SETS OF CONSTANT WIDTH IN FINITE DIMENSIONAL BANACH SPACES

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ABSTRACT

In Euclidean space a set of constant width has the property that it is not a proper subset of any set of the same diameter. The converse implication is also true. Here we show that if Euclidean is replaced by n-dimensional Banach space the direct statement is true, but the converse statement is false. Attention is drawn to the problem of characterising those Banach spaces of finite dimension for which the converse is true.

Introduction. In Euclidean n-dimensional real space, a set X is said to be of constant width, if and only if, it is convex, compact, and the distance apart of any two parallel support hyperplanes is constant (i.e. the same whichever pair of parallel support hyperplanes we consider). A number of equivalent properties are known of which the most important (in Euclidean space) is that a set X is of constant width if and only if it is complete, that is to say, if Y is a set of which X is a proper subset then the diameter of Y exceeds that of X. Instead of "complete", which is used in many other ways, we shall use the phrase "diametrically maximal". [4]

The object of this note is to develop analogous properties in n-dimensional Banach spaces [see 2.3]. In particular we show, in contradiction to accepted belief [see 1], that the concepts "constant width" and "diametrical maximality" are not equivalent in every Banach space. It is always the case that constant width implies diametrical maximality. The reverse implication is not necessarily true even when the unit sphere of the space is both smooth and rotund. It is not easy to see what conditions on the space will ensure equivalence of these two concepts but a simple example can be given of a non Euclidean space in which this equivalence holds. Many of these equivalent conditions have been known previously (See [2]).

Notation. Denote an n-dimensional Banach space by B^n and its unit ball by S. For two parallel hyperplanes π_1, π_2 in B^n the width between π_1 and π_2 is twice the largest number λ such that a set obtained from λS by translation is contained in the strip bounded by π_1 and π_2. A convex compact subset X of B^n is of constant width if and only if the width between each pair of parallel support hyperplanes of X is constant. We shall use the phrase "of constant width" to imply that X

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is convex and compact. The vector domain of a set X, denoted by X_V is the set of all vectors of the form $x_1 - x_2$ where x_1 and x_2 vary independently in X. We use $+$ between sets to denote vector sum. Thus $X_V = X + (-1)X$. Also the width of $A + B$ between two parallel support hyperplanes π_1, π_2 is the sum of the widths of A and of B between corresponding pairs of hyperplanes all parallel to π_1 and π_2. Finally $(A + B)_V = A_V + B_V$.

Except where the contrary is expressly stated every set is assumed to be convex and compact and to contain interior points.

The origin of B^n, the centre of S, is denoted by 0. We use the same symbol for points and for vectors. The symbol xy will be used either for the line xy or the segment xy or the length of the segment xy.

We shall use $B(S)$ to indicate the Banach space with the central convex set S as its unit sphere.

1. Some properties equivalent to "constant width".

(A) *X (assumed to be convex and compact) is of constant width if and only if for some $\lambda > 0$ $X_V = \lambda S$.*

If X is of constant width then so are $(-1)X$ and $X_V = X + (-1)X$. In any case X_V is central thus it is sufficient to prove that a central set of constant width is a sphere. These two properties of X_V mean that for some fixed $\lambda > 0$, λS is supported by each pair of parallel support hyperplanes of X_V. Since a convex compact set is uniquely defined by its support hyperplanes, X_V is λS.

On the other hand if X_V is λS then X_V is of constant width. But the width of X between any pair of parallel support hyperplanes π_1, π_2 is equal to the width of $(-1)X$ between support hyperplanes parallel to π_1 and π_2, and thus is half the width of X_V between two support hyperplanes parallel to π_1 and π_2. Hence X is of constant width.

(B) A subset X of B^n has the support intersection property if, and only if, given any pair of parallel support hyperplanes π_1 and π_2 of X, and one of the support hyperplanes of S, say π, parallel to π_1 and π_2, then to any point $p \in \pi \cap S$ we can find $x_1 \in \pi_1 \cap X$ and $x_2 \in \pi_2 \cap X$ such that line x_1x_2 or x_2x_1 is parallel to the line op.

X is of constant width if and only if it has the support intersection property.

If X is of constant width then for some $\lambda > 0 X_V = \lambda S$. Given π_1, π_2 and π all parallel hyperplanes of which the first two support X and the last S, let $p \in \pi \cap S$. Then $\lambda p \in X_V$ and thus $\lambda p = x_1 - x_2$ where $x_1 \in X$, $x_2 \in X$. Also x_1 and x_2 belong one each to π_1 and π_2 (for λp is a frontier point of X_V and there is a hyperplane of support to X_V at λp parallel to π). Thus X has the support intersection property.

