ISRAEL JOURNAL OF MATHEMATICS, Vol. 28, Nos. 1-2, 1977

THE PLANCHEREL FORMULA FOR PARABOLIC SUBGROUPS

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ABSTRACT

We prove an explicit Plancherel Formula for the parabolic subgroups of the simple Lie groups of real rank one. The key point of the formula is that the operator which compensates lack of unimodularity is given, not as a family of implicitly defined operators on the representation spaces, but rather as an explicit pseudo-differential operator on the group itself. That operator is a fractional power of the Laplacian of the center of the unipotent radical, and the proof of our formula is based on the study of its analytic properties and its interaction with the group operations.

1. Introduction

The Plancherel Theorem for non-unimodular groups has been developed and studied rather intensively during the past five years (see [7], [11], [8], [4]). Furthermore there has been significant progress in the computation of the ingredients of the theorem (see [5], [2]) — at least in the case of solvable groups. In this paper we shall give a completely explicit description of these ingredients for an interesting family of non-solvable groups. The groups we consider are the parabolic subgroups MAN of the real rank 1 simple Lie groups. As an intermediate step we also obtain the Plancherel formula for the exponential solvable groups AN (see also [6]). In that case our results are more extensive than those of [5] since in addition to the "infinitesimal" unbounded operators we also obtain explicitly the "global" unbounded operator on $L_2(AN)$. These global operators turn out to be fractional powers of the Laplacian on certain manifolds — in particular they are pseudo-differential operators.

1a. *The Non-Unimodular Plancherel Theorem.* Let G be a locally compact group with right Haar measure dg. Define the modular function δ as usual by

† Partially supported by NSF Grant 33039 and a NATO fellowship

‡ Partially supported by NSF Grant MPS74 01477

Received October 19, 1976

$$\delta(g)\int_G f(gx)\,dx = \int_G f(x)\,dx.$$

The left and right regular representations of G on $L_2(G) = L_2(G, dg)$ are defined by

$$\lambda_g f(x) = \delta^{-1/2}(g) f(g^{-1}x), \quad \rho_g f(x) = f(xg), \qquad x, g \in G, \quad f \in L_2(G).$$

Write $\hat{G}$ for the unitary equivalence classes of irreducible unitary representations of G with the Mackey Borel structure. We summarize the key points of the Non-Unimodular Plancherel Theorem in

1.1 THEOREM. *Let G be type* I. *Then there exist: a positive standard Borel measure $\mu = \mu_G$ on $\hat{G}$, a μ-measurable field $(\pi_\zeta, \mathcal{H}_\zeta)_{\zeta\in\hat{G}}$ of unitary representations of G such that $\pi_\zeta \in \zeta$ for μ-almost all $\zeta \in \hat{G}$, a μ-measurable field $(D_\zeta)_{\zeta\in\hat{G}}$ of non-zero positive self-adjoint operators such that D_ζ is a semi-invariant of weight δ in $\mathcal{H}_\zeta$ for μ-almost all $\zeta \in \hat{G}$, with the following properties.*

(i) *If $f \in L_1(G) \cap L_2(G)$, then $D_\zeta^{1/2}\pi_\zeta(f)$ is Hilbert–Schmidt for μ-almost all $\zeta \in \hat{G}$. If $f \in C_c^\infty(G)$, then $D_\zeta^{1/2}\pi_\zeta(f)D_\zeta^{1/2}$ is trace class for μ-almost all $\zeta \in \hat{G}$.*

(ii) *The map $f \to D_\zeta^{1/2}\pi_\zeta(f)$ extends to an isometry of $L_2(G)$ onto $\int_{\zeta\in\hat{G}}^{\oplus} \mathcal{H}_\zeta \otimes \bar{\mathcal{H}}_\zeta\, d\mu(\zeta)$ so as to intertwine λ with $\int^{\oplus} \pi_\zeta \otimes 1_\zeta\, d\mu(\zeta)$, and ρ with $\int^{\oplus} 1_\zeta \otimes \bar{\pi}_\zeta\, d\mu(\zeta)$.*

(iii) *The operators D_ζ are unique up to scalars (depending on ζ), and the quantity $D_\zeta^{1/2} d\mu(\zeta)$ is uniquely determined up to a scalar (depending only on the normalization of Haar measure).*

To say that D_ζ is a semi-invariant of weight δ means

$$\pi_\zeta(g) D_\zeta \pi_\zeta(g)^{-1} = \delta(g) D_\zeta, \qquad g \in G, \tag{1.2}$$

which in turn implies

$$\pi_\zeta(f) D_\zeta = D_\zeta \pi_\zeta(\delta f), \qquad f \in C_c^\infty(G). \tag{1.3}$$

Implicit in the statement of Theorem 1.1 are the equations

$$\int_G |f(g)|^2\, dg = \int_{\hat{G}} \|D_\zeta^{1/2}\pi_\zeta(f)\|_2^2\, d\mu(\zeta), \qquad f \in L_1(G) \cap L_2(G), \tag{1.4}$$

$$f(1_G) = \int_{\hat{G}} \mathrm{Tr}(D_\zeta^{1/2}\pi_\zeta(f)D_\zeta^{1/2})\, d\mu(\zeta), \qquad f \in C_c^\infty(G). \tag{1.5}$$

We may use (1.3) to rewrite (1.5) as

$$f(1_G) = \int_{\hat{G}} \mathrm{Tr}(D_\zeta \pi_\zeta(\delta^{1/2} f))\, d\mu(\zeta), \qquad f \in C_c^\infty(G). \tag{1.6}$$

Replacing f by $\delta^{-1/2}f$ in (1.6) we obtain

$$f(1_G) = \int_{\hat{G}} \operatorname{Tr}(D_\zeta \pi_\zeta(f))\, d\mu(\zeta), \qquad f \in C_c^\infty(G). \tag{1.7}$$

We remark at this point that the operators D_ζ have been computed explicitly in [5], [2] for simply connected solvable type I Lie groups.

Now it is known from [7] that the operators D_ζ must be the infinitesimal components of a positive self-adjoint unbounded invertible operator D on $L_2(G)$ which is affiliated with the left ring of G. (In the notation of [7, §6], $D = (M')^{-2}$). Therefore $\pi_\zeta(D)$ is defined for μ-almost all $\zeta \in \hat{G}$, and formula (1.7) may now be rewritten

$$f(1_G) = \int_{\hat{G}} \operatorname{Tr} \pi_\zeta(Df)\, d\mu(\zeta), \qquad f \in C_c^\infty(G) \cap \operatorname{Dom}(D) \cap D^{-1}(L_1(G)). \tag{1.8}$$

It is not hard to show (using part (iii) of Theorem 1.1) that D must be a semi-invariant of weight δ for G, that is $\operatorname{Ad}(g)D = \delta(g)D$, where $[\operatorname{Ad}(g)D]f = \operatorname{Ad}(g)[D(\operatorname{Ad}(g)^{-1}f)]$ and $\operatorname{Ad}(g)f(x) = f(g^{-1}xg)$. But beyond that one can say nothing in general about D. Naturally one would like to find an explicit description of D, and to show in particular that $C_c^\infty(G) \cap \operatorname{Dom}(D) \cap D^{-1}(L_1(G))$ is a "nice big" space — ideally that

$$C_c^\infty(G) \cap \operatorname{Dom}(D) \cap D^{-1}(L_1(G)) = C_c^\infty(G). \tag{1.9}$$

At this time the only result on $\operatorname{Dom}(D)$ is in the solvable case (see [10, lemma 7.4]), and an explicit formula for D is known essentially only for the $ax + b$ group (where D is differentiation with respect to the nilpotent variable). We shall compute D explicitly for the groups AN and MAN. In all cases D will be seen to be a pseudo-differential operator defined by a fractional power of a certain Laplacian, and (1.9) will always be valid.

1b. *Statement of Results.* Let G be a connected simple Lie group of finite center and $\mathbf{R}$-rank 1. Fix an Iwasawa decomposition $G = KAN$ and consider the corresponding minimal parabolic subgroup $P = MAN$. In [6] one of us wrote out the Plancherel formula for the non-unimodular group AN, modulo some technical results which can be found in §2 below. We recall that result in a moment, and indicate its extension to the parabolic subgroup P. In §3 we reformulate (and reprove by a method different from [6]) the Plancherel formula for AN; and in §4 we prove our Plancherel formula for P.

