is not a group only if $\Lambda(X) \subset \mathbb{N}$ (up to a change of sign). Let q_i^0 (resp. p_i^0) denote the component of q_i (resp. p_i) lying in $D_0(X) = C(X)$ under the root space decomposition of A_n defined by $D(X)$. Given $\Lambda(X) \subset \mathbb{N}$, then q_i^0, p_i^0: $i = 1, 2, \cdots, n$, generate a Weyl algebra A_n and so $\operatorname{Dim} C(X) \geqq \operatorname{Dim} A_n = 2n$. This contradicts [8, theor. 1.1] and so proves the lemma.

4.2. Lemma. *Choose $\theta \in \operatorname{End} \operatorname{Fract} A_n$. The commutant $C(\theta(A_n))$ of $\theta(A_n)$ in* Fract A_n *reduces to scalars.*

Set $y_i = \theta(q_i)$, $y_{i+n} = \theta(p_i)$: $i = 1, 2, \cdots, n$. Take $U = \operatorname{Fract} A_n$, $B = C = (\operatorname{Fract} A_n)^{y_1, y_2, \cdots, y_m}$, $\partial = \operatorname{ad} y_{m+1}$ in [13, prop. 3.2]. For each $m \in \{0, 1, 2, \cdots, 2n - 1\}$, one has $B^{y_{m+1}} \neq B$ and so by [13, prop. 3.2] we obtain

$$\operatorname{Dim} \operatorname{gr} (\operatorname{Fract} A_n) > \operatorname{Dim} \operatorname{gr} (\operatorname{Fract} A_n)^{y_1} > \operatorname{Dim} \operatorname{gr} (\operatorname{Fract} A_n)^{y_1, y_2} > \cdots > \operatorname{Dim} \operatorname{gr} C(\theta(A_n)).$$

Since each of the above terms is integer-valued and $2n = \operatorname{Dim} \operatorname{gr} (\operatorname{Fract} A_n)$, it follows that $\operatorname{Dim} \operatorname{gr} C(\theta(A_n)) = 0$. Hence $\operatorname{gr} C(\theta(A_n))$ and so $C(\theta(A_n))$ reduces to scalars.

Remark. This was first proved in my unpublished lecture notes on the Weyl algebras [9, §§2.10, 2.11]. It has the important corollary that Fract A_m cannot be embedded in Fract A_n if $m > n$. (The corresponding assertion for A_n is elementary.)

4.3. From now on we assume that $m = n$ in 2.3 (and $D(X) = A_n$).

Lemma. *For each $x \in C(X)$, $N(x) = C(x)$.*

By definition $N(x) \supset C(x)$ and suppose this inclusion is strict. Then there exists $y \in A_n$ such that $[x, y] \neq 0$, $[x, [x, y]] = 0$. Since $x \in C(X)$ and $D(X) = A_n$, we can assume that $y \in D_\lambda(X)$, for some $\lambda \in \Lambda(X)$, without loss of generality. Then by 2.6 (i), we may write $[x, y] = yz$, with $z \in \operatorname{Fract} C(X)$. By 2.4 (i), this gives $0 = [x, [x, y]] = [x, y]z = yz^2$ and so $z = 0$, which is a contradiction.

4.4. Proposition

(i) $C(X) = S(X)$,

(ii) *For each $\lambda \in \Lambda(X)$, $D_\lambda(X)$ is finitely generated as a module over $S(X)$.*

(iii) *There exists $\varphi \in \operatorname{Aut} \operatorname{Fract} A_n$ such that $\varphi(X) = \bigoplus_{i=1}^n (q_i p_i + \alpha_i)$, for some $\alpha_i \in \mathbf{k}$.*

By 4.1, $\Lambda(X)=\mathbb{Z}^n$, for some basis of X. Let ρ_i: $i=1,2,\cdots,n$, be positive real numbers linearly independent over $\mathbb{Q}$. Given $K=(k_1,k_2,\cdots,k_n)\in\mathbb{Z}^n$ and $y\in D_K(X)$, set

