Level Structure of Dual-Resonance Models (*).

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Summary. — In this paper the level structure of narrow-resonance models with duality is considered. We shall use to this purpose the multiparticle dual amplitudes recently proposed by several authors and shall study the structure of the residue of each pole in what concerns its factorizability. We find that for each energy eigenvalue $E_n = \sqrt{s_n}$ the residue does indeed factorize in a finite number of terms (number of degenerate levels) and that this number increases with n like $\exp[cE_n]$. The physical interpretation of this wild increase is found in the essential many body nature of models consistent with duality. The appearence of states with imaginary coupling follows easily from the covariant, four-dimentional approach that we have taken and that insures absence of kinematical singularities. It is nevertheless found that a cancellation mechanism analogous to the one existing in Q.E.D. (Ward identities) occurs here too. Although the problem of a systematic cancellation of all ghosts in a realistic case has not been solved, we see that the leading and most troublesome ghosts are indeed eliminated in this way.

1. – Introduction and contents of the paper.

Interesting developments have recently taken place in the field of strong interaction physics. The study of superconvergence relations, finite-energy sum rules together with the assumption of straight-line trajectories are leading

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to new ways of analyzing S-matrix theory (1). Of particular importance has been the idea of duality which was originally expressed by DOLEN, HORN and SCHMID (2) as saying that resonance and Regge-pole contributions should not be added coherently but that the Regge-pole term is already an average description of the full amplitude.

A proposal (3) for the construction of a simple Reggeized crossing symmetric amplitude of the form

$$A(s, t) = \text{const}\int_0^1 x^{-\alpha(s)-1}(1-x)^{-\alpha(t)-1}\,\mathrm{d}x + (\text{cyclic terms}) \tag{1.1}$$

allows to put the previous arguments on a more direct and explicit basis.

A prescription that generalizes eq. (1.1) to any number of external spinless particles has been recently suggested (4,5).

It is to be noted that this generalized representation could provide at least a first hint in order to construct rules valid for particles endowed with spin. Indeed it is possible to combine two or many external lines in order to create poles corresponding to particles with arbitrary spin. However, as we shall see later, due to the very complicated factorization properties of each pole, this general problem is not as simple as it looks at first sight.

Let us now consider more in detail eq. (1.1). The first term of this equation can be written in the two equivalent forms

$$A(s, t) = \sum_n \frac{C_n(t)}{s - s_n} = \sum_n \frac{C_n(s)}{t - t_n}. \tag{1.2}$$

Equation (1.2) allows the simplest possible interpretation of duality in terms of *resonances only*. It leads to a complete expansion of the full amplitude either in terms of resonances exchanged in the s-channel or equivalently in terms of t-channel resonances.

This property is of course fully shared by the generalized many-particle

(1) For a general review see W. R. FRAZER: *Proceedings of the XIV International Conference on High Energy Physics, Vienna, 1968* (edited by J. PRENTKI and J. STEINBERGER).

(2) R. DOLEN, D. HORN and C. SCHMID: *Phys. Rev.*, **166**, 1768 (1968).

(3) G. VENEZIANO: *Nuovo Cimento*, **57** A, 190 (1968).

(4) The 5-point function was given by K. BARDAKCI and H. RUEGG: *Phys. Lett.*, **28** B, 342 (1968); and by M. A. VIRASORO: *Phys. Rev. Lett.*, **22**, 37 (1969).

(5) The generalization to an arbitrary N-point function was given by H. M. CHAN: *Phys. Lett.*, **28** B, 425 (1969); H. M. CHAN and TSOU S. TSUN: *Phys. Lett.*, **28** B, 485 (1969); C. GOEBEL and B. SAKITA: *Phys. Rev. Lett.*, **22**, 257 (1969); K. BARDAKCI and H. RUEGG: Berkeley preprint (Dec. 1968).

amplitudes. In conclusion it is thus clear that the models we are discussing can be formulated in terms of resonances only. The validity of Regge behaviour in all channels ensures that all superconvergence relations are automatically satisfied in terms of the discrete levels appearing in the models.

In this paper we wish to discuss in detail the particle aspect of the Reggeized dual-resonance models.

By looking at eq. (1.2) one immediately learns that all resonances must have masses of the form

$$M_n^2 = A + Bn\,, \qquad n = 0, 1, 2 \ldots\,, \tag{1.3}$$

and that for each value of n all angular momenta $J = 0, 1, 2 \ldots n$ are present. The simplest interpretation would be to say that, for each value of n, there are just n particles, one for each possible value of J. However, this simple model looks already suspicious if we think that at $t = 0$ the asymptotic behavior of eq. (1.1) (which of course does satisfy all analyticity constraints) is not the one of a single Lorentz pole, which is prescribed by the simplest factorization [6].

We shall find that each level corresponding to a well-defined choice of n and J is in general degenerate. We shall also see that the level structure is much more complicated than what one could have guessed on the basis of Lorentz-pole considerations. The origin of this high degeneracy lies in the fact that the very stringent constraints of duality and superconvergence in all channels require a number of degrees of freedom which is of a different order of magnitude than that appearing in simple two-bodylike problems.

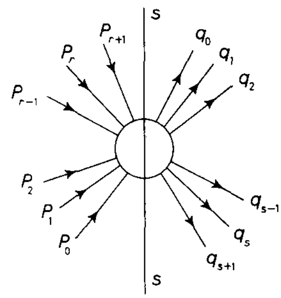

Fig. 1. – The generic multiparticle process considered in this paper. The line s defines the s channel: $s = (\sum p_i)^2 = (\sum q_i)^2$.

Our way of studying the level structure will be straightforward. Starting from the general multiparticle amplitude we shall consider (see Fig. 1) the poles in the variable

$$s = -(p_0 + p_1 + \ldots + p_r + p_{r+1})^2 = -(q_0 + q_1 + q_s + q_{s+1})^2\,. \tag{1.4}$$

The contribution corresponding to the pole at $s = s_n$ will in general be

$$A \underset{s\sim s_n}{=} \frac{R_n(p_0\, p_{r+1};\; q_0 \cdots q_{s+1})}{s - s_n}\,. \tag{1.5}$$

(6) See, for instance, J. B. Bronzan and C. E. Jones: *Phys. Rev. Lett.*, **21**, 564 (1968).