Support next that X (assumed to be compact and convex) has the support intersection property. Let p be a point of the frontier of S, and let the half ray terminating at o and containing p meet the frontier of X_V in q. If the hyperplane

π supports S at p then $\exists\, x_1, x_2$ on the frontier of X such that line x_1x_2 is parallel to op and there exist hyperplanes supporting X at x_1, x_2 and parallel to π. By definition X_V contains the point $x_1 - x_2$ and it is a frontier point of X_V (hence it is q or $-q$) through which passes a hyperplane parallel to π supporting X_V. Thus S and X_V are two central convex sets such that any half ray meets their frontiers at p and q respectively and if p lies on support hyperplane π of S then q lies on a parallel support hyperplane. of X_V. We show that this implies that for some $\lambda > 0$ $X_V = \lambda S$.

It is convenient to consider B^n as represented in Euclidean space E^n with an appropriate metric.

It is sufficient to show that if $o\, p_1q_1$ and $o\, p_2q_2$ are two half rays terminating at o with p_1, p_2 on the frontier of S and q_1q_2 on the frontier of X_V then $o\, p_1/o\, q_1 = o\, p_2/o\, q_2$. The plane τ = plane of $o\, p_1p_2, o\, q_1q_2$ meets S in a set say S' and X_V in a set X'_V. The sets S', X'_V as two dimensional convex sets have the same property as do S and X_V. For if p is any point on the frontier of S' in τ and h is a support line to S' at p in τ, then there exists a support hyperplane π_1 to S at p meeting τ in h. If $o\, p$ meets frontier of X_V in q then there exists a support hyperplane of X_V at q parallel to π_1 and thus meeting τ in a line parallel to h. This line supports X'_V at q. In what follows for the remainder of this paragraph we consider only subsets of τ. Select a fixed half line ox through o and measure angles in a fixed sense from ox. Let $h(\theta)$ be the half line through o making an angle θ with ox. θ is the angle in E^n which with the appropriate metric represents B^n. Let $h(\theta)$ meet the frontier of S' in $p(\theta)$ and that of X'_V in $q(\theta)$. Denote the length (*in* E^n) $op(\theta)$ by $f(\theta)$ and $oq(\theta)$ *by* $g(\theta)$. Then if $|\theta_1 - \theta_2| < \pi$ and $K = \sup f(\theta)$,

$$|f(\theta_1) - f(\theta_2)| \leqq p(\theta_1)p(\theta_2) = \frac{f(\theta_1)\sin|\theta_1 - \theta_2|}{|\sin \underline{/o\, p(\theta_1)\, p(\theta_2)}|} \tag{1}$$

$$\leqq \frac{K|\theta_1 - \theta_2|}{|\sin \underline{/o\, p(\theta_1)\, p(\theta_2)}|}.$$

The angles $\underline{/o\, p(\theta_1)\, p(\theta_2)}$ are such that there exists $\delta > 0$ with the property that $|\theta_1 - \theta_2| < \delta$ implies $|\sin \underline{/o\, p(\theta_1)\, p(\theta_2)}| > \delta$. It follows from (1) that $f(\theta)$ is absolutely continuous. Similarly so is $g(\theta)$ and finally $f(\theta)/g(\theta)$ is absolutely continuous. $f'(\theta)$ and $g'(\theta)$ both exist for almost all θ.

The condition of parallelism of the support lines at $p(\theta)$ and $q(\theta)$ implies that for almost all θ

$$\frac{f'(\theta)}{f(\theta)} = \frac{g'(\theta)}{g(\theta)}$$

i.e. $f(\theta)/g(\theta)$ has a derivative equal to zero almost everywhere. Thus $f(\theta)/g(\theta)$ is a constant.

This implies that $X_V = \lambda S$ i.e. X is of constant width.

(C) *The coincidence normal property*

We say that a line h is normal to a hyperplane π if there is a sphere whose center lies on h, whose frontier passes through p, the unique point of intersection of h with π, and which is supported by π at p. (It is always assumed that h does not lie in π). If p is a frontier point of the compact convex set X then h is a normal to X at p if and only if h is normal to at least one of the support hyperplanes of X at p.

