TWO TOPOLOGICAL PROPERTIES OF TOPOLOGICAL LINEAR SPACES*

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ABSTRACT
A topological and a geometrical-topological property, previously known only for normed linear spaces, are established here for much more general classes of topological linear spaces.

Introduction. Throughout the present paper, E and E' will denote topological linear spaces (real scalars, separation axiom assumed). A *convex body* in E or E' is a convex set which has nonempty interior. Our two main results are as follows.

THEOREM A. *If E is infinite-dimensional and admits a countable family of open (or closed) convex bodies whose intersection consists of a single point, then for each point p of E the spaces E and $E \sim \{p\}$ are homeomorphic.*†

THEOREM B. *Every closed convex body in E is homeomorphic with a closed halfspace or with the product of an n-cell by a closed linear subspace of finite deficiency n in E.*††

These results were first established in [2] for Hilbert space, and were extended in [3] and [1] to arbitrary normed linear spaces.† Note that infinite-dimensionality is required for A but not for B. The topological property expressed in A has a number of interesting consequences; in particular, it implies that E admits a fixed-point-free homeomorphism of period two. Property B is useful in connection with the topological classification of closed convex bodies.

Our methods are analogues or refinements of those employed previously, and as before the notions of gauge functional and characteristic cone will play an important role. When y is an interior point of a convex body U in E, the *gauge functional of U with respect to y* is the real-valued function μ_{Uy} defined as follows for all $x \in E$:

$$\mu_{Uy}(x) = \inf \left\{ \lambda > 0 : \frac{1}{\lambda}(x - y) \in U \right\}.$$

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†, †† See the footnotes on the last page

When y is the origin 0, we speak simply of the *gauge functional* of U and write μ_U rather than μ_{U0}. In several instances below, we shall define a transformation geometrically and leave to the reader the routine but occasionally tedious verification that the transformation is actually a homeomorphism. This can often be accomplished by expressing the transformation in terms of the appropriate gauge functionals and then making use of the well-known continuity of the function $\mu_{Uy}(x) \mid (x, y) \in E \times \operatorname{int} U$.

For $y \in \operatorname{int} U$, we define

$$cc_y U = \{x \in U : \mu_{Uy}(x) = 0\},$$
$$ccU = cc_y U - y, \text{ and } csU = cc(U \cap (y - U)).$$

Thus the sets $y + ccU$ and $y + csU$ are the unions with $\{y\}$ of, respectively, all rays and all lines which issue from y and lie in U. The sets ccU and csU, being independent of the choice of $y \in \operatorname{int} U$, are called respectively the *characteristic cone* and the *characteristic subspace* of U. Note that the convex cone ccU is a linear subspace if and only if $ccU = csU$.

NOTATION

The interior, boundary, closure and convex hull of a set X are denoted by $\operatorname{int} X$, ∂X, $\operatorname{cl} X$ and $\operatorname{con} X$ respectively. The fact that X and Y are homeomorphic is indicated by $X \approx Y$. Set-theoretic addition and subtraction are indicated by $\cup$ and $\sim$ respectively, while $+$ and $-$ are reserved for algebraic operations. The real number field and the set of all positive integers are denoted by $\mathfrak{R}$ and $\mathfrak{N}$ respectively. Equality by definition is indicated by $\cdot =$ or $= \cdot$. When x and y are distinct points of a linear space, the open segment connecting them is denoted by $]x, y[$, the half-open segments by $]x, y]$ and $[x, y[$, and the closed segment by $[x, y]$ The open and closed rays which issue from x and pass through y are denoted by $]x, y($ and $[x, y($ respectively.

1. Three Propositions.

The three propositions of this section are used later in proving Theorems A and B.

1.1. PROPOSITION. *If $(L, \| \ \|)$ is an infinite-dimensional normed linear space, then the linear space L admits norms $| \ |$ and $||| \ |||$ such that*

$$| \ | \leqq \| \ \| \leqq ||| \ |||$$

and the spaces $(L, | \ |)$ and $(L, ||| \ |||)$ are both incomplete.