It is clear that the level will be a single one only if the residue R can be factorized (independently on the number of initial and final external lines) in the form

$$R_n(p, q) = F(p)\cdot F(q) \,. \tag{1.6}$$

In the case in which eq. (1.6) will not be valid, the degeneracy of the level will be obtained by decomposing R in the minimum number of linearly independent factors

$$R_n(p, q) = \sum_{i=1}^{d_n} F_i(p) F_i(q) \,. \tag{1.7}$$

Here again the number d_n of terms in eq. (1.7) should be independent on the number of initial and final lines.

The number of terms d_n will represent the degree of degeneracy of the level. Of course this multiplicity can be experimentally observed as soon as a small perturbation breaks the full degeneracy of the model. It is clear that, in order to give any meaning to the level structure, we must have that d_n is finite for any finite value of n and independent of the process. If a finite decomposition (1.7) is impossible the level structure will be an unphysical peculiarity of eq. (1.1) and will disappear as soon as any perturbation will be applied.

A troublesome question of a great theoretical importance concerns the possibility that the quadratic form (1.7) is a non positive definite one. This would correspond to the presence of «ghosts» which would have imaginary coupling constants (or, if one prefers, indefinite metric) whose presence would give an unphysical flavor to the whole business.

Unfortunately the possibility of ghosts will indeed be found in our investigation. However, even if our understanding of the problem is far from being complete, a mechanism for ghost compensation has been found. As a consequence, the situation in this respect is not at all hopeless.

In Sect. **2** we summarize some of the relevant features of the many-particle amplitudes in a notation appropriate to our program. Section **3** will be devoted to the actual derivation of the factorization properties. We shall see that the degree of degeneracy of each level is finite and increases fast with increasing mass. Ghosts do indeed creep in. In Sect. **4** we shall see that the model provides a natural mechanism of ghost cancellations which bears amusing analogy with the cancellation of timelike photons in quantum electrodynamics. Ward-like identities will be found. This will remove those difficulties at least for the highest trajectories. Finally in Sect. **5** we shall discuss the detailed structure of the low-lying levels and in Sect. **6** the main conclusions are reported together with our interpretation of the results.

A few technical points are discussed in the Appendix.

2. – The multiparticle amplitude.

As we have discussed in the previous Section, we shall use, in order to study factorization properties, the dual-resonance models for interaction of any number of spinless particles proposed in ref. (4,5). Let us first summarize in an appropriate notation those beautiful results.

The M-particle process we consider is described in Fig. 1. The amplitude for such a process as given in ref. (4,5), is

$$A_M(\alpha_i(s_i)) = \int_0^1 \dots \int_0^1 \mathrm{d}u \prod_k u_k^{-\alpha_k(s_k)-1} + (\text{different cyclic terms}) . \tag{2.1}$$

The different cyclic terms correspond to each different ordering of the external lines. For our purposes the analysis of a single term will be enough and all the factorization properties will hold for the whole sum of terms as well (7).

In eq. (2.1) $\alpha_i(s_i)$ is the trajectory function of the i-th channel which corresponds to a set of consecutive lines and carries a total squared energy s_i. The trajectory functions are assumed linear with universal slope. The variables u_j are connected by the constraints of duality or, if we wish, of absence of not allowed coincident poles. These conditions read

$$u_i = 1 - \prod_j u_j , \tag{2.2}$$

where the i-th channel cannot develope a pole when the j-th does. A very interesting property of eq. (2.2) is that they can be used to express all the u_i in terms of M—3 independent ones corresponding to a well-defined choice of compatible poles.

According to our program we wish to separate the external lines into two groups $(1, 2 \dots N)$ and $(N+1 \dots M)$ and to consider explicitly the poles in the corresponding variable

$$s = -(1+2+3 \dots + N)^2 = -((N+1) + \dots + M)^2 . \tag{2.3}$$

For this purpose it is wise to eliminate the dependent variables in eq. (2) in terms of the most convenient set of M—3 independent ones. We want one of the independent variables to be $u_s = z$ the variable conjugate to s. The choice of the other M—4 independent u is *a priori* arbitrary. However for reasons still unclear, at least to the present authors, it turns out that only two choices

(7) The only difference is that, with identical isospinless boson, the symmetrization of eq. (2.1) will wash out some states. This will be removed by isospin factors such as those of J. B. Paton and H. M. Chan: CERN preprint TH. 994 (1969).

give particularly simple expressions. They correspond to the two « multiperipheral configurations » of Fig. 2. In Fig. 2 we have also indicated a

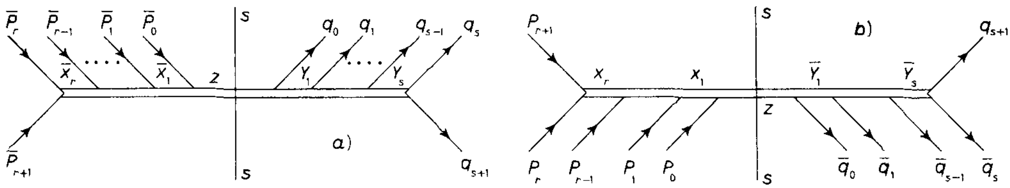

Fig. 2. – The two multiperipheral configurations connected by the R-reflection.

redefinition of the incoming and outgoing momenta, which will prove useful in the following discussions. Conservation of momentum reads

$$\sum_{i=0}^{r+1} \bar{p}_i = \sum_{i=0}^{r+1} p_i = \Pi = \sum_{i=0}^{s+1} q_i = \sum_{i=0}^{s+1} \bar{q}_i \,. \tag{2.4}$$

We have also introduced an operation of reflection of p_i and q_j as

$$p_i \to \bar{p}_i = p_{r+1-i}\,, \qquad q_i \to \bar{q}_i = q_{s+1-i}\,. \tag{2.5}$$

If we perform this « reflection » operation R we essentially pass from configuration a to b) in Fig. 2, since the roles of p_i and $\bar{p}_i$ are interchanged under R.