$$d(y)=\sum_{i=1}^{n}\rho_i k_i.$$

Then d extends to a valuation on A_n and hence defines a filtration on A_n. Let gr denote the associated gradation functor and observe that we can identify $\operatorname{gr}(A_n)$ with A_n. Set $Y=\operatorname{gr}(\mathbf{k}[q_1,q_2,\cdots,q_n])$. Then Y is commutative and each $y\in Y$ is locally ad-nilpotent in A_n. Now given $Z\subset A_n$, set $\Omega(Z)=\{K\in\mathbb{Z}^n$: $\operatorname{gr}(z)\in D_K(X)-\{0\}\}$. Then $\Omega(Y)-\{0\}$ admits a hyperplane of support through the origin in $\mathbb{Z}^n$. Otherwise by Caretheordory's theorem, there exist $y_1,y_2,\cdots,y_r\in Y-\{\mathbf{k}\}$ and positive integers $m_1,m_2,\cdots,m_r$ such that $d(y)=0$, where $y=\Pi_j y_j^{m_j}$. We can assume the y_i to be $\operatorname{ad}x_j$: $j=1,2,\cdots,n$, eigenvectors and then $y\in C(X)$. So by 4.3, $C(y)=N(y)=A_n$, which gives $y\in\mathbf{k}$. Then $y_j\in\mathbf{k}$, for each j, in contradiction to their choice.

Now choose the ρ_i, so that $(\rho_1,\rho_2,\cdots,\rho_n)$ considered as an element of $\Lambda(X)^*$ is normal to a support hyperplane for $\Omega(Y)$. Then $d(y)\geqq 0$, for all $y\in Y$, so the filtration defined by d is bounded from below on Y. This gives $\operatorname{Dim}Y=\operatorname{Dim}\mathbf{k}[q_1,q_2,\cdots,q_n]=n$. Again given $K\in\Omega(Y)$ and $y,z\in Y\cap D_K(X)$, then $y^{-1}z\in\operatorname{Fract}C(X)$. Since y, z commute and are locally ad-nilpotent, the localization $(A_n)_y$ of A_n at y is defined and $y^{-1}z$ is locally ad-nilpotent on $(A_n)_y$. Now 4.3 extends in the obvious fashion to Fract $C(X)$ and by its conclusion, $y^{-1}z$ commutes with the whole of $(A_n)_y$ and is hence scalar. It follows that $n\geqq\dim(\operatorname{lin\,span}\Omega(Y))\geqq\operatorname{Dim}Y$. Then we can choose $y_1,y_2,\cdots,y_n\in Y$ and a basis $\{x_i'\}_{i=1}^n$ for X such that

$$[x_i',y_j]=\delta_{ij}y_j:\quad \delta_{ij}\ \text{the Kronecker delta.} \tag{4.1}$$

One does not in general have $\Lambda(X)=\mathbb{Z}^n$ (identically) in this basis, thus for each $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_n)$ we write $D_\lambda(X')=\{y\in A_n\colon (\operatorname{ad}x_i')y=\lambda_i y\}$. Then for each $K=(k_1,k_2,\cdots,k_n)\in\mathbb{N}^n$, we have

$$\left(\prod_{i=1}^{n}\operatorname{ad}^{k_i}y_i\right)C(X)\subset D_K(X')\subset\left(\prod_{i=1}^{n}y_i^{k_i}\right)\operatorname{Fract}C(X),$$

by 2.6 (i). Since the y_i are locally ad-nilpotent, there exists for each $z\in C(X)$, a $K\in\mathbb{N}^n$, such that

$$0\neq\left(\prod_{i=1}^{n}\operatorname{ad}^{k_i}y_i\right)z\in\left(\prod_{i=1}^{n}y_i^{k_i}\right)(\operatorname{Fract}C(X))^Y=\mathbf{k}\left(\prod_{i=1}^{n}y_i^{k_i}\right),$$

where the last step follows from (4.1) and 4.2. Taking account of (4.1) this gives (i).

For (ii), recall that $\Lambda(X)$ is an additive group and choose $0 \neq y \in D_{-\lambda}(X)$. Define the $S(X)$ module homomorphism $\psi: D_\lambda(X) \to C(X) = S(X)$, through $\psi(x) = xy$. Since A_n is integral, ψ is injective. Then (ii) follows from the fact that $S(X)$ is Noetherian.

Let $C(Y)$ denote the commutant of Y in A_n. Let $\overline{\Omega(C(Y))}$ denote the smallest subgroup of $\mathbb{Z}^n$ containing $\Omega(C(Y))$. We show that $\overline{\Omega(C(Y))} = \mathbb{Z}^n$. Given $K \in \mathbb{Z}^n$, fix $0 \neq c \in D_K(X)$. Through the local ad-nilpotence of the y_i, there exists $L = (\ell_1, \ell_2, \cdots, \ell_n) \in \mathbb{N}^n$ such that

$$0 = \left(\prod_{j=1}^{n} \operatorname{ad}^{\ell_i} y_i\right) c \in C(Y) \cap D_{K+L'}(X), \quad \text{for some } L' \in \Omega(Y)$$

(determined by L and the eigenvalues of the y_i). Then $K + L' \in \Omega(C(Y))$ and so $K \in \overline{\Omega(C(Y))}$, as required.