Proof. We assume that the space $(L, \| \ \|)$ is complete, for otherwise there is nothing to prove. Let B be a Hamel basis for L such that $\| b \| \leqq 1$ for all $b \in B$, and $\inf\{\| b \| : b \in B\} = 0$. Let $||| \ |||$ denote the gauge functional of the set $\operatorname{con}(B \cup -B)$ with respect to the origin 0. Then $||| \ |||$ is a norm for L and $\| \ \| \leqq ||| \ |||$, so the natural mapping τ of $(L, ||| \ |||)$ onto $(L, \| \ \|)$ is continuous. If the space $(L, |||\ |||)$ is complete, then with $(L, \|\ \|)$ also complete, it follows from the open mapping theorem that τ is a homeomorphism. But that is impossible, for the point 0 is in the $\|\ \|$-closure of B but not in the $|||\ |||$-closure of B.

Actually, it is the existence of $|\ |$ rather than $|||\ |||$ which is used in the sequel. To obtain $|\ |$, let M be a separable infinite-dimensional linear subspace of $(L, \|\ \|)$, and use a construction in [3] to produce an unbounded closed convex body V in M such that V is linearly bounded and $V = -V$. Let U denote the unit ball of the space $(L, \|\ \|)$ and let $C = \operatorname{con}(U \cup V)$. Then of course C is a convex body in $(L, \|\ \|)$ and $C = -C$. Now suppose that C contains a line J through 0, and consider an arbitrary point x of J. For each $n \in \mathfrak{N}$ there exist $u_n \in U$, $v_n \in V$ and $\lambda_n \in [0, 1]$ such that

$$nx = \lambda_n u_n + (1 - \lambda_n) v_n.$$

Since the sequence $\{(\lambda_n/n) u_n\}_{n \in \mathfrak{N}}$ is convergent to 0 and since always

$$\frac{1 - \lambda_n}{n} v_n \in V,$$

it follows that $x \in V$. This implies that $J \subset V$, an impossibility since V is linearly bounded. We conclude that the set C is linearly bounded, whence the gauge functional $|\ |$ of C is a norm for L. Clearly $|\ | \leqq \|\ \|$. If the spaces $(L, \|\ \|)$ and $(L, |\ |)$ were both complete, the open mapping theorem would lead to a contradiction as in the preceding paragraph, for the point 0 is in the $|\ |$-closure of the set $\{y \in M : \|y\| = 1\}$ but not in its $\|\ \|$-closure. The proof is complete.

A *strip* is the set of all points lying on or between two parallel hyperplanes. Proof of the following is left to the reader.

1.2 LEMMA. *Suppose that H and H' are closed hyperplanes, S and S' are closed strips, and Q and Q' are closed halfspaces such that*

$$\partial Q = H \subset \partial S \subset S \subset Q \subset E \text{ and } \partial Q' = H' \subset \partial S' \subset S' \subset Q' \subset E'.$$

Then every homeomorphism of H onto H' can be extended to a homeomorphism of E onto E' which carries Q onto Q' and S onto S'. Further, H and H' are homeomorphic if $E = E'$.

1.3. LEMMA. *Suppose that U, V and P are closed convex bodies in E, Q is a closed halfspace, y is a point of $E \sim \{0\}$, and the following four conditions are all satisfied:*

(i) $P \subset \operatorname{int} Q$ *and* $U \subset \operatorname{int} V$;

(ii) $0 \in \partial Q \cap \partial V$;

(iii) $[1, \infty[y \subset \operatorname{int}(P \cap U)$;

(iv) *V does not contain any ray which issues from y, and passes through a point of ∂Q.*

Then there is a homeomorphism of E onto E which carries U onto P and V onto Q.