X is of constant width if every two parallel normals of X coincide.

The coincidence of every two parallel normals of X implies that X has the support intersection property and therefore is of constant width.

The converse of this result is false. Indeed, if S is a non-rotund convex set then there exist non-coincident parallel normals to S in $B(S)$ and S is certainly of constant width in $B(S)$. On the other hand if S is rotund and smooth the reverse implication is true.

If S is rotund and smooth and X is of constant width then every two parallel normals of X coincide.

Since S is rotund and $\lambda S = X_V$ it follows that X is rotund. Let N_1, N_2 be two parallel normals of X. Select p on the frontier of S such that op is parallel to N_1 and N_2. Let π be the support hyperplane of S at p (π is uniquely defined because S is smooth). Since X has the support intersection property we can find two points of X, x_1, x_2 one each on the two support hyperplanes of X parallel to π, such that the line x_1x_2 is parallel to op and thus to N_1 and N_2. If the lines N_1, N_2 do not coincide at least one of them does not coincide with x_1x_2. Suppose that N_1 is distinct from x_1x_2. By the definition of normality and the smoothness of S, N_1 is normal to a hyperplane of support of X parallel to π_1 and π_2, say π_1. But then π_1 contains two distinct points of X and X is not rotund. A contradiction which establishes the required result.

(D) *The spherical intersection property*

Let $S(p, r)$ be the sphere of B^n obtained by applying the translation o to p to the sphere rS.

For any set X (compact and convex) let the diameter of X be $D(X)$ and define the set X_S by

$$X_S = \underset{p \in X}{\Omega} S(p, D(X)).$$

In any case $X_S \supset X$. We shall say that X has the spherical intersection property if and only if $X_S = X$.

If X is of constant width then X has the spherical intersection property

If $p \notin X$ there exists a hyperplane π separating p from X. On the two parallel support hyperplanes of X, π_1, π_2 there is a point q of X on that one of these hyperplanes most distant from X. The strip bounded by $\pi_1\pi_2$ contains a sphere of radius $\frac{1}{2}D(X)$. Thus $S(q, D(X))$ is separated from p by π. Hence $p \notin X_S$ and we have $X_S \subset X$. This establishes the required result.

In Euclidean space the converse implication is known to be true see [4], but this is no longer the case in Banach spaces generally. For example let U be a regular tetrahedron in 3 dimensional space. U_V is a polyhedron with 8 triangular and 6 parallelogram faces. Now let K be a central polyhedron with centre 0, containing U_V but such that the 8 triangular faces of U_V lie in the frontier of K, and such that K is not U_V.

In the Banach space $B(K)$, U is not a set of constant width since $U_V \neq \lambda K$ for any $\lambda > 0$. On the other hand if the vertices of U are a, b, c, d then the set obtained from U_V by the translation 0 to a, contains U and lies so that the face bcd forms the translate of one of the triangular faces of U_V. It follows that

$$U = \underset{p=a,b,c,d}{\Omega} U_V(p, 1)$$

and the same is true with U_V replaced by K. Thus $U \supset U_K$, U has the spherical intersection property in $B(K)$ but is not of constant width in $B(K)$. A particular case is that in which K is a regular octohedron.

It is natural to suppose that this situation arises because K is either not smooth or not rotund or both. A slightly more difficult example shows that this is not the case. Before explaining this example we introduce the last property.

(E) *The diametrically maximal property*

A set X is diametrically maximal if and only if $p \notin X$ implies that $D(p \cup X) > D(X)$.

A set X is diametrically maximal if and only if it has the spherical intersection property.

If $X_S = X$ and $p \notin X$ then $p \notin X_S$ and thus $p \notin S(q, D(X))$ for some $q \in X$. Hence $pq > D(X)$ and $D(p \cup X) > D(X)$.

If X is diametrically maximal and $p \in X_S$ then $D(p \cup X) = D(X) \therefore p \in X$. Thus $X_S \subset X$ and X has the spherical intersection property.

If X is of constant width then X is diametrically maximal. The converse implication is not universally true.

This follows from (D) and the immediately preceding result.

A set X is diametrically maximal if and only if every frontier point of X is distant $D(X)$ from at least one other point of X.

If $X_S = X$ then a frontier point x of X is a frontier point of $S(p, D(X))$ for some $p \in X$ for otherwise by compactness x would be an interior point of X_S.