At this point we are ready for writing the scattering amplitude in the most appropriate form for our work. Following BARDAKCI and RUEGG (ref. [5]) we can rewrite eq. (2.1), by using the configuration a) of Fig. 2, in the form

$$\begin{aligned}
A_{r+s+4}(\alpha_i) = & \int_0^1 \ldots \int_0^1 \mathrm{d}\bar{x}_i \int_0^1 \int_0^1 \mathrm{d}y^i \int_0^1 \mathrm{d}z\, \bar{x}_r^{-\bar{\gamma}(r+1,2)}\, \bar{x}_{r-1}^{-\bar{\gamma}(r-1,2,r-1)} \ldots \cdot \\
& \cdot \ldots \bar{x}_2^{-\bar{\gamma}(r+1,r\ldots 2)}\, \bar{x}_1^{-\bar{\gamma}(r+1,r\ldots 2,1)}\, z^{-\alpha(s)-1}\, y_1^{-\gamma(s+1,\ldots 2,1)}\, y_2^{-\gamma(s+1,\ldots 2)} \ldots \cdot \\
& \cdot \ldots y_{s-1}^{-\gamma(s+1,s,s-1)}\, y_s^{-\gamma(s+1,s)} (1-\bar{x}_r)^{-\bar{\gamma}(r,r-1)} (1-\bar{x}_{r-1})^{\bar{\gamma}(r-1,r-2)} \ldots \cdot \\
& \cdot \ldots (1-\bar{x}_2)^{-\bar{\gamma}(2,1)} (1-\bar{x}_1)^{-\bar{\gamma}(1,0)} (1-z)^{-\delta(0,0)} (1-y_1)^{-\gamma(1,0)} (1-y_2)^{-\gamma(2,1)} \ldots \cdot \\
& \cdot \ldots (1-y_{s-1})^{-\gamma(s-1,s-2)} (1-y_s)^{-\gamma(s,s-a)} (1-\bar{x}_r\bar{x}_{r-1})^{-\bar{\gamma}(r,r-2)} \ldots \cdot \\
& \cdot \ldots (1-\bar{x}_2\bar{x}_1)^{-\bar{\gamma}(2,0)} (1-\bar{x}_1 z)^{-\delta(1,0)} (1-zy_1)^{-\delta(0,1)} (1-y_1y_2)^{-\gamma(2,0)} \ldots \cdot \\
& \cdot \ldots (1-y_{s-1}y_s)^{-\gamma(s,s-2)} \ldots (1-\bar{x}_3\bar{x}_2\bar{x}_1)^{-\bar{\gamma}(3,0)} \ldots (1-y_3y_2y_1)^{-\gamma(3,0)} \ldots \cdot \\
& \cdot \ldots (1-\bar{x}_2\bar{x}_1 z)^{-\delta(2,0)} (1-\bar{x}_1 z y_1)^{-\delta(1,1)} (1-zy_1y_2)^{-\delta(0,2)} \ldots \cdot \\
& \cdot \ldots (1-\bar{x}_i\bar{x}_{i-1}\ldots\bar{x}_{j+1}\bar{x}_j)^{-\bar{\gamma}(i,j-1)} \ldots (1-\bar{x}_i\ldots\bar{x}_1 z y_1 \ldots y_j)^{-\delta(i,j)} \ldots \cdot \\
& \qquad \cdot \ldots (1-y_iy_{i+1}\ldots y_{j-1}y_j)^{-\gamma(j,i-1)} \ldots (1-\bar{x}_r\ldots\bar{x}_1 z y_1\ldots y_s)^{-\delta(r,s)} \,.
\end{aligned} \tag{2.6}$$

The choice of the independent integration variables $\bar{x}_i$, y_j and z are indicated in Fig. 2a). The functions γ, $\bar{\gamma}$, and δ are defined in terms of the $\alpha_i(s_i)$ trajectory functions as follows:

$$(2.7)\quad \begin{cases} \gamma(i, i-1, i-2, \ldots, j+1, j) = \alpha[-(q_i+q_{i-1}+\ldots+q_{j+1}+q_j)^2]+1\,, \\ \gamma(i+1, i) = \gamma(i+2, i+1, i) - \gamma(i+2, i+1) - \gamma(i+1, i) + 1\,, \\ \gamma(i+k, i) = \gamma(i+k, i+k-1, \ldots, i+1, i) + \\ \quad + \gamma(i+k-1, i+k-2, \ldots, i+1) - \\ \quad - \gamma(i+k, i+k-1, \ldots, i+1) - \gamma(i+k-1, \ldots, i+1, i)\,, \qquad (k\geqslant 3)\,. \end{cases}$$

The functions $\bar{\gamma}(i, j)$ have the same definition with $q_i \to \bar{p}_i$ and $\delta(i, j)$ have a similar definition but couple $\bar{p}$-vectors with q-vectors and take into account the fact that the latter are outgoing and the former ingoing momenta.

Equation (2.6) can be rewritten as

$$(2.8)\qquad A_{r+s+4}(\alpha) = \int_0^1 \mathrm{d}\bar{x}\,\varphi(\bar{x}, \bar{p}) \int_0^1 \mathrm{d}y\,\varphi(y, q) \int_0^1 \mathrm{d}z\, z^{-\alpha(s)-1} F(z, \bar{x}, \bar{p}, y, q)\,,$$

where φ and F are reconstructed from (2.6) to be

$$(2.9)\qquad \varphi(\bar{x}, \bar{p}) = \bar{x}_r^{-\bar{\gamma}(r+1,r)} \ldots \bar{x}_1^{-\bar{\gamma}(r+1,r\ldots 1)}(1-\bar{x}_r)^{-\bar{\gamma}(r,r-1)} \ldots (1-\bar{x}_1)^{-\bar{\gamma}(1,0)} \cdot \\ \cdot (1-\bar{x}_r\bar{x}_{r-1})^{-\bar{\gamma}(r,r-2)} \ldots (1-\bar{x}_2\bar{x}_1)^{-\bar{\gamma}(2,0)} \ldots \cdot \\ \cdot \ldots (1-\bar{x}_r\bar{x}_{r-1}\,\bar{x}_j)^{-\bar{\gamma}(i,j-1)} \ldots (1-\bar{x}_r \ldots \bar{x}_1)^{-\bar{\gamma}(r,0)}\,.$$

$$(2.10)\qquad F(z, \bar{x}, \bar{p}, y, q) = (1-z)^{-\delta(0,0)}(1-\bar{x}_1 z)^{-\delta(1,0)}(1-zy_1)^{-\delta(0,1)} \ldots \cdot \\ \cdot \ldots (1-\bar{x}_i \ldots \bar{x}_1 z y_1 \ldots y_j)^{-\delta(i,j)} \ldots (1-\bar{x} \ldots \bar{x}_1 z y_1 \ldots y_s)^{-\delta(r,s)}\,.$$