These Plancherel formulae go as follows. Let Z be the center of the nilpotent group N and set

$$k = \dim Z, \qquad l = \dim N/Z, \qquad q = k + \tfrac{1}{2}l.$$

We will have a positive-definite inner product on the vector group Z and that will give us a Laplacian

$$\Delta = -\sum \frac{\partial^2}{\partial z_i^2}$$

on Z. We will also have particular diffeomorphic splittings of AN and $P = MAN$ in which Z is a factor, and they define pseudo-differential operators

$$D = c_1\Delta^{q/2} \quad \text{on} \quad AN, \qquad E = c_2\Delta^{q/2} \quad \text{on} \quad MAN$$

(see (2.5) below for the constants). The generic irreducible unitary representation classes $[\eta_\lambda]$ of AN are parameterized by the unit sphere S in the dual $\mathfrak{z}^*$ of the Lie algebra $\mathfrak{z}$ of Z. The Plancherel formula for AN says: if $f \in C_c^\infty(AN)$, then (i) $Df \in L_1(AN)$, so that $\eta_\lambda(Df)$ is defined; (ii) $\eta_\lambda(Df)$ is trace class; (iii) $\operatorname{Tr}\eta_\lambda(Df)$ is a C^∞ function of $\lambda \in S$; and (iv) we have

$$f(1_{AN}) = \int_S \operatorname{Tr}\eta_\lambda(Df)\,d\sigma(\lambda) \tag{1.10}$$

where σ is the standard volume element on S.

The Mackey little group method produces unitary representation classes $[\pi_{\lambda,\tau}] \in (MAN)^\wedge$, where λ ranges over a set of representatives of the M-orbits on S and $[\tau]$ ranges over the unitary dual of $M_\lambda = \{m \in M\colon \mathrm{Ad}^*(m)\lambda = \lambda\}$. If $k > 1$ then M is transitive on S; so we fix $\lambda_1 \in S$, write π_τ for $\pi_{\lambda_1,\tau}$ and M_1 for M_{λ_1}, and prove

$$f(1_P) = \sum_{\tau \in \hat{M}_1} (\dim\tau)\operatorname{Tr}\pi_\tau(Ef), \qquad f \in C_c^\infty(P). \tag{1.11}$$

If $k = 1$, then S consists of two M-fixed points and the formula is

$$f(1_P) = \frac{1}{2}\sum_{\tau\in\hat{M}} (\dim\tau)\operatorname{Tr}(\pi_\tau^+(Ef) + \pi_\tau^-(Ef)), \qquad f \in C_c^\infty(P). \tag{1.12}$$

We shall now delineate very explicitly the structure of our groups, their Haar measures and their representations. This will facilitate our proofs of the formulae (1.10), (1.11), (1.12) in §§3, 4.

Let $\mathbf{F}$ be one of the division algebras: $\mathbf{R}$ (real numbers), $\mathbf{C}$ (complex numbers), $\mathbf{Q}$ (quaternions), or Cay (Cayley numbers). $\mathbf{F}^{r,s}$ denotes the right vector space of $(r+s)$-tuples from $\mathbf{F}$ with "hermitian" scalar product $\langle x, y\rangle = -\sum_{j=1}^{r} x_j\bar{y}_j + \sum_{j=r+1}^{r+s} x_j\bar{y}_j$, and $\mathbf{F}^n$ means $\mathbf{F}^{0,n}$, $n \geqq 1$. Up to local isomorphism, G is one of the groups:

$SO(1, n+1)$ = identity component of orthogonal group of $\mathbf{R}^{1,n+1}$;
$SU(1, n+1)$ = special (determinant 1) unitary group of $\mathbf{C}^{1,n+1}$;
$Sp(1, n+1)$ = symplectic (quaternion-unitary) group of $\mathbf{Q}^{1,n+1}$;
$F_{4(-20)}$ = real exceptional group of type F_4 with maximal compact subgroup Spin (9).

K is a maximal compact subgroup of G, A is isomorphic to the multiplicative group $\mathbf{R}_+^*$ of positive reals, and M is the centralizer of A in K. With G as above, $M \subseteq K \subseteq G$ in the various cases is:

$$\begin{array}{rll} SO(n) \subseteq & SO(n+1) & \subseteq SO(1, n+1) \\ \mathbf{Z}_{n+1} \times U(n) \subseteq & U(n+1) & \subseteq SU(1, n+1) \\ Sp(1) \times Sp(n) \subseteq & Sp(1) \times Sp(n+1) & \subseteq Sp(1, n+1) \\ \mathrm{Spin}(7) \subseteq & \mathrm{Spin}(9) & \subseteq F_{4(-20)}. \end{array}$$

The group N is a maximal unipotent subgroup of G. In case $\mathbf{F} = \mathbf{R}$, then $N \cong \mathbf{R}^n$. In the other cases $N \cong \operatorname{Im} \mathbf{F} \oplus \mathbf{F}^n$ with the product

$$(z, x)(z', x') = (z + z' + \tfrac{1}{2}\operatorname{Im}\langle x, x'\rangle, x + x').$$

Thus we have the table:

G	N	Z	k	l	q
$SO(1, n+1)$	$\mathbf{R}^n$	$\mathbf{R}^n$	n	0	n
$SU(1, n+1)$	$\operatorname{Im}\mathbf{C} + \mathbf{C}^n$	$\operatorname{Im}\mathbf{C}$	1	$2n$	$1+n$
$Sp(1, n+1)$	$\operatorname{Im}\mathbf{Q} + \mathbf{Q}^n$	$\operatorname{Im}\mathbf{Q}$	3	$4n$	$3+2n$
$F_{4(-20)}$	$\operatorname{Im}\mathbf{Cay} + \mathbf{Cay}$	$\operatorname{Im}\mathbf{Cay}$	7	8	11

Because we use right Haar measure, it is more convenient to write NA instead of AN, and $P = NAM$ instead of MAN. We adhere to that conyention throughout the rest of the paper.

$A = \{a_r : r \in R_+^*\}$ acts on N by automorphisms $a_r : (z, x) \to (r^2 z, rx)$. M acts on N by automorphisms $m : (z, x) \to (\nu(m)z, \mu(m)x)$ as follows. First, μ is:

the usual representation of $SO(n)$ on $\mathbf{R}^n$;

the representation $\mu(s, t): x \to sx\bar{t}$ of $\mathbf{Z}_{n+1} \times U(n)$ on $\mathbf{C}^n$;

the representation $\mu(s, t): x \to sx\bar{t}$ of $Sp(1) \times Sp(n)$ on $\mathbf{Q}^n$;

the spin representation of Spin (7) on $\mathbf{Cay}^1 \cong \mathbf{R}^8$.

Here $\mathbf{Z}_{n+1} \subseteq \mathbf{C}$ and $\mathrm{Sp}(1) \subseteq \mathbf{Q}$ act as multiplicative subgroups. Second, ν is:

the trivial representation of $SO(n)$ on $\mathrm{Im}\,\mathbf{R} = \{0\}$;

the trivial representation of $\mathbf{Z}_{n+1} \times U(n)$ on $\mathrm{Im}\,\mathbf{C}$;

the representation $\nu(s,t)$: $z \to sz\bar{s}$ of $\mathrm{Sp}(1) \times \mathrm{Sp}(n)$ on $\mathrm{Im}\,\mathbf{Q}$;

the vector representation of $\mathrm{Spin}(7)$ on $\mathrm{Im}\,\mathrm{Cay} \cong \mathbf{R}^7$.

(The reader is referred to [12, lemma 8.8] for these facts in case $G = F_{4(-20)}$.)

Now the parabolic group $P = NAM$ is expressed as $\mathrm{Im}\,\mathbf{F} \times \mathbf{F}^n \times \mathbf{R}_+^* \times M$ with multiplication given by

$$(1.13)\qquad \begin{aligned}&(z, x, a_r, m)(z', x', a_{r'}, m')\\ &= (z + r^2\nu(m)z' + \tfrac{1}{2}\mathrm{Im}\langle x, r\mu(m)x'\rangle, x + r\mu(m)x', a_{rr'}, mm').\end{aligned}$$

If G is replaced by a locally isomorphic group, then M changes but formula (1.13) remains valid.