Proof. Let $W \cdot = (U - y) \cap (y - U) \cap \partial Q$, a closed convex body relative to ∂Q. Then $W = -W$ and $ccW = csW = csU$, where the latter equality depends on (iv) and the fact that $U \subset V$. For each point $q \in \partial Q$, let $c(q) \cdot = (1 + \mu_W(q))y \in \operatorname{int}(P \cup U)$ and let the (unique) points at which the ray $]c(q), q($ intersects the sets ∂U, ∂V and ∂P be denoted by $u(q)$, $v(q)$ and $p(q)$ respectively. The existence of these points follows from conditions (iv), (iii) and (i), and from (i) it follows that $p(q) \in]c(q), q]$ and $u(q) \in]c(q), v(q)[$. Let $t(q) \cdot = \frac{1}{2}c(q) + \frac{1}{2}u(q)$. Since the set $E \sim (csU + [1, \infty[y)$ is simply covered by the family of open rays

$$\{]c(q), q(: q \in \partial Q\},$$

we can define a biunique transformation ξ of E onto E by specifying that ξ is the identity on $csU + [1, \infty[y$ and that for each $q \in \partial Q$, ξ is the identity on the segment $[c(q), t(q)]$, carries the segments $[t(q), u(q)]$ and $[u(q), v(q)]$ affinely onto the segments $[t(q), p(q)]$ and $[p(q), q]$ respectively, and translates the ray $c(q) + [1, \infty[(v(q) - c(q))$ onto the ray $c(q) + [1, \infty[(q - c(q))$. With the aid of the continuity properties of the relevant gauge functionals, it is tedious but not difficult to verify that ξ is the desired homeomorphism of E onto E. This completes the proof of 1.3.

When U and V are subsets of E, we shall write $U \subset\subset V$ to indicate the existence of a neighborhood G of the origin such that $U + G \subset V$. A closed convex body will be said to be of *type Q* provided its characteristic cone is not a linear subspace.

1.4. Lemma. *Suppose that U and V are closed convex bodies of type Q in E, with $U \subset\subset V$. Then there is a homeomorphism of E onto E which carries U and V onto a pair of parallel halfspaces.*

Proof. Clearly $ccU \subset ccV$ and $csU \subset csV$. Suppose first that the set $ccU \sim csV$ is nonempty, and choose $y \in \operatorname{int} U$. Then there is a point $x \in \partial V$ such that $\operatorname{int} U$ contains the ray $x + [1, \infty[(y - x)$. We assume without loss of generality that $x = 0$. Let Q be a closed halfspace such that $0 \in \partial Q$ and $V \subset Q$, and let P be the translate of Q such that $\frac{1}{2}y \in \partial P$. Then the conditions of 1.3 are satisfied and the desired conclusion follows. (Here the full strength of the condition $U \subset\subset V$ was not required; it was sufficient to have $U \subset \operatorname{int} V$.)

In the remaining case, $ccU \subset csV$ and 1.3 does not apply directly. However, we can apply 1.3 in two stages with the aid of a closed convex body J (to be constructed) such that $U \subset \operatorname{int} J$, $J \subset \operatorname{int} U$, and

$$ccU \sim csJ \neq \emptyset \neq ccJ \sim csV.$$

By the preceding paragraph, there are homeomorphisms ξ and η of E onto E such that $(\xi U, \xi J)$ and $(\eta J, \eta V)$ are pairs of parallel halfspaces. Let K be a closed halfspace which is contained in the interior of ηJ and hence is parallel to ηJ. By 1.2, the homeomorphism $\eta\xi^{-1}$ of $\partial\xi J$ onto $\partial\eta J$ can be extended to a homeomorphism ζ of E onto E which carries the strip $cl(\xi J \sim \xi U)$ onto the strip $cl(\eta J \sim K)$. For each point $w \in E$, let

$$\tau(w) \cdot = \begin{cases} \eta(w) & \text{if} \quad w \in E \sim \operatorname{int} J \\ \zeta(\xi(w)) & \text{if} \quad w \in J. \end{cases}$$

Then τ is a homeomorphism of E onto E which carries U onto K and V onto ηV. Thus it remains only to construct the intermediate body J.

