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ON THE PROBLEM OF k STRUCTURE

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ABSTRACT

For every k and every p, $1 < p < 2$, we construct a Banach space having a k structure and being of type p. This is an answer to a question raised by W. Davis and J. Lindenstrauss in [1].

We recall that a Banach space E is said to be of type p, $1 < p \leqq 2$, if there exists a constant c such that

$$\forall x_1, \cdots, x_n \in E, \qquad \int_0^1 \left\| \sum_{i=1}^n \varepsilon_i(t) x_i \right\| dt \leqq c \left(\sum_{i=1}^n \| x_i \|^p \right)^{1/p}$$

where $(\varepsilon_i(t))$ denotes the usual Rademacher functions on [0, 1].

Let E be a Banach space of type p for some $p > 1$ and let us denote by $\| \ \|_1$ its norm.

We shall denote by $E^{(\mathbf{N})}$ the space of vector valued finitely supported functions on the positive integers $\mathbf{N}$.

For any $x = (x(n)) \in E^{(\mathbf{N})}$ we shall denote $h(x) = \sup_{n \in \mathbf{N}} \| x(n) \|_1$ the altitude of the sequence and

$$p(x) = [h(x)]^{-1} \sum_{n \in \mathbf{N}} \| x(n) - x(n+1) \|_1$$

the variation of x, if $x \neq 0$. If $x = (0)$, we put $p(x) = 0$.

Now consider the functional $[\![\ ]\!]$ defined by

$$[\![x]\!] = \inf \left[\sum_{j=1}^m p(x_j) \left[\left(\sum_{l=j}^m h(x_l) \right)^p - \left(\sum_{l=j+1}^m h(x_l) \right)^p \right] \right]^{1/p} \tag{1.1}$$

where the infimum is taken over all the representations of x as $\Sigma_{j=1}^m x_j$.

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Let us denote by

$$\|x\| = \inf \sum_{j=1}^{m} [\![x_j]\!] \tag{1.2}$$

where the infimum is taken over all possible representations $x = \Sigma_{j=1}^m x_j$. It is easy to check it is a norm. The completion of the space $E^{(\mathbf{N})}$ with the norm given by (1.2) will be denoted by $J_\rho(E)$.

For the sake of completeness we shall go on reproducing word by word the arguments of [1] which show that in order to prove that $J_\rho(E)$ is of type q it is enough to prove that we have

$$\int_0^1 \left\| \sum_{i=1}^{k} \varepsilon_i(t) x_i \right\| dt \leqq C(q) k^{1/q} \tag{1.3}$$

for all $(x_i)_{i=1,\cdots,k}$ such that $[\![x_i]\!] = 1 \quad \forall i$.

As observed in [5] to prove that we have

$$\int_0^1 \| x_1\varepsilon_1(t) + \cdots + x_k\varepsilon_k(t) \| \, dt \leqq C(q,\varepsilon) \left(\sum_{i=1}^{k} \| x_i \|^{q-\varepsilon} \right)^{1/q-\varepsilon} \tag{1.4}$$

for all $\varepsilon > 0$ and all $x_1, \cdots, x_k \in J_\rho(E)$, it is enough to prove

$$\int_0^1 \| x_1\varepsilon_1(t) + \cdots + x_k\varepsilon_k(t) \| \, dt \leqq C(q) k^{1/q} \tag{1.5}$$

in the case where $\| x_1 \| = \cdots = \| x_k \| = 1$.

Another simple reduction is obtained by remarking that it is possible to replace in the assumption $\| \quad \|$ by $[\![\quad]\!]$, i.e., it is enough to show that if $(x_i)_{i=1}^k \in J_\rho(E)$ with $[\![x_i]\!] = 1$ for all i, then (1.5) holds. This follows easily from the remark that the unit ball B of $J_\rho(E)$ is the closed convex hull of the set $A = \{x, [\![x]\!] = 1\}$ and that the function φ on

$$B^k = \underbrace{B \times \cdots \times B}_{k \text{ times}} : \varphi(x_1, \cdots, x_k) = \int_0^1 \left\| \sum_i \varepsilon_i(t) x_i \right\| dt$$

is convex.