It is very easy to verify that the $(r+3)$-point function for the process $\bar{p}_0+\bar{p}_1+ +\bar{p}_r+\bar{p}_{r+1}\to\Pi$ is just given by

$$(2.11)\qquad A_{r+3}(\alpha) = \int_0^1 \mathrm{d}\bar{x}\,\varphi(\bar{x}, \bar{p})\,.$$

We could have worked instead with the configuration b) of Fig. 2, ending with the formula

$$(2.12)\qquad A_{r+s+4}(\alpha) = \int_0^1 \mathrm{d}x\,\varphi(x, p) \int_0^1 \mathrm{d}y\,\varphi(\bar{y}, \bar{q}) \int_0^1 \mathrm{d}z\, z^{-\alpha(s)-1} F(z, x, p, \bar{y}, \bar{q})\,.$$

The complete equivalence between eqs. (2.12) and (2.8) follows directly from the fact that both can be obtained from the fundamental symmetric form (2.1). However, checking by direct inspection the equivalence of the two forms is by no means a trivial exercise.

In this paper the existence of the two forms will be exploited to find identities whose direct proof would be at least very involved.

Let us now look more in detail to the two « conjugate » forms (2.8) and (2.12). It is useful to perform a change of variables such that, together with the transformation $p_i \to \overline{p}_i$, it leaves invariant the quantity $\mathrm{d}x\varphi(x, p)$. This is just the transformation from $x \to \overline{x}$ which is given explicitely by

$$\left\{\begin{aligned} &x_1 \to \overline{x}_1 = (1 - x_r x_{r-1} \dots x_2 x_1)\,, \\ &x_2 \to \overline{x}_2 = (1 - x_{r-1} x_{r-2} \dots x_1)(1 - x_r x_{r-1} \dots x_1)^{-1}\,, \\ &x_3 \to \overline{x}_3 = (1 - x_{r-2} \dots x_1)(1 - x_{r-1} \dots x_1)^{-1}\,, \\ &\quad \vdots \\ &x_r \to \overline{x}_r = (1 - x_1)(1 - x_1 x_2)^{-1}\,. \end{aligned}\right. \tag{2.13}$$

The transformation (2.13) is more easily expressed in terms of the quantities $\varrho_i = x_i x_{i-1} \dots x_2 x_1$. In terms of these quantities the transformation (2.13) is a linear one given by

$$\varrho_i \to \overline{\varrho}_i = (1 - \varrho_{s+1-i})\,. \tag{2.14}$$

The same is true for σ_i defined as $\sigma_i = y_i y_{i-1} \dots y_1$. It is immediately verified that eq. (2.14) can be inverted to give the ϱ in terms of the $\overline{\varrho}$ by the same formulas.

With this transformation we can write eq. (2.8) as

$$A_{r+s+4}(\alpha) = \int_0^1 \mathrm{d}x\varphi(x, p) \int_0^1 \mathrm{d}y\varphi(y, q) \int_0^1 \mathrm{d}z z^{-\alpha(s)-1} F(z, \overline{x}, \overline{p}, y, q) \tag{2.15}$$

and (2.12) as

$$A_{r+s+4}(\alpha) = \int_0^1 \mathrm{d}x\varphi(x, p) \int_0^1 \mathrm{d}y\varphi(y, q) \int_0^1 \mathrm{d}z z^{-\alpha(s)-1} F(z, x, p, \overline{x}, \overline{q})\,. \tag{2.16}$$

Considering (2.15) and (2.16) we get the fundamental identity:

$$\int_0^1 \mathrm{d}z z^{-\alpha(s)-1} \int_0^1 \mathrm{d}x\varphi(x, p) \int_0^1 \mathrm{d}y\varphi(y, q)[F(z, x, p, \overline{y}, \overline{q}) - F(z, \overline{x}, \overline{p}, y, \overline{q})] = 0\,. \tag{2.17}$$

We shall finally introduce a useful notation by defining for a function $f(x, p)$ the quantity

$$(2.18)\quad \begin{cases} \langle f(p)\rangle = \int\limits_0^1 \mathrm{d}x\varphi(x, p)f(x, p)\,, \\ \langle f(\overline{p})\rangle = \int\limits_0^1 \mathrm{d}x\varphi(x, p)f(\overline{x}, \overline{p})\,. \end{cases}$$

At this point we have all the machinery we need in order to discuss the form of the poles in the s-variable. This will be obtained in the next Section.

3. - Factorization properties.

Let us consider now a more specific model, in which a few simplifying assumptions about our Regge trajectories will be introduced. Most of the qualitative results of our analysis will not depend upon these assumptions. We shall assume that:

1) The trajectory function is the same in all channels $\alpha(x) = a + bx$.

2) The external particles are also identical and lie on the above trajectory. Consequently, if μ is the common mass of the external scalar particles, we have $\alpha(\mu^2) = a + b\mu^2 = 0$.

In Sect. **5** we shall comment on the modifications encountered when we release either of these two assumptions.

In this simplified situation we have (we use the metric $+1+1+1-1$)

$$(3.1)\quad \begin{cases} \gamma(i+k, i+k-1, ..., i+1, i) = \alpha[-(p_i + ... + p_{i+k})^2] + 1\,, \\ \gamma(i+1, i) = \alpha[-(p_i + p_{i+1})^2] + 1 = a + 2b\mu^2 - 2bp_i\cdot p_{i+1} + 1 = \\ \qquad\qquad = -2bp_i\cdot p_{i+1} + (1-a)\,, \\ \gamma(i+2, i) = -2bp_i\cdot p_{i+2} - a - b\mu^2 = -2bp_i\cdot p_{i+2}\,, \\ k(i+k, i) = -2bp_i\cdot p_{i+k} \qquad\qquad (\gamma \geqslant 3)\,. \end{cases}$$