We now normalize once and for all Haar measures. We choose dz = Lebesgue measure on Z, dx = Lebesgue measure on $\mathbf{F}^n$, $da_r = dr/r$ where dr is Lebesgue measure on R_+^*, and dm = normalized Haar measure on M. Then $dn = dzdx$ is right Haar measure on N (in case $\mathbf{F} = \mathbf{R}$, $dn = dz = dx$), $dnda$ is right Haar measure on NA and $dndadm$ is right Haar measure on $P = NAM$. The modular function of P is easily computed from (1.13)—indeed if $a = a_r$

$$(1.14)\qquad \delta_P(nam) = \delta_{NA}(na) = \delta(a) = \begin{cases} r^k & \mathbf{F} = \mathbf{R} \\ r^{2k+l} & \mathbf{F} \neq \mathbf{R}. \end{cases}$$

We conclude this introductory section with a description of the generic irreducible unitary representations of the groups N, NA and $P = NAM$. Let λ be any non-zero element of $\mathfrak{z}^*$. We shall commit an abuse of notation by writing $\lambda(z) = e^{i\lambda(\log z)}$, so that λ also denotes the corresponding non-trivial unitary character of Z. Associated to each $\lambda \in \mathfrak{z}^* - \{0\}$, there exists an irreducible unitary representation class $[\gamma_\lambda]$ of N, uniquely determined by the property $\gamma_\lambda(zn) = \lambda(z)\gamma_\lambda(n)$, $z \in Z$, $n \in N$. Moreover $\lambda \neq \lambda'$ implies $[\gamma_\lambda] \neq [\gamma_{\lambda'}]$. These are the generic representations of N. The generic representations of NA are obtained by induction. Since

$$a_r \cdot [\gamma_\lambda] = \begin{cases} [\gamma_{r^{-1}\lambda}], & \mathbf{F} = \mathbf{R} \\ [\gamma_{r^{-2}\lambda}], & \mathbf{F} \neq \mathbf{R}, \end{cases}$$

we have that the representations

$$\eta_\lambda = \mathrm{Ind}_N^{NA}\gamma_\lambda, \qquad \lambda \in \mathfrak{z}^* - \{0\}$$

are irreducible; and $[\eta_\lambda] = [\eta_{\lambda'}]$ iff $\lambda = r\lambda'$, $r > 0$. The generic representations of NA are thus parameterized by the unit sphere $S = S^{k-1}$ in $\mathfrak{z}^*$. Finally conjugation by $m \in M$ commutes with induction from N to NA. Thus it sends:

$$[\gamma_\lambda] \to [\gamma_{\mu(m)^*\lambda}], \quad [\eta_\lambda] \to [\eta_{\mu(m)^*\lambda}], \quad \mathbf{F} = \mathbf{R}$$

$$[\gamma_\lambda] \to [\gamma_{\nu(m)^*\lambda}], \quad [\eta_\lambda] \to [\eta_{\nu(m)^*\lambda}], \quad \mathbf{F} \neq \mathbf{R}.$$

The M-stabilizer of both $[\gamma_\lambda]$ and $[\eta_\lambda]$ is

$$M_\lambda = \begin{cases} \{m \in M \colon \mu(m)^*\lambda = \lambda\}, & \mathbf{F} = \mathbf{R} \\ \{m \in M \colon \nu(m)^*\lambda = \lambda\}, & \mathbf{F} \neq \mathbf{R} \end{cases}$$

From the definitions of μ and ν, we see immediately that $M_\lambda \cong \mathrm{SO}(n-1)$, $Z_{n+1} \times \mathrm{U}(n)$, $U(1) \times \mathrm{Sp}(n)$, or Spin (6), respectively, as $\mathbf{F} = \mathbf{R}$, $\mathbf{C}$, $\mathbf{Q}$ or Cay. Also M is transitive on S, except in the case $k = 1$ where S consists of two M-fixed points. Now we need

1.15 Lemma. *The representation η_λ of NA extends to an ordinary representation $\tilde{\eta}_\lambda$ of NAM_λ.*

Proof. If $\mathbf{F} = \mathbf{R}$, γ_λ extends from N to NM_λ by $\gamma_\lambda(x, a_1, m) = \gamma_\lambda(x)$. If $\mathbf{F} \neq \mathbf{R}$, γ_λ extends from N to a representation $\tilde{\gamma}_\lambda$ of NM_λ by the trivial (0-cohomology) case of the argument of [12, prop. 4.16]; in short M_λ preserves everything in the Bargmann–Fock realization of $[\gamma_\lambda]$ on a Hilbert space of holomorphic functions on $\mathbf{C}^{l/2}$, and that gives the extension. Now $\tilde{\eta}_\lambda = \mathrm{Ind}_{NM_\lambda}^{NAM_\lambda}\tilde{\gamma}_\lambda$ satisfies

$$\tilde{\eta}_\lambda \mid_{NA} = (\mathrm{Ind}_{NM_\lambda}^{NAM_\lambda}\tilde{\gamma}_\lambda)\mid_{NA} \cong \mathrm{Ind}_N^{NA}(\tilde{\gamma}_\lambda \mid_N) \cong \mathrm{Ind}_N^{NA}\gamma_\lambda = \eta_\lambda. \qquad \text{q.e.d.}$$

Lemma 1.15 and the M-orbit structure of S combine with Mackey's little group method as follows. If $k > 1$ then we fix $\lambda_1 \in S$, set $M_1 = M_{\lambda_1}$, and define the generic representations

$$\pi_\tau = \mathrm{Ind}_{NAM_1}^{NAM}(\tilde{\eta}_{\lambda_1} \otimes \tau), \qquad [\tau] \in \hat{M}_1.$$

If $k = 1$, M acts trivially on $S = \{\lambda_1, \lambda_{-1}\}$ and we define

$$\pi_\tau^+ = \tilde{\eta}_{\lambda_1} \otimes \tau, \quad \pi_\tau^- = \tilde{\eta}_{\lambda_{-1}} \otimes \tau, \qquad [\tau] \in \hat{M}.$$

The representations $\pi_\tau^{(\pm)}$ are the ones that occur in the Plancherel formulae (1.11) and (1.12).

Finally we wish to thank G. Eskin for a number of helpful conversations on L_1 estimates.

2. Properties of certain pseudo-differential operators

The Plancherel Formulae for NA and NAM involve pseudo-differential operators D and E that are, in effect, fractional powers of the Laplacian Δ of the center $Z \cong \mathbf{R}^k$ of N. Here we give a rigorous definition (§2a) of these operators, and then prove (§2b) that they keep sufficiently differentiable compactly supported functions inside L_1. This fundamental property means that the basic ingredients

$$\eta_\lambda(Df), \quad f \in C_c(NA) \qquad\qquad \pi_\tau^{(\pm)}(Ef), \quad f \in C_c(NAM)$$

of the Plancherel Formulae (1.10), (1.11), (1.12), are in fact defined for f sufficiently differentiable. Then we prove (§2c) some commutation properties of D and E. We employ these and Duflo's factorization theorem [1, pp. 250f] to show in §§3, 4 that the operators

$$\eta_\lambda(Df), \quad f \in C_c^\infty(NA) \qquad\qquad \pi_\tau^{(\pm)}(Ef), \quad f \in C_c^\infty(NAM)$$

are trace class.

Some of the analytic material here will seem familiar to PDE experts. But it is not easily accessible to workers in Lie groups, and there is apparently no satisfactory reference. Thus we felt it worthwhile to write out a reasonably detailed treatment. The PDE expert, of course, can skip much of §2b.

2a. *Definitions of the Operators.* Fix a differentiable manifold $V = Z \times W$ where Z has a fixed identification with an euclidean vector space $\mathbf{R}^k$. In our applications, we will have $V = NA$ or NAM and Z will be the center of N. The euclidean structure on Z defines an operation of partial Fourier transform for functions on V

$$\mathcal{F}(f)(\xi, w) = \int_{\mathbf{R}^k} f(z, w) e^{-i\langle z, \xi\rangle}\, dz; \tag{2.1}$$

and also defines a partial Laplacian on V

$$[\Delta f](z, w) = -\sum_1^k \frac{\partial^2 f}{\partial z_i^2}(z, w) \tag{2.2}$$

where $(z_1, \cdots, z_k)$ is the euclidean coordinate on Z. As usual these are related by

$$\mathcal{F}(\Delta^s f)(\xi, w) = \|\xi\|^{2s}\mathcal{F}(f)(\xi, w)$$

for integral $s \geqq 0$. We define non-negative real powers of Δ by

$$[\Delta^s f](z, w) = (\mathcal{F}^{-1}\{\|\xi\|^{2s}\mathcal{F}(f)\})(z, w) \tag{2.3}$$

for real $s \geqq 0$. Δ^s does not increase the W-projection of the support of f; but for non-integral s it may increase the Z-projection.