In order not to complicate the notations in the proof by using an arbitrary $\delta > 0$ we begin by supposing that the infimum in (1.1) is actually attained.

Hence for every i we have a representation $x_i = \Sigma_{j=1}^{n_i} \tilde{x}_i^j$ in which the infimum in (1.1) is attained. We may assume $\tilde{x}_i^j \neq 0$ for all i and $j = 1, \cdots, n_i$.

In view of the telescopic nature of the right hand side of (1.1), the right side does not change if we break up a term $\tilde{x}_i^j$ into $\lambda \tilde{x}_i^j$ and $(1-\lambda)\tilde{x}_i^j$ where $0 < \lambda < 1$.

We also see, by an easy argument of approximation, that we can also suppose all the $h(\tilde{x}_i^j)$ rational numbers. Hence by breaking up the $\tilde{x}_i^j$ we can suppose that $h(\tilde{x}_i^j)$ is constant, say $h(\tilde{x}_i^j) = 1/N$ for some $N \in \mathbf{N}$. Put $m = \max_{1 \leqq i \leqq k} n_i$, and

$$\begin{aligned} x_i^j &= \tilde{x}_i^{j-(m-n_i)} && \text{if} \quad (m-n_i) < j \leqq m, \\ &= 0 && \text{if} \quad 1 \leqq j \leqq m - n_i. \end{aligned}$$

We see that

$$1 = \sum_{j=1}^{m} p(x_i^j) \left[\left(\frac{m-j+1}{N} \right)^p - \left(\frac{m-j}{N} \right)^p \right].$$

We thus have

$$\text{(1.6)} \qquad \begin{aligned} x_i &= \sum_{j=1}^{m} x_i^j, \\ 1 &= \sum_{j=1}^{m} p(x_i^j) d_j \qquad 1 \leqq i \leqq k \end{aligned}$$

where $d_j = (m - j + 1/N)^p - (m - j/N)^p$. (Note that for every j, there is at least one i with $x_i^j \neq 0$.) Fix now an integer $1 \leqq j \leqq m$ and consider the k vectors $(x_i^j)_{i=1}^k$. Before passing to the proof we shall need two lemmas.

Let us note $\Phi_j^\varepsilon = \sum_{i=1}^k \varepsilon_i x_i^j$ for a given choice of signs and $h(n, j, \varepsilon) = \|\sum \varepsilon_i x_i^j(n)\|_1$. Let us consider the decomposition of $\Phi_j^\varepsilon(n)$ defined (for $\alpha = 1, \cdots, k$) by

$$\Phi_{j\alpha}^\varepsilon(n) = \frac{\sum_{i=1}^{k} \varepsilon_i x_i^j(n)}{Nh(n, j, \varepsilon)} \qquad \text{if} \quad 0 < \alpha \leqq [Nh(n, j, \varepsilon)]$$

$$\Phi_{j\alpha}^\varepsilon(n) = (Nh(n, j, \varepsilon) - [Nh(n, j, \varepsilon)]) \frac{\sum_{i=1}^{k} \varepsilon_i x_i^j(n)}{Nh(n, j, \varepsilon)} \qquad \text{for } \alpha = [Nh(n, j, \varepsilon)] + 1$$

$$\Phi_{j\alpha}^\varepsilon(n) = 0, \qquad 1 + [Nh(n, j, \varepsilon)] < \alpha \leqq k$$

where $[\lambda]$ denotes the integer part of λ.