If $X_S \neq X$ then $\exists p \in X_S$, $p \notin X$. Let $q \in$ interior of X. Then pq meets frontier of X in a point x that necessarily belongs to the interior of X and hence of X_S. Thus x is distant $< D(X)$ from every point of X.

2. **An example of a space $B(S)$ where S is smooth and rotund but there exist diametrically maximal sets that are not of constant width.** We give next an example of $B(S)$ where S is smooth and rotund and yet $B(S)$ contains a subset X which is diametrically maximal without being of constant width.

In three dimensional Euclidean space let a, b, c, d be four points all at the same (Euclidean) distance R from one another. Let X be the intersection of the four balls whose centres are a, b, c, d and whose radii are R. The set X is bounded by four portions of the surfaces of these balls and these four portions meet, in pairs, in six arcs each containing two of the four points a, b, c, d and each of radius $R\sqrt{3}/2$. Of these arcs let that which joins a to b be $\gamma(a, b)$ and that which joins c to d be $\gamma(c, d)$. Let p be the mid point of $\gamma(a, b)$ and let q be the mid point of $\gamma(c, d)$. The points p, q, c, d lie in the plane which bisects perpendicularly the segment ab. Let line pq be perpendicular to the line cd. Thus the plane π_1 perpendicular to the line pq and passing through q meets plane $pqcd$ in a line tangent to $\gamma(cd)$ at q. It follows because of the symmetry of X about the plane $pdqc$ and the convexity of X that π_1 supports X at q. Similarly a parallel plane π_2 supports X at p. Thus p and q are diametrically opposite points of X.

Since the length of segment pq is $R(\sqrt{3} - 1/\sqrt{2})$, and that of pd is R, the fact that $\underline{/dpq} = \pi/6$ implies that $\underline{/pdq} < \pi/2$. Thus the plane through d perpendicular to pd does not separate p from q and the parallel plane which supports X is at a positive distance from q. Since a similar argument applies with d replaced by c we conclude that there are positive numbers ρ, δ such that if the frontier point x of X is within a distance ρ of p then x is diametrically opposite to some point whose distance from q is greater than δ.

Given $x \in FrX$ let $P(x)$ be the set of points diametrically opposite to x and $f(x)$ be the largest value of the acute angle made by pq with xy, $y \in P(x)$. (such largest value existing because $P(x)$ is compact). Write $\chi = \inf f(x)$ $x \in frX$. Our aim is to show that $\chi > 0$. Suppose on the contrary that $\chi = 0$ and that $\{x_n\}$ $\{y_n\}$ are sequences of points such that $y_n \in P(x_n)$ and the angle between pq and $x_n y_n \to 0$ as $n \to \infty$. We can by considering appropriate subsequences assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then xy is parallel to pq. There are parallel support planes to X through x and y. This is possible only if x, y lie on the support planes to X at p and q which in turn implies that x is p and y is q or vice versa. But then if the distance of x_n from p is less than ρ that of y_n from q is greater than δ: this contradicts the fact that $x_n \to p$ and $y_n \to q$. Thus our assumption is false and in fact $\chi > 0$.

Next consider X_V. X_V is rotund since X is rotund. Moreover X_V fails to be smooth if and only if there are two diametrically opposite points s_1, s_2 of X and two pairs of parallel support planes of X π_1, π_2 and τ_1, τ_2 such that s_1 lies on both π_1 and τ_1 whilst s_2 lies on both π_2 and τ_2. Two such points would necessarily lie either at the points a, b, c, d or on the arcs such as $\gamma(a, b)$ which connect these points. If s_1 were a point of arc $\gamma(a, b)$ not a or b, then planes $\pi_1\tau_1$ would intersect plane abq in a line λ_1 tangent to $\gamma(a, b)$ at s_1. Now plane abq meets the plane containing $\gamma(cd)$ in line pq. If s_2 lies in this last plane then since π_2, τ_2 intersect in a line which both lies in this plane and is parallel to λ_1 it follows that λ_1 is parallel to pq. This is impossible because the centre of $\gamma(ab)$ lies on segment pq and its arc length is less than π. Similarly if s_2 lies on the plane containing arc $\gamma(bc)$, then λ_1 and λ_2 are parallel to the line joining b to the median point of triangle acd. This is not possible. Similarly s_2 cannot lie on arc $\gamma(ad)$. It is obvious that s_2 cannot lie on arc $\gamma(ab)$. Thus this case cannot arise and we are left with the case when each of s_1, s_2 is one of a, b, c, d. Suppose for example that s_1 is a and s_2 is b. The planes π_1 and π_2 through a and b respectively and both perpendicular to ab are support planes of X. Consider parallel lines λ_1, λ_2 through a and b respectively lying in π_1 and in π_2. On at least one site of the plane spanned by lines λ_1 and λ_2 there lies an arc $\gamma(ax)$ terminating at a and lying both on the frontier of X and on the frontier of the ball centre b and radius R. The reflection of this arc in the plane that perpendicularly bisects ab also lies in X and terminates at b. But of two parallel planes τ_1, τ_2 one through each of λ_1 and λ_2 one at least must cut one of the arcs $\gamma(ax)$ or its reflection. Thus of these two planes one at least does not support X. Hence a, b do not have the properties required of s_1 and s_2. Thus finally X_V is s mooth.