In constructing J, we assume without loss of generality that $0 \in \operatorname{int} U$. Since U and V are both of type Q, there are points $u, v \in E \sim \{0\}$ such that

(1) $$[0, u(\subset U \not\supset [0, -u(\text{ and } [0, v(\subset V \not\supset [0, -v(.$$

On the other hand, the fact that $ccU \subset csV$ implies that

(2) $$[0, -u(\subset V \text{ and } [0, v(\not\subset U.$$

For each $\lambda > 0$, let L_λ denote the line $-\lambda u + \Re v$. Since U is closed and convex it is easy to derive from (1) and (2) the existence of $\lambda > 0$ such that $L_\lambda \cap U = \varnothing$. But then $L_{2\lambda} \cap 2U = \varnothing$, and by a standard separation theorem there is a closed halfspace Q in E such that $L_{2\lambda} \subset \partial Q$ and $2U \subset Q$; the latter condition implies that $U \subset\subset Q$. Let

$$J \cdot = \tfrac{1}{2}U + \tfrac{1}{2}(V \cap Q).$$

Then $U \subset\subset J$ because $U \subset\subset V \cap Q$, while $J \subset\subset V$ because $U \subset\subset V$ and $V \cap Q \subset V$. From the relevant definitions in conjunction with (1) and (2), it follows that

$$u \in ccU \sim csJ \text{ and } v \in ccJ \sim csV.$$

Thus the proof of 1.4 is complete.

1.5. Proposition. *If V and V' are closed convex bodies in E which have the same characteristic cone or are both of type Q, then there is a homeomorphism of E onto E which carries V onto V' and ∂V onto $\partial V'$.*

Proof. Suppose first that $ccV = ccV'$, and let τ and τ' be translations of E such that

$$0 \in W \cdot = \operatorname{int} \tau V \cap \operatorname{int} \tau' V'.$$

Of course, $cc\tau V = cc\tau' V'$. For each point $v \in \partial(\tau V)$, let the points w_v and v' be defined by the conditions that

$$w_v \in \partial W \cap [0, v(\text{ and } v' \in \partial(\tau V) \cap [0, v(.$$

Let η be the identity transformation on W, and for each $v \in \partial(\tau V)$ let η map the segment $[w_v, v]$ affinely onto the segment $[w_v, v']$ and translate the ray $[1, \infty[v$ onto the ray $[1, \infty[v'$. Then η is a homeomorphism of E onto E which carries τV onto $\tau' V'$, and the transformation $\tau'^{-1}\eta\tau$ has the properties desired in 1.5.

In view of 1.2, it suffices for the other case in 1.5 to show that an arbitrary closed convex body V of type Q in E can be carried onto a closed halfspace by means of a homeomorphism of E onto E. But this is a special case of 1.4, for if $0 \in \operatorname{int} V$ then the set $U \cdot = \frac{1}{2} V$ is a convex body of type Q with $U \subset\subset V$.

A finite or infinite sequence $V_1, V_2, \cdots$ of closed convex bodies in E will be called *nested* provided $V_1 \neq E$ and one of the following two conditions is satisfied:

(*) the convex bodies V_i all have the same characteristic cone; $V_{i+1} \subset \operatorname{int} V_i$ $(i = 1, 2, \cdots)$;

(**) the convex bodies V_i are all of type Q; $V_{i+1} \subset\subset V_i$ $(i = 1, 2, \cdots)$.

1.6. PROPOSITION. *Suppose that E and E' are topological linear spaces, that $V_1, V_2, \cdots$ is a nested sequence of convex bodies in E, that $V_1', V_2', \cdots$ is a nested sequence of convex bodies in E', and that the two sequences are of the same length. Then every homeomorphism of ∂V_1 onto $\partial V_1'$ can be extended to a homeomorphism of $E \sim \bigcap_i \operatorname{int} V_i$ onto $E' \sim \bigcap_i \operatorname{int} V_i'$ which (for each i) carries ∂V_i onto $\partial V_i'$.*

(Presumably, the proposition remains valid when "$V_{i+1} \subset\subset V_i$" is replaced by "$V_{i+1} \subset \operatorname{int} V_i$" in condition (**). However, the replacement seems to entail some technical complications and the present form of the proposition is adequate for our needs.)