By 2.4, $C(Y)$ is a commutative algebra of locally ad-nilpotent elements. Thus for each $y \in C(Y) - \{0\}$, the localizations $C(Y)_y$ and $(A_n)_y$ can be defined ([4, §1.4]). Since $\overline{\Omega(C(Y))} = \mathbb{Z}^n$, there exists $y \in C(Y) - \{0\}$, such that $\Omega(C(Y)_y) = \mathbb{Z}^n$. Let $1_i \in \mathbb{Z}^n$, denote the n-tuple with one in the ith entry and zeros elsewhere and choose non-zero $p'_i \in C(Y)_y \cap D_{1_i}(X)$. Then for all $K = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n$, we have by 2.6 (i) and the local ad-nilpotence of the p'_i on $(A_n)_y$ that

$$D_K(X) \subset \left(\prod_{i=1}^{n} (p'_i)^{k_i}\right) C(X).$$

Set $q'_i = x_i p'^{-1}_i \in (A_n)_y$. Then the q'_i, p'_i are the required canonical generators.

4.5. 4.4 (iii) is weaker than Dixmier's result in the special case $n = 1$, since we require the bigger automorphism group Aut Fract A_n instead of Aut A_n. An attempt to obtain this refinement in the general case motivated much of Section 3 and posed the following

Conjecture. *Suppose* $N(X) = A_n$. *Then* $C(X)$ *is a unique factorisation domain* (UFD).

This holds for $n = 1$, and in fact $C(X) = \mathbf{k}[q_1]$, up to Aut A_1 [5, §8.9 (i)] (but this stronger assertion fails if $n = 2$ [11]). Again given $A_n \supsetneq N(X) \supsetneq C(X)$, then $C(X)$ is not necessarily a UFD even when $n = 1$. For example, set $y = p^{-1} - qpq$. Then $y^m \in A_1$: $m = 2, 3, \cdots$, and $C(y^2) = \mathbf{k}[y^2, y^3]$, since there exists [7, sect. 10] $\varphi \in$ Aut Fract A_1 such that $y = \varphi(q)$. Furthermore $p \in N(y^2)$, yet $p \notin C(y^2)$.

To see how the analysis might carry through given a positive reply to this conjecture, we resolve the analogue problem for the Poisson algebra in the next section.

5. The Poisson algebra

5.1. Let P_n denote $\mathbf{k}[X_1, X_2, \cdots, X_n, Y_1, Y_2, \cdots, Y_n]$ considered both as a polynomial algebra and as a Lie algebra through the classical Poisson bracket

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial X_i} \frac{\partial g}{\partial Y_i} - \frac{\partial g}{\partial X_i} \frac{\partial f}{\partial Y_i} \right): \quad f, g \in P_n. \tag{5.1}$$

Although P_n and A_n are not isomorphic (even as Lie algebras) they have many similar properties. In particular, for each $x \in P_n$ we can define a derivation ad x of P_n through $(\operatorname{ad} x)y = \{x, y\}$ and then $S(X)$, $C(X)$, $F(X)$, $N(X)$, $D(X)$ can be defined in a manner analogous to 2.3. A simplification in studying P_n derives from the fact that it is a UFD. This gives, as in [14, §3.5],

LEMMA (Notation 2.3; $m = 1, 2, \cdots, n$). *Suppose $N(X) = A_n$. Then*
(i) *The irreducible elements of $C(X)$ are irreducible in P_n.*
(ii) *$C(X)$ is a UFD.*

5.2. By $\operatorname{Aut} P_n$ we shall mean the subgroup of $\operatorname{Aut} \mathbf{k}[Y_1, Y_2, \cdots, Y_n, Z_1, Z_2, \cdots, Z_n]$ which preserves Poisson bracket. The main result of this section is the following. Take $m = n$ in 2.3 and set $X_i = Y_i Z_i$. Assume $D(X) = P_n$.

THEOREM. *There exists $\varphi \in \operatorname{Aut} P_n$, such that $\varphi(X) = \bigoplus_{i=1}^{n} (X_i + \alpha_i)$, for some $\alpha_i \in \mathbf{k}$.*

This is established below. In the proof we shall assume that the results of Sections 2 and 4 can be applied here, leaving the reader to fill in the necessary details. It is a general rule that the same argument applies and even with a little simplification. However one should note that P_n as a Lie algebra admits an outer derivation (see for example [7, theor. 3.1]), and we do not know if 2.6 holds for P_n. In the special case when $F(X) = P_n$ the latter result obtains from [13, §2.5], and 4.1 above.