Next construct for every $\eta > 0$ a compact, smooth, rotund, convex, central set K_η such that $K_\eta \supset X_V$ and the frontier of K_η contains the whole of the frontier of X_V except possibly some of the points whose distance from $p-q$ or $q-p$ is less than η. Also suppose that K_η is distinct from X. This is possible because of the smoothness and rotundity of X_V.

Now if η is sufficiently small every frontier point of X is distant $D(X)$ from at least one other point of X where distance is measured in $B(K_\eta)$. (Because for some $\chi > 0$ every frontier point x of X is diametrically opposite a point y of X such that the acute angle made by xy with greater than χ). But then X is diametrically maximal in $B(K_\eta)$ without being of constant width.

3. **Miscellaneous remarks.** We say that a Banach finite dimensional space $B(S)$ has (i) property (A) if and only if every set in $B(S)$ that is diametrically maximal is necessarily of constant width, (ii) property (B) if and only if every set in $B(S)$ that is diametrically maximal is necessarily a sphere, (iii) property (C) if and only if every set in $B(S)$ that is of constant width is necessarily a sphere.

It is not easy to see what simple property distinguishes the spaces $B(S)$ that have property (*A*) from those which do not have property (*A*), and in this paragraph a number of results are given which partially elucidate this problem.

Let $B(S_1)$ and $B(S_2)$ be n-dimensional and m-dimensional Banach spaces respectively and let $B(S)$ be the $m+n$ dimensional Banach space whose unit sphere $B(S)$ is the cartesian product of S_1 and of S_2. We prove the following lemma.

LEMMA. *If X is a diametrically maximal subset of $B(S)$ then X is the cartesian product of two sets X_1, X_2 such that X_1 is a diametrically maximal subset of $B(S_1)$ and X_2 is a diametrically maximal subset of $B(S_2)$.*

Suppose (as we may without loss of generality) that $D(X) = 1$. We have by the spherical intersection property

$$X = X_S = \underset{x \in X}{\Omega} S(x, 1)$$

$$= \underset{x_1 \times x_2 = x \in X}{\Omega} S_1(x_1, 1) \times S_2(x_2, 1)$$

$$= (\Omega\, S_1(x_1, 1)) \times (\Omega\, S_2(x_2, 1)),\ x_1 \times x_2 = x \in X,$$

$$= \underset{x_1 \in X_1}{\Omega} S_1(x_1, 1) \times \underset{x_2 \in X_2}{\Omega} S_2(x_2, 1)$$

where X_1 is the projection of X on $B(S_1)$ and X_2 is the projection of X on $B(S_2)$. Now for any $x \in X$, $S(x, 1) \supset X$ and thus for $x_1 \in X_1$ $S_1(x_1, 1) \supset X_1$. Thus

$$\underset{x_1 \in X_1}{\Omega} S_1(x_1, 1) \supset X_1 \text{ and similarly } \underset{x_2 \in X_2}{\Omega} S_2(x_2, 1)) \supset X_2.$$

To complete the proof of the lemma we have only to show that $X \subset X_1 \times X_2$. Bur this is trivial since X_1 and X_2 are two projections of X.

The lemma is proved.

COROLLARY 1. *If both of $B(S_1)$ and $B(S_2)$ have any one of the properties (A) (B) or (C) then $B(S)$ has the same property.*

This is because the cartesian product of two sets X_1 in $B(S_1)$ and X_2 in $(B(S_2)$ which are both of constant width (or both spheres) is a set X in $B(S)$ that is of constant width (or a sphere).