Proof. It suffices to consider the case in which the sequences involve only V_1, V_2 and V_1', V_2' respectively. For if this is known, then by its use the given homeomorphism of ∂V_1 onto $\partial V_1'$ can be extended to a homeomorphism η_1 of $V_1 \sim \operatorname{int} V_2$ onto $V_1' \sim \operatorname{int} V_2'$ such that $\eta_1(\partial V_2) = \partial V_2'$. The restriction of η_1 to ∂V_2 can then be extended to a homeomorphism η_2 of $V_2 \sim \operatorname{int} V_3$ onto $V_2' \sim \operatorname{int} V_3'$ such that $\eta_2(\partial V_3) = \partial V_3'$. Then the restriction of η_2 to ∂V_3 can be extended$\cdots$. Proceeding in this way to obtain a sequence $\eta_1, \eta_2, \cdots$ of homeomorphisms, we find that $\bigcup_i \eta_i$ is the homeomorphism required in the statement of 1.6.

Now if $ccV_1 = ccV_2$, $ccV_1' = ccV_2'$, and ξ is a homeomorphism of ∂V_1 onto $\partial V_1'$, a straightforward application of gauge functionals extends ξ_1 to a homeomorphism of $E \sim \operatorname{int} V_2$ onto $E' \sim \operatorname{int} V_2'$. (See 1.1 of [1].) The other cases can be reduced to this one, for if V_1 and V_2 are both of type Q then 1.4 guarantees the existence of a homeomorphism of E onto E which carries V_1 and V_2 onto a nested pair of closed halfspaces, and of course these halfspaces have the same characteristic cone.

2. **Proof of Theorem A.** The following result provides some alternative characterizations of the spaces for which Theorem A will be established.

2.1. PROPOSITION *If E is a topological linear space and $\aleph$ is an infinite cardinal number, then the following four assertions are equivalent:*

(i) *E contains $\aleph$ closed convex bodies whose intersection is $\{0\}$;*

(ii) *E contains $\aleph$ closed convex bodies whose characteristic cones have intersection $\{0\}$;*

(iii) *E contains $\aleph$ open convex bodies whose intersection is $\{0\}$;*

(iv) *E contains $\aleph$ open convex bodies whose characteristic cones have intersection $\{0\}$.*

In each case, the convex bodies may be chosen so that the origin 0 is interior to all of them and so that all are of type Q or are linearly bounded and centrally symmetric (about 0). In the latter circumstance, we may take $\aleph = \aleph_0$.

When $\aleph = \aleph_0$, we may require that the $\aleph_0$ convex bodies are arranged in a sequence $C_1, C_2, \cdots$ such that $C_{n+1} + C_{n+1} \subset C_n$ for all $n \in \mathfrak{N}$.

Proof. (i) $\rightarrow$ (iii). Suppose that $\mathscr{B}$ is a family of $\aleph$ closed convex bodies such that $\bigcap \mathscr{B} = \{0\}$. For each $B \in \mathscr{B}$, choose a point $p_B \in \operatorname{int} B$ and then for each $n \in \mathfrak{N}$ let $B_n \cdot = \operatorname{int} B - (1/n)p_B$. Then each B_n is an open convex body and $0 \in \bigcap_{n \in \mathfrak{N}} \operatorname{cl} B_n \subset B$. The family $\{B_n: B \in \mathscr{B}, n \in \mathfrak{N}\}$ has intersection $\{0\}$, and with $\aleph \geqq \aleph_0$ it has the same cardinality as $\mathscr{B}$.