Now consider n_0' such that $n_0' > \sup\{n, n \in \text{support } x_i^j \text{ for some } i, j\} + 1$. This is possible because the x_i^j are finitely supported. Let y_1 be a vector of norm $1/N$ belonging to E and let $y = (y(n))$ be defined by

$$\begin{aligned} y(n) &= 0 \qquad \text{if } n \neq n_0', \\ y(n_0') &= y_1. \end{aligned}$$

Let us define

$$\Phi^{\varepsilon}_{j\alpha} = (\Phi^{\varepsilon}_{j\alpha}(n)) \qquad \text{if } h(\Phi^{\varepsilon}_{j\alpha}) = \frac{1}{N}, \text{ or } h(\Phi^{\varepsilon}_{j\alpha}) = 0,$$

$$\Phi^{\varepsilon}_{j\alpha} = (\Phi^{\varepsilon}_{j\alpha}(n)) + y \quad \text{if } 0 < h(\Phi^{\varepsilon}_{j\alpha}) < \frac{1}{N},$$

$$\Phi^{\varepsilon}_{j0} = -y$$

if there exists an α such that $0 < h(\Phi^{\varepsilon}_{j\alpha}) < N^{-1}$. (Note there is at most one such α). We easily check that either $h(\Phi^{\varepsilon}_{j\alpha}) = N^{-1}$ or $h(\Phi^{\varepsilon}_{j\alpha}) = 0$ and that $\Sigma^{k}_{\alpha=0}\Phi^{\varepsilon}_{j\alpha}(n) = \Phi^{\varepsilon}_{j}(n)$.

We claim that we have

LEMMA 1. $$\sum_{\alpha=0}^{k} p(\Phi^{\varepsilon}_{j\alpha}) \leqq 8 \cdot \sum_{i=1}^{k} p(x^{j}_{i}).$$

Indeed the remark that for any two vectors u, v belonging to a Banach space we have

$$\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| \leqq 2\frac{\|u-v\|}{\|v\|}$$

followed by a simple computation shows us that we have

$$\forall n \neq n'_0 \qquad \sum_{\alpha=1}^{k} \|\Phi^{\varepsilon}_{j\alpha}(n) - \Phi^{\varepsilon}_{j\alpha}(n+1)\| \leqq 4\|\Phi^{\varepsilon}_{j}(n) - \Phi^{\varepsilon}_{j}(n+1)\|.$$

Indeed in $\Sigma^{k}_{\alpha=1}\|\Phi^{\varepsilon}_{j\alpha}(n) - \Phi^{\varepsilon}_{j\alpha}(n+1)\|$ we have, if we suppose for example $h(n+1, j, \varepsilon) \leqq h(n, j, \varepsilon)$, $[Nh(n+1, j, \varepsilon)]$ terms of the form $N^{-1}(u/\|u\| - v/\|v\|)$ which give us a contribution smaller than

$$\frac{2h(n+1,j,\varepsilon)}{h(n+1,j,\varepsilon)}\|\Phi^{\varepsilon}_{j}(n) - \Phi^{\varepsilon}_{j}(n+1)\| \leqq 2\|\Phi^{\varepsilon}_{j}(n) - \Phi^{\varepsilon}_{j}(n+1)\|,$$

and one term of the form

$$\frac{1}{N}\left\| \frac{u}{\|u\|} - \frac{\alpha v}{\|v\|} \right\|$$

where $\alpha = Nh(n+1, j, \varepsilon) - [Nh(n+1, j, \varepsilon)]$ if $[Nh(n, j, \varepsilon)] > [Nh(n+1, j, \varepsilon)]$ which gives us a contribution smaller than

$$\frac{1}{N}\left\| \frac{u}{\|u\|} - \frac{v}{\|v\|} \right\| + \frac{(1-\alpha)}{N},$$

but in this case

$$(1-\alpha) \leqq h(n,j,\varepsilon) - h(n+1,j,\varepsilon) \leqq \| \Phi_j^\varepsilon(n) - \Phi_j^\varepsilon(n+1) \|,$$

the contribution of the other terms is

$$(Nh(n,j,\varepsilon) - [Nh(n+1,j,\varepsilon)] + 1)\frac{1}{N} \leqq h(n,j,\varepsilon) - h(n+1,j,\varepsilon)$$

$$\leqq \| \Phi_j^\varepsilon(n) - \Phi_j^\varepsilon(n+1) \|,$$

and we get the result in this case. The computation in the other cases is the same. Hence we get