The operators D on NA and E on NAM are special cases of the above construction. Recall that $k = \dim Z$, $l = \dim N/Z$, $q = k + \frac{1}{2}l$ and that we have diffeomorphic splittings

$$\begin{array}{ll} Z = \mathbf{R}^n,\ NA = Z \times \mathbf{R}_+^*,\ NAM = Z \times (\mathbf{R}_+^* \times M) & \mathbf{F} = \mathbf{R} \\ Z = \operatorname{Im}\mathbf{F},\ NA = Z \times (\mathbf{F}^n \times \mathbf{R}_+^*),\ NAM = Z \times (\mathbf{F}^n \times \mathbf{R}_+^* \times M) & \mathbf{F} \neq \mathbf{R}. \end{array} \tag{2.4}$$

Now let $\omega_{k-1} = 2\pi^{k/2}/\Gamma(k/2)$, the volume of the unit sphere $S^{k-1} \subseteq \mathbf{R}^k$. Let $2^{\operatorname{sgn}(l)}$ denote 1 if $l = 0$ or 2 if $l > 0$. Then our operators are defined, relative to (2.4), by

$$D = 2^{\operatorname{sgn}(l)}(2\pi)^{-q}\Delta^{q/2} \quad \text{on} \quad NA, \tag{2.5a}$$

$$E = \omega_{k-1}2^{\operatorname{sgn}(l)}(2\pi)^{-q}\Delta^{q/2} \quad \text{on} \quad NAM. \tag{2.5b}$$

Fix a positive Radon measure dw on W. That defines a positive Radon measure on $V = Z \times W$ by $dv = dzdw$, where dz is Lebesgue measure on Z. If V is NA or NAM, then right Haar measure is of this form.

2.6 Proposition. *View Δ^s, $s \geqq 0$, as an operator on $L_2(V, dv)$ with domain* $\operatorname{Dom}(\Delta^s) = C_c^\infty(V)$. *Then Δ^s is symmetric, and its closure is a positive self-adjoint operator.*

This result is standard on $\mathbf{R}^k$, i.e. in the case $Z = V$. In our general case, $L_2(V, dv) = L_2(Z, dz) \otimes L_2(W, dw)$ and Δ^s splits as an operator of the form $\Delta_Z^s \otimes 1_W$. The proposition follows.

2b. *L_1 Properties of the Operators.* Retain the notation $V = Z \times W$ and $dv = dzdw$ as above. $C_c^{(m)}(V)$ denotes the space of compactly supported complex functions on V that are m times continuously differentiable. We are going to prove

2.7 Theorem. *If $s \geqq 0$ and $f \in C_c^{(m)}(V)$ with $m > 2s + k$, then* $\Delta^s f \in L_1(V, dv)$.

Then in particular we will have

2.8 COROLLARY. *Let G be a connected simple Lie group of* $\mathbf{R}$*-rank* 1, *NAM a minimal parabolic subgroup of G, Z the center of N, and Δ the Laplacian on Z. If $s \geqq 0$ and m is an integer greater than $2s + \dim Z$, then for $f \in C_c^{(m)}(NA)$ and $F \in C_c^{(m)}(NAM)$, we have $\Delta^s f \in L_1(NA)$ and $\Delta^s F \in L_1(NAM)$.*

In view of this, whenever $f \in C_c(NA)$ and $F \in C_c(NAM)$ are sufficiently differentiable, the operators $\eta_\lambda(Df)$, $\pi_\tau^{(\pm)}(EF)$ are well-defined.

The heart of the proof of Theorem 2.7 is

2.9 PROPOSITION. *Let $h \in C^\infty(\mathbf{R}^k)$ and $s \geqq 0$ be such that $\|z\|^m h(z)$ is bounded for some integer $m > k + 2s$. Then*

$$\|\xi\|^{2s+k}\mathcal{F}\{\|z\|^{2s}h(z)\}(\xi) \tag{2.9a}$$

is bounded.

PROOF. It is clearly a matter of examining the behavior of (2.9a) when ξ is near ∞. We fix a C^∞ partition of unity on $\mathbf{R}^k$, $1 = \beta_1 + \beta_2$ where the β_j are non-negative and $\beta_1(z) = 0$ for $\|z\| \geqq 2$ and $\beta_2(z) = 0$ for $\|z\| \leqq 1$. Then $\mathcal{F}\{\|z\|^{2s}h(z)\}(\xi) = I_1(\xi) + I_2(\xi)$ where

$$I_j(\xi) = \int \|z\|^{2s}\beta_j(z)h(z)e^{-i\langle z,\xi\rangle}\,dz \qquad j = 1,2.$$

As $\|z\|^{2s}\beta_2(z)h(z)$ is C^∞, its Fourier transform decays to 0 at infinity faster than the reciprocal of any polynomial — in particular $\|\xi\|^{2s+k}I_2(\xi)$ is bounded. Set $y = \|\xi\|z$. Then $I_1(\xi) = \|\xi\|^{-(2s+k)}I_3(\xi)$ where

$$I_3(\xi) = \int \|y\|^{2s}\beta_1(y/\|\xi\|)h(y/\|\xi\|)e^{-i\langle y,\xi/\|\xi\|\rangle}\,dy.$$

Then we need only prove that $I_3(\xi)$ is bounded.

Using the partition of unity again $I_3(\xi) = I_4(\xi) + I_5(\xi)$ where

$$I_{3+j}(\xi) = \int \|y\|^{2s}\beta_j(y)\beta_1(y/\|\xi\|)h(y/\|\xi\|)e^{-i\langle y,\xi/\|\xi\|\rangle}\,dy \qquad j = 1,2.$$

Here $I_4(\xi)$ is bounded — indeed

$$|I_4(\xi)| \leqq \int_{\|y\|\leqq 2} \|y\|^{2s}|h(y/\|\xi\|)\,dy \leqq (\text{const.})\|h\|_\infty < \infty.$$

Also since $\xi/\|\xi\|$ is a unit vector, $\Delta_y = -\Sigma\,\partial^2/\partial y_i^2$ sends $e^{-i\langle y,\xi/\|\xi\|\rangle}$ to its negative. Now for every integer $r \geqq 0$, integration by parts gives

$$I_5(\xi) = (-1)^r \int \Delta_y^r \{\| y \|^{2s} \beta_2(y) \beta_1(y/\|\xi\|) h(y/\|\xi\|)\} e^{-i\langle y, \xi/\|\xi\| \rangle} dy.$$

In view of the $\beta_2(y)\beta_1(y/\|\xi\|)$ term, the integration here is over the ring $1 \leqq \|y\| \leqq 2\|\xi\|$.

Since $u(y) = \|y\|^{2s}\beta_2(y)$ and $v(z) = \beta_1(z)h(z)$ are C^∞ functions, the expression $\Delta_y^r\{u(y)v(y/\|\xi\|)\}$ is a finite sum of terms of the form $a(y)b(y/\|\xi\|)\|\xi\|^{-c}$ where a and b are C^∞ and $c \geqq 0$. Thus in $I_5(\xi)$, the integration over $1 \leqq \|y\| \leqq 2$ gives a bounded function of ξ. Since $\beta_2(y) = 1$ for $\|y\| \geqq 2$, we now need only prove that

$$I_6(\xi) = \int_{2 \leqq \|y\| \leqq 2\|\xi\|} \Delta_y^r \{\|y\|^{2s} v(y/\|\xi\|)\} e^{-i\langle y, \xi/\|\xi\|\rangle} dy, \qquad v = \beta_1 h$$

is bounded.

A direct calculation for $r = 1$ and then recursion shows that $\Delta_y^r\{\|y\|^{2s}v(y/\|\xi\|)\}$ is of the form

$$\sum_{j=0}^{r} \|y\|^{2(s-j)} \|\xi\|^{-2(r-j)} f_j(y/\|\xi\|), \qquad f_j \in C^\infty(\mathbf{R}^k).$$

From that we conclude

$$\begin{aligned} |I_6(\xi)| &\leqq \sum_{j=0}^{r} \|\xi\|^{-2(r-j)} \int_{2 \leqq \|y\| \leqq 2\|\xi\|} \|y\|^{2(s-j)} |f_j(y/\|\xi\|)| \, dy \\ &\leqq \sum_{j=0}^{r} \|\xi\|^{-2(r-j)} c_j (\|\xi\|^{2(s-j)+k} + c_j') \\ &\leqq c\|\xi\|^{2s+k-2r} + c', \end{aligned}$$

which is bounded (as $\xi \to \infty$) when we take $r \geqq \frac{1}{2}(2s + k)$. q.e.d.

Remark. It is obvious that Proposition 2.9 is equally valid if the Fourier transform $\mathcal{F}$ is replaced by its inverse $\mathcal{F}^{-1}$.