(iii) $\Rightarrow$ (iv). Note that if $0 \in B$ and B is open, then $ccB \subset B$.

(iv) $\Rightarrow$ (ii). Suppose that $\mathscr{B}$ is a family of $\aleph$ open convex bodies such that $\bigcap \{ccB : B \in \mathscr{B}\} = \{0\}$. For each $B \in \mathscr{B}$, choose a point $p_B \in B$ and let $B' \cdot = \frac{1}{2} cl(B - p_B)$. Then B' is a closed convex body and $ccB' = ccB$, so $\bigcap \{ccB' : B \in \mathscr{B}\} = \{0\}$.

(ii) $\Rightarrow$ (i). Suppose that $\mathscr{B}$ is a family of $\aleph$ closed convex bodies such that $\bigcap \{ccB : B \in \mathscr{B}\} = \{0\}$. For each $B \in \mathscr{B}$, choose a point $p_B \in \operatorname{int} B$ and then for each $n \in \mathfrak{N}$ let $B_n \cdot = (1/n)(B - p_B)$. Then each B_n is a closed convex body and $\bigcap_{n \in \mathfrak{N}} B_n = ccB$, so the desired conclusion follows.

In the above discussion, the point 0 is always interior to the sets B_n and B'; further, the sets B_n and B' are linearly bounded or of type Q if and only if the same is true of the set B. Thus in restricting the type of the convex bodies in question, it suffices to consider condition (iii). Suppose, then, that $\mathscr{B}$ is a family of open convex bodies in E whose intersection is $\{0\}$. If some member B of $\mathscr{B}$ is linearly bounded, then the same is true of the centrally symmetric sets $(1/n)(B \cap -B)$, and of course $\bigcap_{n \in \mathfrak{N}} (1/n)(B \cap -B) = \{0\}$. Suppose, on the other hand, that no member of $\mathscr{B}$ is linearly bounded. Then for each $B \in \mathscr{B}$ there exists $q_B \in E \sim \{0\}$ such that $[0, \infty[q_B \subset B$. Since $\bigcap \mathscr{B} = \{0\}$, there exists $C_B \in \mathscr{B}$ such that $-q_B \notin C_B$, and then by the separation theorem there is an open half space $Q_B \supset C_B$ such that $-q_B \notin Q_B$. But then $[0, \infty[q_B \subset Q_B$, and with $B'' \cdot = B \cap Q_B$ we have $ccB'' \neq csB''$. The family $\{B'' : B \in \mathscr{B}\}$ has intersection $\{0\}$ and its members are all open convex bodies of type Q.

For the last assertion of 2.1, it suffices to observe that if $G_1, G_2, \cdots$ is a sequence of convex sets whose intersection is $\{0\}$, then the same is true of the sequence $C_1, C_2, \cdots$ where $C_j \cdot = 2^{-j} \bigcap_{i=0}^{j} G_i$. Further, $C_{n+1} + C_{n+1} \subset C_n$ for all $n \in \mathfrak{N}$.

2.2. THEOREM A. *Suppose that E is an infinite-dimensional topological linear space which admits a countable family of open (or closed) convex bodies whose intersection consists of a single point. Then for each point p of E, the spaces E and $E \sim \{p\}$ are homeomorphic.*

Proof. By 2.1, there exists a sequence $C_1, C_2, \cdots$ of closed convex bodies in E such that the following four conditions are all satisfied:

(1) $\bigcap_{n \in \mathfrak{N}} C_n = \{0\}$;
(2) $0 \in \operatorname{int} C_n$ for all $n \in \mathfrak{N}$;
(3) $C_{n+1} + C_{n+1} \subset C_n$ for all $n \in \mathfrak{N}$;
(4) each set C_n is of type Q, or each set C_n is linearly bounded and centrally symmetric about 0.