COROLLARY 2. *If S is a parallelopiped then $B(S)$ has property (B).* This follows from Corollary 1 because a linear Banach space has property (*B*).

If the subset X is of constant width in $B(S)$ and if P_π denotes projection orthogonally (in the Euclidean sense) onto a two dimensional space π then $P_\pi(X)$ is of constant width in $B(P_\pi(S))$. The converse of this is true if $P_\pi(X)$ is of constant width in $B(P_\pi(S))$ for every plane π then X is of constant width in $B(S)$.

However a set X can be diametrically maximal in $B(S)$ without it being true that $P_\pi(X)$ is diametrically maximal in $B(P_\pi(S))$ for every plane π. For, as we shall see, in any plane property (*A*) holds and thus if the above statement were false, property (A) would hold for any finite dimensional Banach space.

Thus a space $B(S)$ fails to have property (A) if and only if there exists a subset X of $B(S)$ such that (i) for every $x \in X$ the frontier of $S(x, D(X))$ meets X and (ii) for some $x \in X$ and some plane π $P_\pi(S(x, D(X)) \cap X)$ is contained in the relative interior of $P_\pi(X)$, whilst x belongs to the frontier of $P_\pi(X)$.

Any two dimensional Banach space has property (A)

We show first that if X is a diametrically maximal subset of the two dimensional Banach space $B(S)$ then the set of points of X that are diametrically opposite to x (denoted by $P(x)$) either consist of a single point or form an arc of the frontier of X. This is because

(i) for each $x \in FrX$, $P(x) \neq \emptyset$, and

(ii) if $y \in P(x)$ then there exist parallel support lines of X, one through each of x and y (for otherwise X would contain a segment parallel to and longer than segment xy, which is impossible as the length of segment xy is $D(X)$).

Suppose then that y, $z \in P(x)$ and w lies on the arc yz of FrX that does not contain x. Since x lies on lines of support of X parallel to lines of support through y and z it follows that, every support line, parallel to a support line of X through w, must pass through x. Thus $P(w) \subset x$ and by (i) $P(w) = x$. Thus $w \in P(x)$. Since $P(x)$ is closed the stated result is proved. To complete the proof of the lemma let λ_1, λ_2 be any pair of parallel support lines of X. Take $x \in \lambda_1 \cap X$. If $P(x)$ did not meet $\lambda_2 \cap X$ we could find a point w of the frontier of X lying in that arc between $\lambda_2 \cap X$ and $P(x)$ which does not contain x. Then by an argument similar to that above $w \in P(x)$. It follows that $P(x)$ does in fact meet $\lambda_2 \cap X$. Thus on λ_1, λ_2 there are points of X whose distance apart is $D(X)$. This implies that the support lines of X_V coincide with those of $D(X).S$. Thus finally X_V is $D(X).S$. i.e. X is of constant width.

The lemma is proved.

In the first example of a space that was a finite dimensional Banach space and which did not have property (A), we considered $B(K)$ where K was a central convex set obtained by modifying the vector domain T_V of a regular tetrahedron T. In fact the space $B(T_V)$ itself does not have the property (A) because it has at least one vertex lying on four distinct edges of T_V. This follows from the lemma below.

LEMMA. *The space $B(S)$ does not have property (A) when S is a 3 dimensional polyhedron of whose vertices at least one lies on at least 4 edges of S.*

Let a vertex as described be x. Then there are two 2 dimensional faces of S say π_1 and π_2 both containing x and meeting in a line λ which meets S in the point x only. There is a support plane π of S such that $\pi \supset \lambda$ and $\pi \cap S$ is the single point x. A plane $\pi(\eta)$ parallel to π, distant η from π and lying on the same side of π as S meets S in a polygon of which two sides are parallel to λ (provided η is sufficiently small). Select points x_1 and x_2 on the relative interiors of these two sides. Let S_1 be the set obtained from S by the translation $x_1 - x_2$. The set $S_1 \cap S$ has $\pi(\eta)$ as a support plane and in this plane a segment s parallel to λ. The segment s together with the points 0, $x_1 - x_2$ form a set of diameter 1 and therefore a subset of a diametrically maximal set Y. Now Y contains 0 and $x_1 - x_2$. Thus Y is contained in S and in S_1. Thus $\pi(\eta)$ is a support plane of Y. Hence Y_V meets the support plane parallel to π in a set of dimension at least 1. Thus $Y_V \neq \lambda S$ and $B(S)$ does not have property (A).

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