ON DILWORTH'S THEOREM IN THE IN FINITE CASE

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ABSTRACT

The following theorem is proved: Let c be an infinite cardinal. There exists a partially ordered set of cardinal c, which contains no infinite independent subset, and which is not decomposable into less than c chains.

Let P be a partially ordered set, and k a natural number. Dilworth's theorem states that if the maximal number of mutually incomparable elements in P is k, then P is the union of k chains. (The terminology is explained in the preceding note [2].) P. Erdös raised the question, whether Dilworth's theorem can be extended to the case where the cardinals of independent subsets of P are not bounded. The following theorem shows that such an extension is impossible.

THEOREM. *For every infinite cardinal $\aleph_\alpha$ there exists a partially ordered set T_α, such that*

(A) $|T_\alpha| = \aleph_\alpha$;

(B) *T_α contains no infinite independent subset;*

(C) *T_α is not decomposable into less than $\aleph_\alpha$ chains.*

($|M|$ denotes the cardinal of the set M.)

Proof. Let ω_α be the smallest ordinal power $\aleph_\alpha$. Let T_α be the set of all ordered pairs (ξ,η) of ordinals $\zeta < \omega_\alpha, \eta < \omega_\alpha$. Define $(\xi_1,\eta_1) \leqq (\xi_2,\eta_2)$ iff $\xi_1 \leqq \xi_2$ and $\eta_1 \leqq \eta_2$. The relation $\leqq$ is a partial ordering. If $\alpha < \beta$, then $T_\alpha \subset T_\beta$, and the partial order relation of T_β is an extension of the partial order of T_α. Clearly $|T_\alpha| = \aleph_\alpha^2 = \aleph_\alpha$.

(ξ_1,η_1) and (ξ_2,η_2) are incomparable iff $\xi_1 < \xi_2$ and $\eta_2 < \eta_1$, or $\xi_2 < \xi_1$ and $\eta_1 < \eta_2$. Therefore, if S is an independent subset of T_α, then distinct elements of S have different first coordinates, and S may be well-ordered according to the magnitude of the first coordinates of its elements. If S were infinite, there would be an infinite sequence $\{(\xi_i,\eta_i) \mid 0 \leqq i < \omega\}$ of elements of S, such that $\xi_i < \xi_{i+1}$, and therefore $\eta_i > \eta_{i+1}$, for $0 \leqq i < \omega$, which is impossible. This proves (B).

In order to prove (C), consider three cases.

CASE 0. $\alpha = 0$. T_0 is not decomposable into less that $\aleph_0$ chains, since it contains, for every $n < \omega$, an independent subset $S_n = \{(i, n-i) \mid 0 \leqq i \leqq n\}$ having $n+1$ elements.

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CASE 1. $\alpha = \beta + 1$. Assume that $\mathscr{C} = \{C_\nu \mid \nu < \omega_\gamma\}$ is a system of $\aleph_\gamma$ totally ordered subsets of T_α (chains), where $\gamma < \alpha$, i.e. $\gamma \leqq \beta$. Let C_ν^* be the set of all distinct second coordinates of elements of C_ν $(\nu < \omega_\gamma)$. Define

$$I_1 = \{\nu \mid \nu < \omega_\gamma, \ |C_\nu^*| < \aleph_\alpha\},$$
$$I_2 = \{\nu \mid \nu < \omega_\gamma, \ |C_\nu^*| = \aleph_\alpha\},$$
$$D = \bigcup \{C_\nu^* \mid \nu \in I_1\}.$$

Clearly, $I_1 \cup I_2 = \{\nu \mid \nu < \omega_\gamma\}$, $I_1 \cap I_2 = \emptyset$, and $|D| \leqq \aleph_\gamma \cdot \aleph_\beta = \aleph_\beta < \aleph_\alpha$. Let $\eta^* = \sup\{\eta \mid \eta \in D\} + 1$. $\eta^* > \eta$ for all $\eta \in D$, and $\eta^* < \omega_\alpha$, since all elements of D are smaller than ω_α.

Now, if $\nu \in I_2$, then $|C_\nu| = \aleph_\alpha$, hence there exists an element $(\xi_\nu, \eta_\nu) \in C_\nu$ with $\eta_\nu > \eta^*$. Choose such a pair (ξ_ν, η_ν) for each $\nu \in I_2$, and let $\xi^* = \sup\{\xi_\nu \mid \nu \in I_2\} + 1$. Since $|I_2| \leqq \aleph_\gamma$ and $\xi_\nu < \omega_\alpha$ for every $\nu \in I_2$, we have also $\xi^* < \omega_\alpha$, and $(\xi^*, \eta^*) \in T_\alpha$. But (ξ^*, η^*) cannot belong to any C_ν with $\nu \in I_1$, since η^* is greater than any $\eta \in C_\nu$. Also, (ξ^*, η^*) cannot belong to any C_ν with $\nu \in I_2$, since (ξ^*, η^*) and (ξ_ν, η_ν) are incomparable ($\zeta^* > \xi_\nu$ and $\eta^* < \eta_\nu$). Thus (ξ^*, η^*) does not belong to any $C_\nu \in \mathscr{C}$, and therefore $\bigcup\{C_\nu \mid \nu < \omega_\gamma\} \neq T_\alpha$.

CASE 2. α is a limit number, $\alpha \neq 0$. Assume that $\mathscr{C} = \{C_\nu \mid \nu < \omega_\gamma\}$ is a system of $\aleph_\gamma$ totally ordered subsets of T_α, such that $\gamma < \alpha$ (and therefore also $\gamma + 1 < \alpha$), and $\bigcup\{C_\nu \mid \nu < \omega_\gamma\} = T_\alpha$. Let $\mathscr{C}' = \{C_\nu \cap T_{\gamma+1} \mid \nu < \omega_\gamma\}$; then $\mathscr{C}'$ is a system of at most $\aleph_\gamma$ totally ordered subsets of $T_{\gamma+1}$, and

$$\bigcup\{C_\nu \cap T_{\gamma+1} \mid \nu < \omega_\gamma\} = T_\alpha \cap T_{\gamma+1} = T_{\gamma+1}$$

which is impossible by case 1. Q.E.D.

REMARK. T_α is a distributive lattice. We may obtain a *complete* distributive lattice T'_α satisfying the requirements of the theorem by defining

$$T'_\alpha = \{(\xi \ \eta) \mid \xi \leqq \omega_\alpha, \eta \leqq \omega_\alpha\}.$$

BIBLIOGRAPHY

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