Similarly for $\bar{\gamma}$ and δ except for some minus sign corresponding to the fact that q momenta are outgoing. We now rewrite eq. (2.10) as

$$F = \exp[G]\,, \tag{3.2}$$

$$\begin{aligned}G = -[&\log(1-z)\delta(0,0) + \log(1-\bar{x}_1 z)\delta(1,0) + \log(1-zy_1)\delta(0,1) + \\ &+ \log(1-\bar{x}_2\bar{x}_1 z)\delta(2,0) + \log(1-\bar{x}_1 z y_1)\delta(1,1) + \\ &+ \log(1-zy_1y_2)\delta(0,2) + \dots \log(1-\bar{x}_i \dots \bar{x}_1 zyt \dots y_j)\delta(i,j) + \\ &+ \dots \log(1-\bar{x}_r\bar{x}_{r-1}\dots\bar{x}_1 zy_1y_2\dots y_s)\delta(r,s)]\,.\end{aligned} \tag{3.3}$$

Expanding now all the $\log(1-z)$ in power series of z and collecting the powers in z we obtain

$$\log F = G = \sum_{n=1}^{\infty} \frac{z^n}{n} G_n\,, \tag{3.4}$$

where, using eq. (2.7) for the δ's, we obtain

$$G_n = \bar{P}^{(n)}_\mu Q^{(n)}_\mu + (1-a)\,. \tag{3.5}$$

From now on we shall redefine all our momenta by multiplying them by $\sqrt{2b}$. The expression for $\bar{P}^{(n)}$ and $Q^{(n)}$ is then

$$\bar{P}^{(n)} = \bar{p}_0 + \sum_{i=1}^{r} \bar{\varrho}_i \bar{p}_i\,, \qquad Q^{(n)} = q_0 + \sum_{i=1}^{s} \sigma_i q_i\,, \tag{3.6}$$

where ϱ_i and σ_i are defined in Sect. 2.

The advantage of eq. (3.5) is that it contains the quantities that depend on the left variables ($\bar{p}$ and $\bar{x}$) separated and factorized with respect to the right variables (q and y). At this point the explicit computation of the poles in s is just a matter of expanding F in power series of z, *i.e.*

$$F(z, \bar{x}, \bar{p}, y, q) = \sum_{n=0}^{\infty} z^n F_n(\bar{p}, \bar{x}, q, y)\,, \tag{3.7}$$

which by a term by term integration over z gives

$$A_{r+s+4}(\alpha) = \sum_{n=1}^{\infty} \frac{R_n}{\alpha(s)-n}\,, \tag{3.8}$$

with:

$$R_n = \int \mathrm{d}\bar{x} \int \mathrm{d}y\, \varphi(\bar{x}, \bar{p})\, \varphi(y, q) F_n(\bar{x}, \bar{p}, y, q)\,. \tag{3.9}$$

Using eq. (3.5) we get F as a combination of factorized terms. The first ones are easily computed to be

$$\begin{cases} F_0 = 1\,, \qquad F_1 = \bar{P}^{(1)}_\mu Q^{(1)}_\mu + (1-a)\,, \\ F_2 = \frac{1}{2}\bar{P}^{(1)}_\mu \bar{P}^{(1)}_\nu Q^{(1)}_\mu Q^{(1)}_\nu + (1-a)\bar{P}^{(1)}_\mu Q^{(1)}_\mu + \frac{1}{2}\bar{P}^{(2)}_\mu Q^{(2)}_\mu\, \frac{1}{2}(1-a)(2-a)\,. \end{cases} \tag{3.10}$$

In general we can write

$$F = (1-z)^{(-1-a)} \exp\left[\sum_{n=1}^{\infty} \frac{z^n}{n} \bar{P}^{(n)}_\mu Q^{(n)}_\mu\right]. \tag{3.11}$$

It is now easy to obtain the expressions for R_i. Using the bracket notation of eq. (2.18) and defining $c = 1 - a = 1 + b\mu^2 > 0$, we find

$$\begin{cases} R_0 = S_0 = \langle 1\rangle\langle 1\rangle\,, \qquad R_1 = S_1 + cS_0\,, \qquad S_1 = \langle \bar{P}^{(1)}_\mu\rangle\langle Q^{(1)}_\mu\rangle\,, \\ R_2 = S_2 + cS_1 + \dfrac{c(c+1)}{2} S_0\,, \qquad S_2 = \dfrac{1}{2}\langle \bar{P}^{(1)}_{\mu\nu}\rangle\langle Q^{(1)}_{\mu\nu}\rangle + \dfrac{1}{2}\langle \bar{P}^{(2)}_\mu\rangle\langle Q^{(2)}_\mu\rangle\,, \\ R_3 = S_3 + cS_2 + \dfrac{c(c+1)}{2} S_1 + \dfrac{c(c+1)(c+2)}{2!} S_0\,, \\ S_3 = \dfrac{1}{3!}\langle \bar{P}^{(1)}_{\mu\nu\varrho}\rangle\langle Q^{(1)}_{\mu\nu\varrho}\rangle + \dfrac{1}{2}\langle \bar{P}^{(1)}_\mu \bar{P}^{(2)}_\nu\rangle\langle Q^{(1)}_\mu Q^{(2)}_\nu\rangle + \dfrac{1}{3}\langle P^{(3)}_\mu\rangle\langle Q^{(3)}_\nu\rangle\,, \end{cases} \tag{3.12}$$

where

$$\bar{P}^{(k)}_{\mu_1\mu_2\ldots\mu_i} = \bar{P}^{(k)}_{\mu_1} \bar{P}^{(k)}_{\mu_2} \ldots \bar{P}^{(k)}_{\mu_i}\,. \tag{3.13}$$

In general

$$R_k = S_k + cS_{k-1} + \ldots \frac{c(c+1)\ldots(c+i-1)}{i!} S_{k-i} + \\ + \ldots \frac{c(c+1)\ldots(c+k-1)}{k!} S_0 = \sum_{i=0}^{k} S_{k-i} \frac{\Gamma(c+i)}{\Gamma(i+1)\Gamma(c)}\,, \tag{3.14}$$

where $S_0 = \langle 1\rangle\langle 1\rangle$ and

$$S_l = \sum_{\{l_2, l_1, l_i \ldots\}} \frac{\langle \bar{P}^{(1)}_{\mu_1\ldots\mu_{l_1}} \bar{P}^{(2)}_{\nu_1\ldots\nu_{l_2}} \ldots\rangle\langle Q^{(1)}_{\mu_1\ldots\mu_{l_1}} Q^{(2)}_{\nu_1\ldots\nu_{l_2}} \ldots\rangle}{\Pi_j (j^{l_j} j!)}\,. \tag{3.15}$$

The sum in (3.15) is extended to the integer non negative solutions of the equation (partitions equation)

$$l_1 + 2l_2 + 3l_3 + \ldots + il_i + \ldots = l\,. \tag{3.16}$$

At this point we are in a position to discuss the level structure of our amplitudes. We see that the residue of each pole does indeed factorize in a finite number of terms. The number of terms is independent of the number of initial and final external lines and therefore can be used as a basis for level counting.