$$\begin{aligned}\sum_{\alpha=0}^{k} p(\Phi_{j\alpha}^\varepsilon) &= N \sum_{\alpha=0}^{k} \sum_{n\in\mathbf{N}} \| \Phi_{j\alpha}^\varepsilon(n) - \Phi_{j\alpha}^\varepsilon(n+1) \|_1 \\ &\leqq 2 + 4N \sum_{n\in\mathbf{N}} \sum_{\alpha=1}^{k} \| \Phi_j^\varepsilon(n) - \Phi_j^\varepsilon(n+1) \|_1 + 2 \\ &\leqq 4 + 4N \sum_{n\in\mathbf{N}} \left\| \sum_{i=1}^{k} \varepsilon_i [x_i^j(n) - x_i^j(n+1)] \right\|_1 \\ &\leqq 4 + 4 \sum_{i=1}^{k} \sum_{n\in\mathbf{N}} N \| x_i^j(n) - x_i^j(n+1) \|_1 \\ &\leqq 4 + 4 \sum_{i=1}^{k} p(x_i^j).\end{aligned}$$

But $p(x_i^j) \geqq 1$ for at least one $i \in \{1, \cdots, k\}$. Hence we have

$$\sum_{\alpha=0}^{k} p(\Phi_{j\alpha}^\varepsilon) \leqq 8 \sum_{i=1}^{k} p(x_i^j)$$

and Lemma 1 is proved.

It is a well known fact, by a result of Kahane [4, p. 17], that there exists a constant c such that, $\forall x_1, \cdots, x_n \in E$, E Banach space, satisfying $\int_0^1 \| \Sigma_{i=1}^n \varepsilon_i(t) x_i \| \, dt \leqq 1$, we have $\int_0^1 \exp(c \| \Sigma_{i=1}^n \varepsilon_i(t) x_i \| \, dt \leqq e$. Hence we have, if E is of type p and $x_1, \cdots, x_k$ are elements of E such that $\| x_i \|_1 = 1$, applying the Tchebyschev inequality,

$$p\left\{ t, \left\| \sum_{i=1}^{k} \varepsilon_i(t) x_i \right\| \geqq \frac{\mu k^{1/p}}{D} \right\} \leqq \exp(-\mu) \tag{1.7}$$

where D^{-1} is equal to $1/c$ multiplied by twice the type p constant of the space. We are now able to prove:

Lemma 2. *If $\alpha \geqq \mu k^{1/p} D^{-1} + 1$ we have $Ep(\Phi_{j\alpha}^\varepsilon) \leqq 14 \exp(-\mu)(\Sigma_{i=1}^k p(x_i^j))$, where $E(f)$ denotes the mean value $\int_0^1 f(t)\,dt$.*

Indeed, by assumption we have

$$N \sum_{n\in\mathbf{N}} \left(\sum_{i=1}^{k} \| x_i^j(n) - x_i^j(n+1) \|_1 \right) = \sum_{i=1}^{k} p(x_i^j).$$

We define by induction:
$n_0 = 0$,

n_1 is the least integer such that $N \sum_{n=0}^{n_1} \left(\sum_{i=1}^{k} \| x_i^j(n) - x_i^j(n+1) \|_1 \right) > 1$,

n_2 is the least integer such that $N \sum_{n_1+1}^{n_2} \left(\sum_{i=1}^{k} \| x_i^j(n) - x_i^j(n+1) \|_1 \right) > 1$.

We thus determine m integers $n_1, \cdots, n_m$ and it is a trivial fact we have $m \leqq \sum_{i=1}^{k} p(x_i^j)$. Let us denote $I_0, I_1, \cdots, I_m$ the following interval of integers:

$$I_j = \{ n_j + 1, \cdots, n_{j+1} - 1 \} \qquad \text{if} \quad n_{j+1} > n_j + 1,$$

$$I_{m+1} = \{ n_m, \cdots, n_0' - 1 \}.$$

By construction on I_j the following holds:

$$N \left[\sum_{n\in I_j} \left(\sum_{i=1}^{k} \| x_i^j(n) - x_i^j(n+1) \| \right) \right] \leqq 1$$