Proof of Theorem 2.7. The partial Fourier transform $\mathcal{F}(f)(\xi, w)$ is C^∞ in ξ and $C^{(m)}$ in (ξ, w), and $\|\xi\|^m \mathcal{F}(f)(\xi, w)$ is bounded because $f \in C_c^{(m)}(V)$. Thus the functions $h_w(\xi) = \mathcal{F}(f)(\xi, w)$ on $\mathbf{R}^k$ satisfy the hypothesis of Proposition 2.9. Applying the Proposition together with the remark after its proof, we conclude

$$\|z\|^{2s+k} \mathcal{F}^{-1}\{\|\xi\|^{2s} h_w(\xi)\}(z)$$

is bounded. Moreover examining the proof of Proposition 2.9 we see that the bound may be taken in $C_c^{(m)}(W)$. Thus

$$\| z \|^{2s+k} | \Delta^s f(z, w) | \leqq \psi(w)$$

for some $\psi \in C_c^{(m)}(W)$. In particular $\| \Delta^s f(\cdot, w) \|_{L_1(Z, dz)}$ exists and is in $L_1(W, dw)$. Finally by Fubini's Theorem $\Delta^s f \in L_1(V, dv)$.

2c. *Algebraic Properties of the Operators.* If H is a Lie group, its Lie algebra $\mathfrak{h}$ acts on functions from the left by

$$(\xi * f)(y) = \frac{d}{dt} f(\exp(-t\xi)y) \,|_{t=0}, \qquad \xi \in \mathfrak{h}, \quad y \in H;$$

and also from the right by

$$(f * \xi)(y) = \frac{d}{dt} f(y \exp(-t\xi)) \,|_{t=0}, \qquad \xi \in \mathfrak{h}, \quad y \in H.$$

The left action extends to the natural isomorphism of the universal enveloping algebra $\mathfrak{H}$ of $\mathfrak{h}_c$ with the algebra of right-invariant differential operators on H. The right action extends to an anti-isomorphism (reversing order of products) of $\mathfrak{H}$ with the algebra of left invariant differential operators on H. Since we are dealing with right Haar measure in this paper, group convolution is defined by

$$(f_1 * f_2)(h') = \int_H f_1(h'h^{-1}) f_2(h) \, dh$$

with dh = right Haar measure. Several straightforward computations show that

(2.10a) $\quad \Omega * (f_1 * f_2) = (\Omega * f_1) * f_2 \qquad (f_1 * f_2) * \Omega = f_1 * (f_2 * \Omega)$

(2.10b) $\quad \Omega_1 * (f * \Omega_2) = (\Omega_1 * f) * \Omega_2$

for $\Omega, \Omega_1, \Omega_2 \in \mathfrak{H}$.

2.11 LEMMA. *Let H be NA or NAM, $\{\xi_1, \cdots, \xi_k\}$ an orthonormal basis of the Lie algebra $\mathfrak{z}$ of Z, and $\Xi = -\Sigma \xi_i^2 \in \mathfrak{H}$. Define*

$$\alpha : H \to \mathbf{R}_+^* \qquad\qquad \alpha(z, x, a_r, m) = r.$$

Then if $f \in C^{(2)}(H)$ we have

(i) $\Xi * f = \Delta(f)$,

(ii) $$f * \Xi = \begin{cases} \alpha^2 \times \Delta(f) & \mathbf{F} = \mathbf{R} \\ \alpha^4 \times \Delta(f) & \mathbf{F} \neq \mathbf{R}. \end{cases}$$

PROOF. We check the case $\mathbf{F} \neq \mathbf{R}$; the case $\mathbf{F} = \mathbf{R}$ is even easier. $Z = \operatorname{Im} \mathbf{F}$ has orthonormal basis $\{e_1, \cdots, e_k\}$ with $\exp(\Sigma c_i \xi_i) = \Sigma c_i e_i$. Therefore

$$(\Xi * f)(z,x,a_r,m) = -\sum \frac{d^2}{dt^2} f((-te_i,0,1,1)(z,x,a_r,m))\,|_{t=0}$$

$$= -\sum \frac{d^2}{dt^2} f(z-te_i,x,a_r,m)\,|_{t=0}$$

$$= \Delta(f)(z,x,a_r,m).$$

On the other hand

$$(f * \Xi)(z,x,a_r,m) = -\sum \frac{d^2}{dt^2} f((z,x,a_r,m)(-te_i,0,1,1))\,|_{t=0}$$

$$= -\sum \frac{d^2}{dt^2} f(z-tr^2\nu(m)e_i,x,a_r,m)\,|_{t=0}$$

$$= r^4\Big[-\sum (\nu(m)\xi_i)^2 * f\Big](z,x,a_r,m)$$

$$= r^4\Delta(f)(z,x,a_r,m). \qquad \text{q.e.d.}$$

In order to simplify notation, let us agree to write for $s \geqq 0$

$$(2.12) \quad \Delta^s * f = \Delta^s(f) \quad \text{and} \quad f*\Delta^s = \begin{cases} \alpha^{2s}\times\Delta^s(f) & \mathbf{F}=\mathbf{R} \\ \alpha^{4s}\times\Delta^s(f) & \mathbf{F}\neq\mathbf{R}. \end{cases}$$

Observe: if $f \in C_c^\infty(H)$, so is $\alpha^p \times f$ for any $p \geqq 0$. Thus both $\Delta^s * f$ and $f * \Delta^s$ are well-defined in $L_2(H)$.

2.13 PROPOSITION. *Let H denote NA or NAM, $s \geqq 0$. Then the pseudo-differential operators*

$$f \to \Delta^s * f \qquad f \to f * \Delta^s$$

have $L_2(H)$-closures, from the domain $C_c^\infty(H)$, that are positive self-adjoint operators satisfying

$$(2.13a) \quad \Delta^s * (f_1 * f_2) = (\Delta^s * f_1) * f_2 \quad (f_1 * f_2) * \Delta^s = f_1 * (f_2 * \Delta^s)$$

$$(2.13b) \qquad \Delta^s * (f * \Delta^t) = (\Delta^s * f) * \Delta^t.$$

PROOF. The essential self-adjointness and positivity follow from Proposition 2.6, the definition (2.12), and the fact that "differentiation" by Δ^s commutes with multiplication by α^p. Lemma 2.11 says that $f \to \Delta^s * f$ is right-invariant for $s = 1$; so now the entire 1-parameter semigroup is right invarinat, thus proving $\Delta^s * (f_1 * f_2) = (\Delta^s * f_1) * f_2$. Similarly $(f_1 * f_2) * \Delta^s = f_1 * (f_2 * \Delta^s)$. Equation (2.13b) holds by the previous established invariance properties; or by the observation that both sides are equal to $\alpha^{pt} \times \Delta^{s+t}(f)$, $p = 2$ or 4 depending on $\mathbf{F}$. q.e.d.

Next recall the usual involution $f \to f^*$ of $L_1(H)$ given by

$$f^*(h) = \bar{f}(h^{-1})\delta_H(h)^{-1}.$$

If we put $p = 2\dim_{\mathbf{R}} \operatorname{Im} \mathbf{F} + \dim_{\mathbf{R}} \mathbf{F}^n$, then by (1.14) we have $\delta_{NAM}(z, x, a, m) = \delta_{NA}(z, x, a) = \alpha^p(a)$. Now we come to a key identity.