The left and right tensors appearing in eqs. (3.12)-(3.13) are four-dimensional tensors. This reflects the relativistic invariance of the whole approach together with the lack of kinematic singularities. It is well known that those requirements have led people to classify Regge trajectories in Lorentz families. Since the present model can be simply discussed in terms of resonances, we can simply talk about « Lorentz particles ». The presence of those « Lorentz particles », although esthetically beautiful is a cause of worry. Because of the unpleasant fact that the Lorentz metric has a minus sign, the time components of our tensors in the c.m. system can give rise to ghosts with imaginary coupling constants (or equivalently with negative metric!) The problem thus arises of whether some mechanism for compensation of these ghosts can be found. We shall see in the next Section that this will indeed be the case.

Let us go back to eq. (3.15) and stress the new important feature that, although for each $\alpha(s) = n$ a finite number of factorized terms exists, this number is (for large n) much larger than what might be intuitively suggested by the two-body problem or by elementary counting of Regge or Toller trajectories. It is readily seen that for two initial lines all vectors $P^{(1)}$, $P^{(2)}$, $P^{(r)}$ coincide and are simply given by p_0. Therefore the essential difference among tensors created with different $P^{(i)}$ gets lost.

Before being able to discuss more in detail the main features of our level structure, we have to answer the fundamental question of whether our general tensors are really linear independent. Our general tensor can be expanded in the following way:

$$\langle P^{(1)}_{\mu_1\ldots\mu_{l_1}} P^{(2)}_{\nu_1\ldots\nu_{l_2}} \ldots\rangle = \sum_{i_1,i_2} \langle \varrho_1^{i_1} \varrho_2^{i_2} \varrho_3^{i_3} \ldots\rangle P^{(i_1,i_2)}_{\mu_1\ldots\mu_{l_1},\nu_1\ldots\nu_{l_2}}\,, \tag{3.17}$$

where $P^{(i_1,i_2\ldots)}$ are tensor products of p_0, p_1, p_2 etc. More symbolically

$$\langle P^{(A)}_M\rangle = \sum_I \langle \varrho^A_I\rangle P^I_M\,. \tag{3.18}$$

Since the functions $\langle \varrho^A_I\rangle$ are scalar functions of the invarients made with the P_i vectors and are linearly independent of each other (at least if we choose a sufficiently large number of external lines), the only possibility for linear relations comes from compensation of different terms of the expansion (3.18) through the kinematic factors P^l_M.

We have not yet succeded in giving a general treatment of the problem. However, on the basis of several examples, we are convinced that the only possible linear relation between tensors should involve saturation by the total momentum Π_μ. In other words of we consider the c.m. system in which $\Pi_\mu = (0, 0, 0, \sqrt{s})$ all space components of the tensors will be independent and only the time components (which could generate ghosts) are related to lower tensors. Examples of such relations are given in Sect. **4**.

Once we get convinced that the space components of our tensors are linearly independent, the question arises of how many tensors we have for a fixed energy s_n. We notice that each term in which the residue R_n at $s = s_n$ factorizes can be defined by the nonnegative integer numbers $l_1, l_2 \dots l_i$ characterizing a term of the sum (3.15) plus an extra l_0 which is just the index i of the summation of eq. (3.14) and corresponds to the order of the polynominal in c in that equation. Thus, a part from a reduction in O_3 (little group) a state in our scheme is given by a sequence of quantum numbers:

(3.19) $$|l_0, l_1, l_2 \dots l_i \dots\rangle = |\lambda\rangle\,.$$

The squared mass of the state is given by

(3.20) $$\alpha(M_n^2) = n = l_0 + \sum_{i=1}^{\infty} i l_i\,.$$

When decomposed in its spin components the state $|\lambda\rangle$ will contain all spins up to

(3.2) $$J_{\max} = \sum_{i=1}^{\infty} l_i \leqslant n\,.$$

It is easily seen at this point that the main new feature in these dual models is the great richness of levels. We could indeed consider the integers l_i as the values of the different quantum numbers identifying each level. Since we have infinitely many l_i our level structure is characteristic of a system possessing an infinite number of degrees of freedom.

In order to have a rough idea of the multiplicity, let us estimate the number ν_n of different tensors appearing for a given large values s_n. The solution of this mathematical problem is ([8])

$$\log \nu_n \underset{n\,\text{large}}{\simeq} a\sqrt{n} \qquad \left(a = \frac{2\pi}{\sqrt{6}}\right).$$

([8]) G. H. Hardy and S. Ramanujan: *Proc. Math. Soc.*, **17**, 75 (1917). See eqs. (1.34), (1.35).

Of course, in order to estimate the number of levels one should take into account that more than one level corresponds to each tensor. It is not hard to get convinced that the number of levels will also follow the exponential law (3.22), possibly with a different value for a. The physical meaning of this enormous multiplicity will be discussed in more detail in Sect. **6**.

4. – Divergence conditions.

Let us now investigate in detail the explicit form of the linear relations among the tensors defined in the previous Section. As already discussed, the only relations we expect to find should relate the divergence of a tensor (*i.e.* a tensor where one index is saturated with Π_μ) to tensors of lower rank.

We shall indeed obtain Ward-like identities which will be extremely useful in order to obtain at least a partial compensation of the effects of ghosts.

The starting point for our derivation is the important fact that our factorization can be performed equally well in both multiperipheral configurations discussed in Sect. **2**. The equivalence between the final formulae obtained in the two ways will lead to the identities which will be discussed in this Section.