so that if for some $n \in I_j$ we have

$$\left\| \sum_{i=1}^{k} \varepsilon_i x_i^j(n) \right\| \geqq \frac{k'+1}{N},$$

then for all the integers belonging to I_j we have

$$\left\| \sum_{i=1}^{k} \varepsilon_i x_i^j(n) \right\|_1 \geqq \frac{k'}{N}.$$

By the proof of Lemma 1 we have

$$\sum_{n\in I_j} \left(\sum_{\alpha=1}^{k} \| \Phi_{j\alpha}^{\varepsilon}(n) - \Phi_{j\alpha}^{\varepsilon}(n+1) \| \right) \leqq 4 \sum_{n\in I_j} \| \Phi_j^{\varepsilon}(n) - \Phi_j^{\varepsilon}(n+1) \|$$

$$\leqq \frac{4}{N}.$$

Let $n_l^j \in I_l$. Now the reader will convince himself (not so easily), after having remarked that for $\alpha \geqq k'+1$ we have the equality:

$$\Big(\sum_{n\in I_j}\|\Phi^{\varepsilon}_{j\alpha}(n)-\Phi^{\varepsilon}_{j\alpha}(n+1)\|\Big)1\Big\{\Big\|\sum\varepsilon_i x^j_i(n'_j)\Big\|>k'/N\Big\}$$

$$=\sum_{n\in I_j}\|\Phi^{\varepsilon}_{j\alpha}(n)-\Phi^{\varepsilon}_{j\alpha}(n+1)\|,$$

that we have

$$p(\Phi^{\varepsilon}_{j\alpha})\leqq 8\sum_{l=1}^{m+1}1\Big\{\Big\|\sum_{i=1}^{k}\varepsilon_i x^j_i(n'_l)\Big\|\geqq\frac{k'}{N}\Big\}$$

$$+4\sum_{l=1}^{m+1}1\Big\{\Big\|\sum_{i=1}^{k}\varepsilon_i x^j_i(n_l)\Big\|\geqq\frac{k'}{N}\Big\},$$

where $1(A)$ denotes the characteristic function of the set A. (We have multiplied the constants by 2 to take into account the fact that for at most one α we have $\Phi^{\varepsilon}_{j\alpha}=(\Phi^{\varepsilon}_{j\alpha}(n))+y$. By a more precise computation we could have obtained better constants, but it is of no importance since the estimate of Lemma 2 is already far better than the one of Lemma 1.) Hence we get for $\alpha\geqq\mu k^{1/p}D^{-1}+1$,

$$Ep(\Phi^{j}_{\varepsilon\alpha})\leqq 8\sum_{j=1}^{m+1}p\Big(\Big\|\sum_{i=1}^{k}\varepsilon_i x^j_i(n'_j)\Big\|_1\geqq\frac{\mu k^{1/p}}{DN}\Big)$$

$$+4\sum_{j=1}^{m+1}p\Big(\Big\|\sum_{i=1}^{k}\varepsilon_i x^j_i(n_j)\Big\|_1\geqq\frac{\mu k^{1/p}}{DN}\Big),$$

and by (1.7) we have

$$Ep(\Phi^{j}_{\varepsilon\alpha})\leqq\exp(-\mu)12(m+1)$$

$$\leqq 24\Big(\sum_{i=1}^{k}p(x^j_i)\Big)\exp(-\mu),$$

and Lemma 2 is proved.

Let us denote

$$\Phi^{\varepsilon}_{\alpha}=\sum_{j=1}^{m}\Phi^{\varepsilon}_{j\alpha}.$$

We have by the definition of $\|\ \|$, $[\![\ ]\!]$, and the remark that the function $f(t)=(t+N^{-1})^p-(t)^p$ is increasing which shows that we increase the value of $[\![x]\!]$ if in a representation of $x\,\Sigma\,x_i$ we allow dummy summands; counting their altitudes for $1/N$ and their variation for 0

$$\|\Phi^{\varepsilon}_{\alpha}\|^p\leqq\sum_{j=1}^{m}p(\Phi^{\varepsilon}_{j\alpha})d_j$$