2.14 Proposition. $(\Delta^s * f)^* = f^* * \Delta^s$.

Proof. Once again we consider the case $\mathbf{F} \neq \mathbf{R}$, the case $\mathbf{F} = \mathbf{R}$ being similar. In fact it suffice to prove

$$\Delta^s * f^* = \alpha^{-4s} \times [\Delta^s(f)]^*. \tag{2.15}$$

For (2.12) and (2.15) combine to give

$$\begin{aligned} f^* * \Delta^s &= \alpha^{4s} \times (\Delta^s * f^*) = \alpha^{4s} \times \alpha^{-4s} \times [\Delta^s(f)]^* \\ &= (\Delta^s * f)^*. \end{aligned}$$

We proceed to the proof of (2.15). First note that $(z, x, a_r, m)^{-1} = (-r^{-2}\nu(m)^{-1}z, \ -r^{-1}\mu(m)^{-1}x, \ a_r^{-1}, \ m^{-1})$. Calculating as in Lemma 2.11,

$$\begin{aligned} (\Delta * f^*)(z, x, a_r, m) &= -\sum \frac{d^2}{dt^2} f^*(z - te_i, x, a_r, m)\,|_{t=0} \\ &= -\sum \frac{d^2}{dt^2} \bar{f}(-r^{-2}\nu(m)^{-1}(z - te_i), \ -r^{-1}\mu(m)^{-1}x, \ a_r^{-1}, \ m^{-1})r^{-p}\,|_{t=0} \\ &= -r^{-4}\sum \frac{d^2}{dt^2} \bar{f}(-r^{-2}\nu(m)^{-1}z + t\nu(m)^{-1}e_i, \ -r^{-1}\mu(m)^{-1}x, \ a_r^{-1}, \ m^{-1})r^{-p}\,|_{t=0} \\ &= -r^{-4}\left\{\sum (\nu(m)^{-1}\xi_i)^2 * \bar{f}\right\}((z, x, a_r, m)^{-1})r^{-p} \\ &= \alpha(a_r)^{-4}\overline{(\Delta * f)}((z, x, a_r, m)^{-1})\alpha(a_r)^{-p} \\ &= \{\alpha^{-4} \times \Delta(f)^*\}(z, x, a_r, m). \end{aligned}$$

That proves (2.15) for $s = 1$ and it follows for $s \geqq 0$ as in Proposition 2.13. q.e.d.

We now specialize to the operators D on NA and E on NAM defined in (2.5). They are positive multiples of $\Delta^{q/2}$ where $q = k + \frac{1}{2}l$. The modular function satisfies

$$\delta_{NAM}(z, x, a, m) = \delta_{NA}(z, x, a) = \begin{cases} \alpha(a)^q & \mathbf{F} = \mathbf{R} \\ \alpha(a)^{2q} & \mathbf{F} \neq \mathbf{R}. \end{cases}$$

Therefore equation (2.12) specializes to

$$(2.16)\quad f * D = \delta_{NA} \times (D * f) \text{ on } NA \qquad f * E = \delta_{NAM} \times (E * f) \text{ on } NAM.$$

It follows by an approximate identity argument that the operators D and E are semi-invariants of weight δ in the sense described in §1a.

Finally let us note that D and E behave as expected under restriction of a function from NAM to NA. First right-invariance and a glance at (2.5) give $(E^s * f)|_{NA} = (\omega_{k-1})^s D^s * (f|_{NA})$. Now by (2.16) and (2.13b)

$$\begin{aligned}\{E^s * f * E^t\}|_{NA} &= (E^s * \{\delta^t \times (E^t * f)\})|_{NA} \\ &= (\omega_{k-1})^s D^s * \{\delta^t \times (E^t * f)\}|_{NA} \\ &= (\omega_{k-1})^{s+t} D^s * \{\delta^t \times (D^t * (f|_{NA}))\} \\ &= (\omega_{k-1})^{s+t} D^s * f|_{NA} * D^t.\end{aligned}$$

In particular for $s = t = \frac{1}{2}$, we have

$$(2.17)\qquad \{E^{1/2} * f * E^{1/2}\}|_{NA} = \omega_{k-1} D^{1/2} * f|_{NA} * D^{1/2},$$

which we need for §4.

3. Plancherel formula for NA

In this section we use the pseudo-differential operator D to derive the Plancherel formula for the group NA. We begin with the following fact.

3.1 LEMMA. *Let $0 \neq \lambda \in \mathfrak{z}^*$ and $[\gamma_\lambda]$ the corresponding class in $\hat{N}$. Then*

$$(3.1a)\quad \operatorname{Tr} \gamma_\lambda(f) = (2\pi)^{l/2} \|\lambda\|^{-l/2} \int_Z f(z)\lambda(z)\,dz, \qquad f \in C_c^\infty(N).$$

This formula is certainly known to specialists (see e.g. [9, p. 9]). But we wish to use a slightly more general result, and for that reason we recall the proof. One realizes γ_λ as a monomial representation (induced from a real polarization), obtains $\gamma_\lambda(f)$ as a kernel operator, then computes $\operatorname{Tr} \gamma_\lambda(f)$ by integrating down the diagonal. The formula (3.1a) follows then by a simple calculation. But this calculation is formal — one must prove separately that for $f \in C_c^\infty(N)$, $\gamma_\lambda(f)$ is in fact trace class. This can be done in various ways. One method is to use the fact that there exists an element $\Omega \in \mathfrak{N}$ such that $\gamma_\lambda(\Omega)^{-1}$ is trace class for all $\lambda \in \mathfrak{z}^* - \{0\}$. Then $\gamma_\lambda(f) = \gamma_\lambda(\Omega)^{-1}\gamma_\lambda(\Omega f)$ is also trace class. But we see then that the assumption $f \in C_c^\infty(N)$ is excessive. On one hand (3.1a) is valid for any function on N which is sufficiently differentiable and sufficiently rapidly decreasing at infinity. On the other hand if f is a continuous positive-definite integrable function whose restriction to Z is also integrable, then it follows already from the formal calculation that the *positive* operator $\gamma_\lambda(f)$ must be trace class and (3.1a) is again valid. It is in this latter extended sense that we shall apply Lemma 3.1.

In the remainder of this section expressions such as $D^{1/2} * f * D^{1/2}$ and $D^{1/2} * f * f^* * D^{1/2}$ will occur. They are well defined by Proposition 2.13.

Here is the key lemma of §3.

3.2 LEMMA. *Let $0 \neq \lambda \in \mathfrak{z}^*$ and $\eta_\lambda = \mathrm{Ind}_N^{NA}\gamma_\lambda$. If $\phi \in C_c^\infty(NA)$, then the operator $\eta_\lambda(D^{1/2} * \phi * D^{1/2})$ is trace class and*

$$\text{(3.2a)} \qquad \mathrm{Tr}\,\eta_\lambda(D^{1/2} * \phi * D^{1/2}) = (2\pi)^{-k}\int_0^\infty \hat{\phi}_0\left(r\frac{\lambda}{\|\lambda\|}\right) r^{k-1}\,dr,$$

where $\phi_0 = \phi\mid_Z$ and $\hat{\phi}_0(\lambda) = \int_Z \phi(z)\lambda(z)\,dz$.

PROOF. By Duflo's factorization theorem [1, pp. 250f] we may write any $\phi \in C_c^\infty(NA)$ as a linear combination of functions of the form $\psi * \psi^*$, where $\psi \in C_c^{(p)}(NA)$ and p is as large as we please. Thus it suffices to prove the lemma under the assumption that $\phi = \psi * \psi^*$, $\psi \in C_c^{(p)}(NA)$, p large. Indeed by Proposition 2.14 and Theorem 2.7, we can choose p large enough so that

$$\begin{aligned} D^{1/2} * \phi * D^{1/2} &= D^{1/2} * \psi * \psi^* * D^{1/2} \\ &= (D^{1/2} * \psi) * (D^{1/2} * \psi)^* \end{aligned}$$

is a continuous positive-definite integrable function.

Put $f = D^{1/2} * \phi * D^{1/2}$, $\phi = \psi * \psi^*$. We realize the *positive* operator $\eta_\lambda(f)$ as a kernel operator. To prove that it is trace class, it is enough — by the positivity — to show that the integral down the diagonal is finite. This we do by an explicit evaluation. It is possible to achieve this by realizing η_λ as a monomial representation (see e.g. [6]). It is easier and more direct, however, to use the realization $\eta_\lambda = \mathrm{Ind}_N^{NA}\gamma_\lambda$, Lemma 3.1 and the formula for the trace of an induced representation found in [7]. Indeed, a direct application of [7, theor. 3.2] yields

$$\mathrm{Tr}\,\eta_\lambda(f) = \int_A \delta(a)^{-1}\mathrm{Tr}\left[\int_N f(a^{-1}na)\gamma_\lambda(n)\,dn\right]da.$$

(The hypothesis of compact support in [7] is unnecessary; continuity and integrability will suffice.) Now the function $n \to f(a^{-1}na)$ is one to which the comments on Lemma 3.1 apply. Therefore we obtain

$$\operatorname{Tr}\eta_\lambda(f)=\int_A \delta(a)^{-1}(2\pi)^{l/2}\|\lambda\|^{-l/2}\int_Z f(a^{-1}za)\lambda(z)\,dz\,da.$$

If $\mathbf{F}=\mathbf{R}$ we compute

$$\begin{aligned}\operatorname{Tr}\eta_\lambda(f)&=\int_A\int_Z f(z)\lambda(a^{-1}za)\,dz\,da\\&=\int_0^\infty \hat{f}_0(r\lambda)\frac{dr}{r}\\&=\int_0^\infty \hat{f}_0\left(r\frac{\lambda}{\|\lambda\|}\right)\frac{dr}{r}\\&=(2\pi)^{-k}\int_0^\infty \hat{\phi}_0\left(r\frac{\lambda}{\|\lambda\|}\right)r^{k-1}\,dr\end{aligned}$$

using the fact that

$$\begin{aligned}\hat{f}_0(y)&=(D^{1/2}*\phi*D^{1/2})\hat{}_0(y)=(D\phi)\hat{}_0(y)\\&=(2\pi)^{-k}(\Delta^{k/2}\phi)\hat{}_0(y)=(2\pi)^{-k}\|y\|^k\hat{\phi}_0(y)\end{aligned}$$

(see (2.5a) and (2.16)).