We shall find it useful to work in the special case in which we have only two final momenta q_0 and q_1 with

$$(4.1)\qquad \begin{cases} q_0 = \frac{1}{2}(\Pi + \Delta)\,, \\ q_1 = \frac{1}{2}(\Pi - \Delta)\,, \end{cases}$$

defining the four-vector Δ_μ. The two final particles have arbitrary « masses » q_0^2 and q_1^2. The process is illustrated in Fig. 3. We then write down for this process the two equivalent forms (2.15) and (2.16). We do not imply that this is the correct way of going physically off the mass shell, but just observe that these two mathematical expressions do indeed satisfy all the required properties and in particular the reflection symmetry.

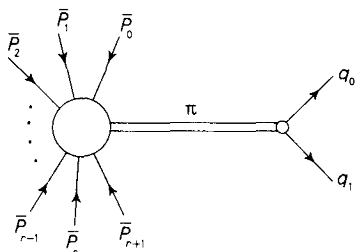

Fig. 3. – The particular multiparticle process used to derive Ward-like identities. The final momenta q_0 and q_1 are off the mass shell.

This reflection symmetry transforms the set $P^{(n)}$ into the set $\bar{P}^{(n)}$ defined in eq. (3.6).

The transformation law as given in the Appendix is simply

$$(4.2)\qquad P^{(n)} \underset{R}{\rightarrow} \bar{P}^{(n)} = \sum_{i=0}^{n} d_i^{(n)} P^{(i)}\,,$$

where

(4.3) $$d_i^{(n)} = (-1)^i \binom{n}{i}.$$

It is very useful to introduce new vectors $V^{(i)}$ defined by (see again the Appendix)

(4.4) $$\begin{cases} V^{(n)} = \sum_{i=0}^{n} C_i^{(n)} P^{(i)}, \\ C_i^{(n)} = (-1)^i \left(\frac{1}{2}\right)^{n-i} \binom{n}{i}, \end{cases}$$

which have the simple transformation law

(4.5) $$V^{(i)} \underset{R}{\rightarrow} \overline{V}^{(i)} = (-1)^i V^{(i)}.$$

Let us now consider the equivalence between the two factorized forms. This leads to the identity

(4.6) $$A_{r+4} = \left\langle \int_0^1 z^{-\alpha(s)-1} dz \exp\left[\sum_{n=1}^{\infty} \frac{z^n}{n} [(P^{(n)} \cdot q_1) + (1 - q_1^2/2)]\right] \right\rangle = \\ = \left\langle \int_0^1 dz\, z^{-\alpha(s)-1} \exp\left[\sum_{n=1}^{\infty} \frac{z^n}{n} [(\overline{P}^{(n)} \cdot q_0) + (1 - q_1^2/2)]\right] \right\rangle.$$

We now substitute eq. (4.1) in (4.6). After a few manipulations and using the fact that eq. (4.6) is valid for any value of s we get

(4.7) $$\left\langle \exp\left[\overline{W}_\mu \frac{(\Pi + \Delta)}{2} {}_\mu\right] \right\rangle = \left\langle \exp\left[W_\mu \frac{(\Pi - \Delta)}{2} {}_\mu\right] \right\rangle,$$

where

(4.8) $$W_\mu = \sum_{n=1}^{\infty} \mathscr{P}_\mu^{(n)} \frac{z^n}{n}, \qquad \overline{W}_\mu = \sum_{n=1}^{\infty} \overline{\mathscr{P}}_\mu^{(n)} \frac{z^n}{n}$$

and

(4.9) $$\mathscr{P}_\mu^{(n)} = P_\mu^{(n)} - \Pi_\mu/2.$$

At this point it is important to exploit the different relations connecting $P^{(n)}$, $\overline{P}^{(n)}$, $V^{(n)}$ and $\overline{V}^{(n)}$. This can be done by using eqs. (A.23)-(A.27) obtained in the Appendix, which enable us to write for W and $\overline{W}$

$$(4.10)\qquad W=\sum_n \frac{y^n}{n}\mathscr{V}^{(n)}\,,\qquad \overline{W}=\sum_n \frac{(-y)^n}{n}\mathscr{V}^{(n)}\,,$$

where

$$(4.11)\qquad \frac{1}{y}+\frac{1}{z}=\frac{1}{2}$$

and

$$(4.12)\qquad \begin{cases} \mathscr{V}^{(n)}=V^{(n)} & \text{for } n \text{ odd}\,,\\ \mathscr{V}^{(n)}=V^{(n)}-2^{-n}\Pi & \text{for } n \text{ even}\,. \end{cases}$$

In terms of the $P^{(i)}$ we have

$$(4.13)\qquad W=\sum\frac{z^n}{n}\mathscr{P}^{(n)}\,;\qquad \overline{W}=\sum\frac{z'^n}{n}\mathscr{P}^{(n)}\,,$$

where

$$(4.14)\qquad \frac{1}{z'}-\frac{1}{y}=\frac{1}{2}\,,\quad\text{or also}\quad \frac{1}{z}+\frac{1}{z'}=1\,.$$

The linear relations we are looking for are obtained by using the fact that eq. (4.7) should be valid for any value of Δ_μ and of z (or y).

Expanding eq. (4.7) in power series of Δ_μ we get immediately

$$(4.15)\qquad \left\langle \exp\left[\frac{\Pi W}{2}\right]\right\rangle=\left\langle \exp\left[\frac{\Pi \overline{W}}{2}\right]\right\rangle\,,$$

$$(4.16)\qquad \left\langle W_\mu \exp\left[\frac{\Pi W}{2}\right]\right\rangle=-\left\langle \overline{W}_\mu \exp\left[\frac{\Pi \overline{W}}{2}\right]\right\rangle$$

and in general

$$(4.17)\qquad \left\langle W_{\mu_1}W_{\mu_2}\dots W_{\mu_i}\exp\left[\frac{\Pi \overline{W}}{2}\right]\right\rangle=(-1)^i(W\to\overline{W})\,.$$

Using for instance (4.10) we have to equate to 0 all powers of y which would give the wrong symmetry under reflection (from (4.18) $\overline{W}(y)=W(-y)$). We shall limit ourselves to write explicitely the equations for the first few states and trajectories which will be used in the next Section. To lowest order in y we have

$$(4.18)\qquad \left\langle\left(1+\frac{\Pi\mathscr{V}^{(1)}}{2}y+\dots\right)\left(y\mathscr{V}^{(1)}_{\mu_1}+\frac{y^2}{2}\mathscr{V}^{(2)}_{\mu_1}\right)\cdot\right.$$
$$\left.\cdot\left(y\mathscr{V}^{(1)}_{\mu_2}+\frac{y^2}{2}\mathscr{V}^{(2)}_{\mu_2}\right)\right\rangle=(-1)^i(y\to-y)\,.$$