and hence we get by Lemma 1

$$
\begin{aligned}
\sum_{\alpha=0}^{k} E\,\|\Phi_\alpha^\varepsilon\|^\rho &\leqq \sum_{\alpha=0}^{k}\sum_{j=1}^{m} Ep(\Phi_{j\alpha}^\varepsilon)d_j \\
&\leqq \sum_{j=1}^{m}\left(\sum_{\alpha=0}^{k} Ep(\Phi_{j\alpha}^\varepsilon)\right)d_j \\
&\leqq 8\sum_{j=1}^{m}\sum_{i=1}^{k} p(x_i^j)\,d_j \qquad (1.8)\\
&\leqq 8k.
\end{aligned}
$$

Now, if $\alpha \geqq \mu k^{1/\rho}D^{-1}+1$, we get by Lemma 2

$$
\begin{aligned}
E\,\|\Phi_\alpha^\varepsilon\| &\leqq \left[\int_0^1 \|\Phi_\alpha^\varepsilon\|^\rho\,dt\right]^{1/\rho} \\
&\leqq \left[\sum_{j=1}^{m}\left[\int p(\Phi_{\alpha j}^\varepsilon)\,dt\right]dj\right]^{1/\rho} \\
&\leqq 24^{1/\rho}\exp\left(-\frac{\mu}{\rho}\right)\left[\sum_{j=1}^{m}\left(\sum_{i=1}^{k} p(x_i^j)\right)dj\right]^{1/\rho} \\
&\leqq 24^{1/\rho}k^{1/\rho}\exp\left(-\frac{\mu}{\rho}\right),
\end{aligned}
$$

and so we have

$$
\sum_{\alpha \geqq \mu k^{1/\rho}/D+1} E\,\|\Phi_\alpha^\varepsilon\| \leqq 24^{1/\rho}k^{1+1/\rho}\exp\left(-\frac{\mu}{\rho}\right). \tag{1.9}
$$

We now deduce from (1.8) and (1.9) by applying Hölder's inequality

$$
\begin{aligned}
\sum_{\alpha=0}^{k} E\,\|\Phi_\alpha^\varepsilon\| &= \sum_{\alpha \leqq \mu k^{1/\rho}/D} E\,\|\Phi_\alpha^\varepsilon\| + \sum_{\alpha \geqq \mu k^{1/\rho}/D+1} E\,\|\Phi_\alpha^\varepsilon\| \\
&\leqq \frac{\mu^{1/\rho'}k^{1/\rho\rho'}}{G^{1/\rho'}}\left(\sum_{\alpha \leqq \mu k^{1/\rho}/D} E\,\|\Phi_\alpha^\varepsilon\|^\rho\right)^{1/\rho} + \sum_{\alpha \geqq \mu k^{1/\rho}/D+1} E\,\|\Phi_\alpha^\varepsilon\|,
\end{aligned}
$$

where $\rho'^{-1} = 1-\rho^{-1}$. Hence we get

$$
\sum_{\alpha=0}^{k} E\,\|\Phi_\alpha^\varepsilon\| \leqq K\mu^{1/\rho'}k^{1/\rho\rho'}k^{1/\rho} + K'k^{1+1/\rho}\exp\left(-\frac{\mu}{\rho}\right)
$$

for some constant K and K'.

The choice of $\mu = \rho(1+1/\rho)\log k$ gives us

$$
\sum_{\alpha=0}^{k} E\,\|\Phi_\alpha^\varepsilon\| \leqq K''k^{1/\rho\rho'+1/\rho}(\log k)^{1/\rho'} \tag{1.10}
$$

for some constant K'' depending only on ρ. But as we have

$$\left\| \sum_{i=1}^{k} \varepsilon_i x_i \right\| = \left\| \sum_{\alpha=0}^{k} \Phi_\alpha^\varepsilon \right\| \leq \sum_{\alpha=0}^{k} \| \Phi_\alpha^\varepsilon \|,$$

we thus have proved

$$\int_0^1 \left\| \sum_{i=1}^{k} \varepsilon_i(t) x_i \right\| dt \leq K'' k^{1/p\rho' + 1/\rho} (\log k)^{1/\rho'}$$

and so we have proved:

THEOREM 1. *Whenever E is a Banach space of type p, $J_\rho(E)$ is of type q for all q such that $q^{-1} > (p\rho')^{-1} + \rho^{-1}$.*

It is easy to check that for the norm on $J_\rho(E)$ we have, if $x = (x(n)) \in J_\rho(E)$,

$$\| x \| \geq \sup_{n \in \mathbf{N}} \| x(n) \|_1.$$

Indeed it is sufficient to check that $[\![x]\!] \geq \sup \| x(n) \|_1$. But if we have a representation of $x = \sum_{i=1}^m x_i$ we do not change $[\![x]\!]$ if we eliminate the x_i's such that $x_i = 0$. Hence we get that in (1.1) we can suppose $p(x_i) \geq 1$. We thus have

$$[\![x]\!] \geq \sum_{i=1}^{m} h(x_i) \geq h\left(\sum_{i=1}^{m} x_i \right) = \sup_{n \in \mathbf{N}} \| x(n) \|_1 .$$

Taking the trivial representation we see that if $x = (x(n))$, where $x(1) = x(2) = \cdots = x(n_0) = x_0$, $x(n) = 0$ if $n > n_0$, we have

$$\| x \| = \| x_0 \|_1.$$

We shall now prove:

THEOREM 2. *Assume E has a k structure, then $J_\rho(E)$ has a $(k+1)$ structure.*

Indeed let $x_{i_1,\cdots,i_k} \subset E$, $f_{i_1,\cdots,i_k} \subset E^*$ be a biorthogonal system such that

$$\| f_{i_1,\cdots,i_k} \|_1 \leq M$$

$$\left\| \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_k=1}^{r_k} x_{i_1}, \cdots, x_{i_k} \right\|_1 \leq M,$$

for all $r_1, \cdots, r_k$.

Let us denote by $x_{i_1,\cdots,i_k,i_{k+1}}$ the element of $J_\rho(E)$ defined by

$$x_{i_1,\cdots,i_k,i_{k+1}} = (x(n))$$

where

$$x(n) = \delta_n^{i_{k+1}} \cdot x_{i_1,\cdots,i_k}.$$

Then we have, $\forall \alpha_{i_1,\cdots,i_k,i_{k+1}} \in \mathbf{R}$,

$$\left\| \sum \alpha_{i_1,\cdots,i_k,i_{k+1}} x_{i_1,\cdots,i_k,i_{k+1}} \right\| \geqq \sup_{i_{k+1}\in \mathbf{N}} \left\| \sum \alpha_{i_1,\cdots,i_k,i_{k+1}} x_{i_1,\cdots,i_k} \right\|$$

$$\geqq \sup_{i_{k+1}\in \mathbf{N}} \frac{1}{M} \sup \left| \alpha_{i_1,\cdots,i_k,i_{k+1}} \right| = \frac{1}{M} \sup_{i_1,\cdots,i_k,i_{k+1}} \left| \alpha_{i_1,\cdots,i_{k+1}} \right|$$

(by the assumption of k structure).

Hence by Helly's theorem there exists $f_{i_1,\cdots,i_k,i_{k+1}} \in J_\rho(E)^*$ of norm less than M such that $(f_{i_1,\cdots,i_{k+1}}; x_{i_1,\cdots,i_{k+1}})$ is a biorthogonal system. Moreover it is easy to check, using $\| x \| \leqq \Sigma \| x(n) - x(n+1) \|_1$,

$$\left\| \sum_{i_1=1}^{r_1} \sum_{i_2=1}^{r_2} \cdots \sum_{i_{k+1}=1}^{r_{k+1}} x_{i_1,\cdots,i_{k+1}} \right\| \leqq M,$$

which proves Theorem 2.

In view of the results of [1] which assert that for every $p < 2$ there exists a Banach space of type p and having a 1 structure (in fact it is $J_\rho(\mathbf{R})$), we thus get:

THEOREM 3. *For every $p < 2$, there exists a Banach space of type p and having a k structure.*

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