Suppose first that each set C_n is of type Q, and let u be a point of $\operatorname{int} C_1$ such that $[0, \infty[u \subset C_1$ but $-u \notin C_1$. Let $C_1' = C_1$, and for $n > 1$ let $C_n' = (1/n)C_1 + nu$. Then each set C_n is of type Q, $\bigcap_{n \in \mathfrak{N}} C_n' = \varnothing$, and the sequence $C_1', C_2', \cdots$ is nested. The sequence $C_1, C_2, \cdots$ is nested by conditions (2) and (3). Let ξ be the identity mapping on $E \sim \operatorname{int} C_1 = E \sim \operatorname{int} C_1'$. Since $\bigcap_{n \in \mathfrak{N}} C_n = \{0\}$ while $\bigcap_{n \in \mathfrak{N}} C_n' = \varnothing$, it follows from 1.6 that ξ can be extended to a homeomorphism of $E \sim \{0\}$ onto E.

In the remaining case, the set C_1 is centrally symmetric and linearly bounded. Let $\| \ \|$ denote the gauge functional of C_1, and use 1.1. to obtain a norm $| \ |$ for E such that $| \ | \leqq \| \ \|$ and the space $(E, | \ |)$ is incomplete. Let $\tilde{E}$ denote the completion of E with respect to the norm $| \ |$ and let $p \in \tilde{E} \sim E$ with $|p| < \frac{1}{2}$. For n > 1, let $C_n' = \{x \in E \colon \|x - p\| \leqq 2^{-n}\}$, and let $C_1' = C_1$. Then $C_1, C_2, \cdots$ is a nested sequence of closed convex bodies whose intersection is $\{0\}$ and $C_1', C_2', \cdots$ is a nested sequence of closed convex bodies whose intersection is $\varnothing$, so the use of 1.6 leads to a homeomorphism of $E \sim \{0\}$ onto E.

2.3. COROLLARY. *An infinite-dimensional topological linear space E has property (A) if it satisfies any of the following conditions:*

(i) *E contains a linearly bounded convex body;*
(ii) *E admits a countable separating family of continuous linear forms;*
(iii) *E is separable, metrizable and locally convex.*

Note that property (A) is not possessed by every infinite-dimensional locally convex topological linear space. Indeed, let U be an uncountable set and let E be the space of all bounded real functions on U, in the topology of pointwise convergence. (That is, E is a subspace of $\mathfrak{R}^U$.) Then E is σ-compact, but (with $p \in E$) $\{p\}$ is not a G_δ set in E and consequently $E \sim \{p\}$ is not σ-compact.

3. Proof of Theorem B.

3.1. THEOREM B. *Suppose that U is a closed convex body in a topological linear space E. If the characteristic cone ccU is not a linear subspace or is a linear subspace of infinite deficiency, then U is homeomorphic with a halfspace in E. If ccU is a linear subspace of finite deficiency n, then U is homeomorphic with the product $ccU \times [0,1]^n$.*

Proof. The case in which ccU is not a linear subspace is handled immediately by 1.5. If ccU is a linear subspace of finite deficiency n, then E is topologically and algebraically the direct sum of ccU and an n-dimensional linear subspace L of E. It is clear that U is homeomorphic with the product $ccU \times (U \cap L)$ and that $U \cap L$ is homeomorphic with $[0,1]^n$. There remains only the case in which ccU is a linear subspace of infinite deficiency. With $0 \in ccU = csU$, we have

$$cc(U \cap -U) = cs(U \cap -U) = csU.$$

In view of 1.5, we may assume (in treating the remaining case) that the closed convex body U is centrally symmetric. Choose $u_0 \in \partial U$ and let f be a linear form on E such that $f(u_0) = 1$ and $fU \subset]-\infty, 1]$. Let Q denote the halfspace $\{x : f(x) \geqq 0\}$, whence

$$Y \cdot = csU \subset \partial Q = \{x : f(x) = 0\}.$$

Let

$$U' = \{x : (\mu_U(x - f(x)u_0))^2 + f(x)^2 \leqq 1\}.$$

Then U' is a closed convex body with $ccU' = ccU$, whence $U \approx U'$ by 1.5 and also $\partial U \approx \partial U'$. For each $x \in \partial U \sim (u_0 + Y)$, let

$$\tau(x) = u_0 + \frac{1}{1 - f(x)}(u_0 - x),$$

so that τ is a "stereographic projection" translated by the vector u_0. It can be verified that τ is a homeomorphism of $\partial U \sim (u_0 + Y)$ onto $\partial Q \sim Y$. Thus to complete the proof it suffices to prove the following:

(1) $\partial Q \approx \partial Q \sim Y$;
(2) $\partial U \approx \partial U \sim (u_0 + Y)$;
(3) $U \approx U \times]0,1]$.
(4) $Q \approx \partial Q \times]0,1]$.

Assertion (4) is obvious. In connection with the others, let us prove:

($*$) *There is a homeomorphism of E onto $E \sim Y$ which carries U onto $U \sim Y$ and is the identity on $E \sim \operatorname{int} U$.*

The gauge functional μ_U is a seminorm on E. Let $([E], \| \ \|)$ be the normed linear space corresponding to (E, μ_U) in the usual way, and for each $x \in E$ let $[x]$ denote the corresponding element of $[E]$. Since Y is of infinite deficiency, the space $[E]$ is infinite-dimensional and hence by 1.1 $[E]$ admits a norm $| \ | \leqq \| \ \|$ such that the space $([E], | \ |)$ is incomplete. Let $(\tilde{E}, | \ |)$ be the completion of the space $([E], |\ |)$, and choose $q \in \tilde{E} \sim [E]$ with $|q| < \frac{1}{2}E$. Define $E' \cdot = E$, $V_n \cdot = nU$ for $n \in \mathfrak{N}$, $V_1' \cdot = U$, and

$$V_n' = \{x \in E : |[x] - q| \leqq 1/n\} \qquad (n = 2, 3, \cdots)$$

Then the conditions of 1.6 are all satisfied and the desired conclusion follows from 1.6.

Now (1) follows by applying $(_*)$ with the sets E and U replaced by ∂Q and $U \cap \partial Q$ respectively. (3) also follows easily from $(_*)$, for $U \sim Y$ is simply covered by the segments $]0, u]$ with $u \in \partial U$. It remains only to establish (2).

Let W denote the set $\{x : f(x) \leqq 0\}$. Then with the aid of $(_*)$ it is easy to see that $\partial W \approx \partial W \sim Y$. And of course $\partial W \approx \partial U$ by 1.5. But $\partial W \sim Y$ is the image of $\partial U \sim (u_0 + Y)$ under the homeomorphism σ given by

$$\sigma(x) = \frac{1}{2}u_0 + (\mu_{Uu_0/2}(x))^{-1}(x - \frac{1}{2}u_0).$$

This completes the proof.

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† (Added in proof.) Let us say that a subset X of a topological space E is *negligible* provided the spaces E and $E \sim X$ are homeomorphic. There are now several results which assert the negligibility of certain sets in various infinite-dimensional topological linear spaces. Theorem A above involves the most general class of spaces, but asserts the negligibility only of onepointed sets. It is also known that X is a negligible subset of E if any of the following conditions is satisfied: (i) X is compact and E is an infinite-dimensional normed linear space; (ii) X is compact and E is a topological linear space which admits a Schauder basis whose closure does not include the origin; (iii) X is weakly compact and E is a nonreflexive Banach space; (iv) X is weakly compact and E is a Banach space which admits a Schauder basis; (v) X is σ-compact and $E = R^{\aleph_0}$. The results (i) and (iii) appear in [3] and [2] respectively; (ii) and (iv) are in a recently completed paper of R.D. Anderson ("On a theorem of Klee"), and (v) is in another paper of Anderson ("Topological properties of the Hilbert cube and the infinite product of open intervals").

†† Theorem B is used in the authors' recently completed paper entitled "Every non-normable Frechet space is homeomorphic with all of its closed convex bodies."