5.3. LEMMA. *There exist $y_1, y_2, \cdots, y_n \in P_n$, and a basis $\{x_i\}_{i=1}^{n}$ of X with the following properties*:
(i) $\{x_i, x_j\} = \{y_i, y_j\} = 0$, $\{x_i, y_j\} = \delta_{ij} y_j$ (notation (4.1)),
(ii) *The y_i are irreducible,*

(iii) $N(\tilde{Y}) = P_n$, *where* $\tilde{Y}$ = lin span y_i,

(iv) *For all* $K = (k_1, k_2, \cdots k_n) \in \mathbb{Z}^n$, *one has*

$$D_K(X) = P_n \cap \left(\prod_{i=1}^{n} y_i^{k_i}\right) S(X).$$

One has $\Lambda(X) = \mathbb{Z}^n$ as in 4.1 and we define Y, $C(Y)$ as in 4.4. Clearly $(\operatorname{ad} x)C(Y) \subset C(Y)$ for all $x \in X$. Then given $y, z \in D_K(X) \cap C(Y)$, we have $y^{-1}z \in \operatorname{Fract} C(Y) \cap \operatorname{Fract} C(X) = \mathbf{k}$, by (4.1) and 4.2. Taking 5.1 into account, it follows that $C(Y)$ satisfies all the conditions of [14, §4.2] and is hence a polynomial algebra on generators y_i: $i = 1, 2, \cdots, n$, which are irreducible. This gives (i)–(iii). From 2.6(i), 4.4(i) and the result $\overline{\Omega(C(Y))} = \mathbb{Z}^n$, established in 4.4, we obtain

$$D_K(X) \subset \left(\prod_{i=1}^{n} y_i^{k_i}\right) \operatorname{Fract} S(X), \text{ for all } K = (k_1, k_2, \cdots, k_n) \in \mathbb{Z}^n.$$

Combined with (iii), this gives (iv).

5.4 (Notation 5.3, 4.4).

LEMMA. *For each* $i = 1, 2, \cdots, n$, *there exists* $\alpha_i \in \mathbf{k}$ *such that* $y_i^{-1}(x_i + \alpha_i) \in P_n$.

Consider $D_{-1_i}(X)$. By 5.3(iv), there exists for each i a non-zero polynomial f_i such that $y_i^{-1} f_i(x_1, x_2, \cdots, x_n) \in P_n$. By 5.3(i), repeated application of the ad y_j: j $(\neq i) = 1, 2, \cdots, n$, to this expression gives a non-zero polynomial g_i for which

$$\left(\prod_{j(\neq i)=1}^{n} y_j^{m_j}\right) y_i^{-1} g_i(x_i) \in P_n: \quad m_j \in \mathbb{N}.$$

By 5.3(ii), y_i divides $g_i(x_i)$ and hence one of its linear factors.

5.5 (Notation 5.4). Set $z_i = -y_i^{-1}(x_i + \alpha_i)$.

LEMMA. *For all* $i, j = 1, 2, \cdots, n$,

(i) $\{z_i, z_j\} = 0$, $\{y_i, z_j\} = \delta_{ij}$,

(ii) *the* z_i *are locally ad-nilpotent on* P_n.

(i) follows from 5.3(i) and (ii) from 5.3(iv) and (i).

To establish the theorem, it remains to show that the y_i, z_j generate P_n over $\mathbf{k}$. By 5.3(iii) and 5.4, they generate a Heisenberg Lie algebra whose elements form locally ad-nilpotent derivations of P_n. Then a standard argument using the so-called Taylor lemma (see [9, lemme 1.5], for example) establishes the required assertion.

5.6. In the more general situation when $\mathbf{k}$ is not algebraically closed, passage to the algebraic closure $\mathbf{k}^-$ and use of 4.1 shows that $X \otimes_{\mathbf{k}} \mathbf{k}^-$ is spanned by elements x satisfying $\Lambda(x) = \alpha \mathbf{Z}$: $\alpha \in \mathbf{k}^-$. It follows easily that $\alpha^2 \in \mathbf{k}$, and hence 5.2 holds with $X_i = \beta_i Y_i^2 + Z_i^2$, for suitable non-zero $\beta_i \in \mathbf{k}$.

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