In the other case $\mathbf{F}\neq\mathbf{R}$, we have

$$\begin{aligned}\operatorname{Tr}\eta_\lambda(f)&=\int_A \alpha(a)^{-2q}(2\pi)^{l/2}\|\lambda\|^{-l/2}\int_Z f(a^{-1}za)\lambda(z)\,dz\,da\\&=\int_A \alpha(a)^{-l}(2\pi)^{l/2}\|\lambda\|^{-l/2}\int_Z f(z)\lambda(aza^{-1})\,dz\,da\\&=\int_0^\infty r^{-l}(2\pi)^{l/2}\|\lambda\|^{-l/2}\hat{f}_0(r^2\lambda)\frac{dr}{r}\\&=\frac{1}{2}\int_0^\infty r^{-l/2}(2\pi)^{l/2}\|\lambda\|^{-l/2}\hat{f}_0(r\lambda)\frac{dr}{r}\\&=\frac{1}{2}\int_0^\infty r^{-l/2}(2\pi)^{l/2}\hat{f}_0\left(r\frac{\lambda}{\|\lambda\|}\right)\frac{dr}{r}\\&=(2\pi)^{-k}\int_0^\infty \hat{\phi}_0\left(r\frac{\lambda}{\|\lambda\|}\right)r^{k-1}\,dr,\end{aligned}$$

this time using the fact that

$$\begin{aligned}\hat{f}_0(y)&=(D\phi)\hat{}_0(y)=2(2\pi)^{-(k+\frac{1}{2}l)}(\Delta^{q/2}\phi)\hat{}_0(y)\\&=2(2\pi)^{-(k+\frac{1}{2}l)}\|y\|^q\hat{\phi}_0(y)\end{aligned}$$

(see (2.5b)).

To finish the proof it suffices to observe that, if p is chosen large enough, the right side of (3.2a) is an absolutely convergent integral.

It is now a simple matter to derive the main result of the section, the Plancherel formula for NA.

3.3 THEOREM. *Let D be the pseudo-differential operator on NA defined in* (2.5a). *Then for any $\phi \in C_c^\infty(NA)$ we have*

$$\phi(1_{NA}) = \int_{S^{k-1}} \operatorname{Tr} \eta_\lambda(D^{1/2} * \phi * D^{1/2})\, d\sigma(\lambda), \tag{3.3a}$$

where σ is the standard volume element on S^{k-1}.

PROOF. We know by Lemma 3.2 that the integrand in (3.3a) exists — in fact it is a C^∞ function of λ in S^{k-1}. Thus the integral is absolutely convergent, and we may compute

$$\begin{aligned}
&\int_{S^{k-1}} \operatorname{Tr} \eta_\lambda(D^{1/2} * \phi * D^{1/2})\, d\sigma(\lambda) \\
&\quad = \int_{S^{k-1}} (2\pi)^{-k} \int_0^\infty \hat{\phi}_0(r\lambda) r^{k-1}\, dr d\sigma(\lambda) \\
&\quad = (2\pi)^{-k} \int_{\mathbf{R}^k} \hat{\phi}_0(y)\, dy \\
&\quad = \phi_0(0) = \phi(1_{NA}).
\end{aligned}$$

q.e.d.

Before concluding this section we recast formula (3.3a) into the shape of formula (1.8). Indeed by (2.16) we can rewrite it

$$\phi(1_{NA}) = \int_{S^{k-1}} \operatorname{Tr} \eta_\lambda(D(\delta^{1/2}\phi))\, d\sigma(\lambda), \qquad \phi \in C_c^\infty(NA).$$

Replacing ϕ by $\delta^{-1/2}\phi$ we get finally

$$\phi(1_{NA}) = \int_{S^{k-1}} \operatorname{Tr} \eta_\lambda(D\phi)\, d\sigma(\lambda), \qquad \phi \in C_c^\infty(NA), \tag{3.4}$$

which is the Plancherel formula given in [6]. In [6], the facts that $D\phi \in L_1(NA)$ and $\eta_\lambda(D\phi)$ is trace class were refered ahead to this paper. The first, of course, comes out of Corollary 2.8. For the second, $\eta_\lambda(D\phi) = \eta_\lambda(D^{1/2} * \delta^{-1/2}\phi * D^{1/2})$ and $\delta^{-1/2}\phi \in C_c^\infty(NA)$, so Lemma 3.2 shows that it is in fact true. Either of (3.3a) or (3.4) may be considered to be "the" Plancherel formula for NA: (3.3a) is perhaps preferable because of its similarity to the abstract Plancherel formula (1.5); on the other hand, one is probably more "comfortable" working with (3.4).

4. Plancherel formula for NAM

We now use the technique of group extension representations (as in [7, §4]) to derive the Plancherel formula for NAM from that of NA. We shall see that the pseudo-differential operators D and E are well-suited to this technique. We go immediately to the statement of the main result.

4.1 Theorem. *Let E be the pseudo-differential operator on NAM defined in* (2.5b). *Then for any $\phi \in C_c^\infty(NAM)$ we have*

$$\phi(1_{NAM}) = \begin{cases} \displaystyle\sum_{\tau \in \hat{M}}^{\oplus} \tfrac{1}{2}\operatorname{Tr}(\pi_\tau^+ \oplus \pi_\tau^-)(E^{1/2} * \phi * E^{1/2}) \dim \tau & k = 1 \\ \displaystyle\sum_{\tau \in \hat{M}_1}^{\oplus} \operatorname{Tr} \pi_\tau(E^{1/2} * \phi * E^{1/2}) \dim \tau & k > 1. \end{cases} \tag{4.1a}$$

Proof. We assume that $k > 1$ in the following. First let $\psi \in C_c^\infty(NAM)$ and suppose $\phi = \psi * \psi^*$. Let $\theta = \phi\,|_{NA}$. We perform the following series of calculations, the justifications for which we provide later.

$$\int_{NAM} |\psi|^2 = \phi(1_{NAM}) = \theta(1_{NA})$$

$$= \int_{S^{k-1}} \operatorname{Tr} \eta_\lambda(D^{1/2} * \theta * D^{1/2})\, d\sigma(\lambda) \tag{4.2}$$

$$= \int_M \operatorname{Tr} \eta_{m \cdot \lambda_1}(D^{1/2} * \theta * D^{1/2}) \omega_{k-1}\, dm \tag{4.3}$$

$$= \operatorname{Tr} \operatorname{Ind}_{NA}^{NAM} \eta_{\lambda_1}(E^{1/2} * \phi * E^{1/2}) \tag{4.4}$$

$$= \operatorname{Tr} \sum_{\tau \in \hat{M}_1}^{\oplus} \dim \tau\, \pi_\tau(E^{1/2} * \phi * E^{1/2}) \tag{4.5}$$

$$= \sum_{\tau \in \hat{M}_1} \dim \tau \operatorname{Tr} \pi_\tau(E^{1/2} * \phi * E^{1/2}) \tag{4.6}$$

$$= \sum_{\tau \in \hat{M}_1} \| \pi_\tau(E^{1/2} * \psi) \|_2^2 \dim \tau. \tag{4.7}$$

Assume for the moment we have justified all the steps (4.2)–(4.7). It follows then that for any $\psi \in C_c^\infty(NAM)$ we have

$$\int_{NAM} |\psi|^2 = \sum_{\tau \in \hat{M}_1} \| \pi_\tau(E^{1/2} * \psi) \|_2^2 \dim \tau.$$

Now the test functions are L_2-dense in the domain of $E^{1/2}$, and therefore the same equation holds for any compactly supported, sufficiently differentiable function ψ on NAM. But any $\phi \in C_c^\infty(NAM)$ can (as usual) be factored into a linear combination of convolutions of such functions — and thus we conclude that (4.1a) is valid for all test functions $\phi \in C_c^\infty(NAM)$.

It remains to substantiate (4.2)–(4.7). Equation (4.2) is merely a quotation of Theorem 3.3. Equation (4.3) is valid because (in case $k > 1$) M acts transitively on S^{k-1} and leaves the volume element invariant. The constant ω_{k-1} occurs because dm is normalized, whereas $d\sigma(\lambda)$ is not. Equation (4.6) is a simple consequence of the positivity of all the operators $\pi_\tau(E^{1/2} * \phi * E^{1/2})$. This comes about because $E^{1/2} * \phi * E^{1/2} = (E^{1/2} * \psi) * (E^{1/2} * \psi)^*$ is positive-definite (Proposition 2.14 again). Equation (4.7) is obvious. That leaves (4.4) and (4.5).

Equation (4.5) is a consequence of the representation-theoretic fact:

$$\operatorname{Ind}_{NA}^{NAM} \eta_{\lambda_1} \cong \sum_{\tau \in \hat{M}_1}^{\oplus} (\dim \tau)\pi_\tau.$$

(A proof of this can be found in [7, p. 470], where the unimodularity assumption is unnecessary.) Finally (4.4) follows from the trace formula of [7]. In fact if we use $\delta_{NAM}\,|_{NA} = \delta_{NA}$, $\delta_{NAM}\,|_M \equiv 1$ and the equation (2.17), then an application of [7, theor. 3.2] yields

$$\begin{aligned}
&\operatorname{Tr} \operatorname{Ind}_{NA}^{NAM} \eta_{\lambda_1}(E^{1/2} * \phi * E^{1/2}) \\
&\quad = \int_M \operatorname{Tr}\left[\int_{NA} (E^{1/2} * \phi * E^{1/2})(m^{-1}nam)\eta_{\lambda_1}(na)\,dnda\right] dm \\
&\quad = \int_M \operatorname{Tr}\left[\int_{NA} \omega_{k-1}(D^{1/2} * \theta * D^{1/2})(na)\eta_{\lambda_1}(mnam^{-1})\,dnda\right] dm \\
&\qquad\qquad = \int_M \operatorname{Tr} \eta_{m\cdot\lambda_1}(D^{1/2} * \theta * D^{1/2})\omega_{k-1}\,dm.
\end{aligned}$$

That concludes the proof in case $k > 1$. The case $k = 1$ is proved in an entirely analogous fashion, using that M acts trivially on the two point space S^0; we leave the details to the reader. q.e.d.

It is of interest to examine the operators $\eta_\lambda(D)$ and $\pi_\tau(E)$ as these are the unbounded operators D_ζ in the abstract formulation of the Plancherel Theorem 1.1. First of all they must be defined — at least generically. This is because the operators $f \to Df$ on NA and $f \to Ef$ on NAM commute with right translations. As such they are affiliated with the left ring and have "infinitesimal values" almost everywhere with respect to Plancherel measure. Second they can actually be computed as follows. Since $D^2 \in \mathfrak{N}\mathfrak{A}$, $\eta_\lambda(D^2)$ may be computed in the usual manner (i.e. by differentiating the representation and extending to the enveloping algebra). The result is a positive essentially self-adjoint operator on the space of η_λ for which we can easily compute the square root. Similarly with $\pi_\tau(E)$.

All the computations are straightforward; here are the results. Let $0 \neq \lambda \in \mathfrak{z}^*$ and denote $\mathcal{H}(\lambda)$ a Hilbert space on which γ_λ acts. Then $\eta_\lambda = \mathrm{Ind}_N^{NA}\gamma_\lambda$ may be realized on

$$\mathcal{H}(\eta_\lambda) = \left\{f\colon NA \to \mathcal{H}(\lambda),\ f(ng) = \gamma_\lambda(n)f(g),\ n \in N,\ g \in NA,\right.$$
$$\left.\int_A \|f(a)\|^2\,da < \infty\right\}.$$

A simple computation yields

$$\eta_\lambda(D)f(na) = \lambda_*(D)\delta(a)f(na), \qquad f \in \mathcal{H}(\eta_\lambda)$$

where λ_* is defined as follows: It's given on $\mathfrak{z}$ by $\lambda(\exp\xi) = e^{\lambda_*(\xi)}$, $\xi \in \mathfrak{z}$, and then extended by the functional calculus to continuous functions on $\mathfrak{z}$. More precisely if $\xi_1,\cdots,\xi_k$ is an orthonormal basis of $\mathfrak{z}$, $e_i = \exp\xi_i$, and $-\Delta = \partial^2/\partial e_1^2 + \cdots + \partial^2/\partial e_k^2$, then

$$\lambda_*(D) = 2^{\operatorname{sgn}(l)}(2\pi)^{-q}(-\lambda_*(\xi_1)^2 - \cdots - \lambda_*(\xi_k)^2)^{q/2}.$$

Notice that $\lambda_*(D)$ is a positive number and that $\eta_\lambda(D)$ is a positive operator. From the general theory [4, theor. 6] one knows that if we realize η_λ as induced from an irreducible representation of the kernel of the modular function (as we have done here), then in that realization $\eta_\lambda(D)$ must — up to a constant depending on λ — be multiplication by the modular function. Thus in our realization the constant turned out to be $\lambda_*(D)$. Note also that if $\lambda \in S^{k-1}$, then $\lambda_*(D)$ is the absolute constant $c_{k,l} = 2^{\operatorname{sgn}(l)}(2\pi)^{-k}$.

Similarly if we realize the space of π_τ as

$$\mathcal{H}(\pi_\tau) = \{f: NAM \to \mathcal{H}(\eta_{\lambda_1}) \otimes \mathcal{H}(\tau), f(nam_1g) = [\tilde{\eta}_{\lambda_1}(nam_1) \otimes \tau(m_1)]f(g),$$
$$nam_1 \in NAM_1, g \in NAM, \textstyle\int_M \|f(m)\|^2\,dm < \infty\},$$

then one readily computes (of course using the M-invariance of D) that

$$\pi_\tau(E)f(nam) = \omega_{k-1}c_{k,l}\delta(a)f(nam), \qquad f \in \mathcal{H}(\pi_\tau).$$

It is worthwhile to compare these data with the D_ξ found in [5], [2].

Finally let us conclude the paper by discussing the question of uniqueness for D and E. First consider NA. D is uniquely determined by our normalization of

Haar measure on the group and by the specific choice (within its equivalence class) of the volume element on S^{k-1}. If we allow these measures to vary, then one knows from the general theory [7, theor. 6.4] that D can be modified by any positive self-adjoint invertible operator C which is affiliated with both the left and right rings of the group. Indeed if C is any such operator, then $\eta_\lambda(C) = c_\lambda I$, $c_\lambda > 0$ and the Plancherel formula becomes

$$\phi(1_{NA}) = \int_{S^{k-1}} \operatorname{Tr} \eta_\lambda(CD\phi)\, d\sigma'(\lambda)$$

where $d\sigma'(\lambda) = c_\lambda^{-1} d\sigma(\lambda)$.

Now unfortunately there may be many such C's. For instance let D_1 be a *differential* operator on NA given along Z by *any* homogeneous differential operator of degree $k + \frac{1}{2}l$, such as $(\partial/\partial e_1)^{k+\frac{1}{2}l}$. By virtually the same computation as that of $\eta_\lambda(D)$, we find that

$$\eta_\lambda(D_1)f(na) = \lambda_*(D_1)\delta(a)f(na), \qquad f \in \mathcal{H}(\eta_\lambda).$$

Thus $D_1 = CD$ where $\eta_\lambda(C) = \lambda_*(D_1)/\lambda_*(D)$. In particular then one can replace D in the Plancherel formula of NA by an element of the enveloping algebra.

On the other hand, when we pass to the group NAM, operators such as D_1 above are not M-invariant and so cannot be semi-invariants of the group. Thus we are left with a question: *Can* one change E to a differential operator $E_1 = CE$ in the Plancherel formula? It is easy to see that, in the cases $\mathbf{F} = \mathbf{R}$, n even, and $\mathbf{F} = C$, n arbitrary, one can. We think it unlikely that such a change could be effected in the remaining cases — but we have not succeeded in proving that.

REMARK. There already is an example of a non-unimodular Lie group G for which no element D can be found in $\mathfrak{G}$ to satisfy (1.8). That such a phenomenon occurs can be deduced from [3, §5.8] where a simply connected solvable Lie group G is constructed having the property that no polynomial function on $\mathfrak{g}^*$ is semi-invariant of weight δ_G — indeed not even of weight δ_G^p, for any positive integer p.

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