The first nontrivial identity is obtained by looking at the coefficient of y^{i+1} which should be set to zero. We get

$$(4.19)\qquad \Pi_\nu\langle \mathscr{V}_\nu^{(1)} \mathscr{V}_{\mu_1}^{(1)} \mathscr{V}_{\mu_2}^{(1)} \dots \mathscr{V}_{\mu_i}^{(1)}\rangle = -\sum_{j=1}^{i} \langle \mathscr{V}_{\mu_1}^{(1)} \mathscr{V}_{\mu_2}^{(1)} \dots \mathscr{V}_{\mu_j}^{(2)} \dots \mathscr{V}_{\mu_i}^{(1)}\rangle .$$

The first two of these identities read

$$(4.20)\qquad \Pi_\nu\langle \mathscr{V}_\nu^{(1)}\rangle = 0 ,$$

or

$$(4.21)\qquad \Pi_\nu\langle P_\nu^{(1)}\rangle = \frac{\Pi^2}{2}\langle 1\rangle$$

and

$$(4.22)\qquad \Pi_\nu\langle \mathscr{V}_\nu^{(1)} \mathscr{V}_\mu^{(1)}\rangle = -\langle \overline{W}_\mu^{(2)}\rangle .$$

It is clear that the Ward identities are written most easily in terms of the vectors $\mathscr{V}^{(i)}$ which are eigenvectors of the R-reflection. On the other hand the fundamental form appearing in the factorized residue is simple in terms of the vectors $P^{(n)}$ and $\overline{P}^{(n)}$. If one wants to obtain the Ward identities in terms of these vectors one has to expand W with eq. (4.13) and equate the coefficients of the expansion in z.

Let us now discuss the significance of our results. The identities we have just obtained imply relations between the divergence of one tensor and a combination of lower-rank tensors. The divergences of tensors correspond to time components in the c.m. system and consequently to unphysical states of negative norm.

Our identities show that, at least in certain cases, ghosts are accompanied by « well behaved » particles which are coupled in the same way to all channels. Hence what one sees experimentally is only the algebraic sum of the particle plus ghost contribution and if this sum turns out to be positive the presence of the ghost has no observable consequence. This situation is well illustrated (as we shall see in more detail in the next Section) in the case $n = 1$.

The Ward-like identity of eq. (4.21),

$$\Pi_\mu\langle P_\mu^{(1)}\rangle \approx \langle 1\rangle ,$$

tells us that the c.m. time component of $\langle P_\mu^{(1)}\rangle$ does couple to all channels in the same way as the scalar object $\langle 1\rangle$ which corresponds to a well-behaved particle. It will turn out that the positive residue will always be larger than the negative one, thus compensating completely all bad effects of $\langle P_0^{(1)}\rangle$.

We want to notice that the appearance and cancellation of ghosts in our contest has striking analogies with what happens in quantum electrodynamics. As already pointed out in both cases ghosts appear because of the explicit covariance of the formalism. In our case the ghosts are related to the presence of Lorentz particles which lie on Lorentz trajectories needed to have an S-matrix free of kinematic singularities. In electrodynamics a covariant treatment requires to represent the photon by means of the four-dimentional e.m. potential A_μ. The time component A_0 leads to an unphysical time like photon with negative metric. It is well known that the presence of Ward identities, which follow from current conservation, implies that the negative contribution from time like photons is *always* exactly cancelled by the positive contribution of the longitudinal photon.

This is indeed very similar to what happens here in the Ward like identity (4.21).

Let us finally point out that the equations we have derived in this Section are not the most general divergence conditions. Because of the special choice of a two-particle « final state » we have only obtained an important subset of all possible equations.

The complete study of Ward-like identities together with a more general investigation of ghost compensation is deferred to further work.

5. – Some simple examples.

In this Section we consider explicitely in structure of the lowest-energy levels. Defining $N = \alpha(S_N)$, where S_N is the squared energy of the level, we have:

$N = 0$. This is a level containing just a scalar singlet. In general all the resonances on the leading trajectory factorize in the simplest way as observed already by several authors ([5]).

$N = 1$. This level was discussed already in the previous Section. The residue at the s-pole can be written as

$$R_1 = \langle \overline{V}_\mu^{(1)} \rangle \langle U_\mu^{(1)} \rangle + \frac{\Pi^2}{4} \langle 1 \rangle \langle 1 \rangle + c \langle 1 \rangle \langle 1 \rangle \,. \tag{5.1}$$

Since

$$\frac{\Pi^2}{4} = -\frac{bs}{2} = -\frac{1}{2}(1 + b\mu^2) = -\frac{2}{c}\,,$$

we simply get

$$R_1 = \langle \overline{V}_\mu^{(1)} \rangle \langle U_\mu^{(1)} \rangle + \frac{c}{2} \langle 1 \rangle \langle 1 \rangle \,. \tag{5.2}$$

$V_\mu^{(1)}$ and $U_\mu^{(1)}$ are divergenceless tensors. Hence we have just a spin-one and a spin-zero meson and no ghost appears.

Before leaving this simple example it is amusing to illustrate the Ward identity we have used, in the simple case of 3 incoming momenta. Equation (4.21) is then given by

$$\Pi_\mu[p_{0\mu}\langle 1\rangle + p_{1\mu}\langle x\rangle] = \frac{\Pi^2}{2}\langle 1\rangle\,, \tag{5.3}$$

which leads to

$$(\pi\cdot p_0 - \pi^2/2)\langle 1\rangle = -(p_1\cdot\pi)\langle x\rangle\,. \tag{5.4}$$

Since we know the explicit form of the 4-point function in terms of ratios of Γ-functions it is easy to verify eq. (5.4) [9].

In particular, since $\langle x\rangle$ is regular for $(p_1\pi)=0$, from (5.4) the scalar amplitude $\langle 1\rangle$ has to have a zero for $p_1\pi\to 0$. This is indeed the zero used by LOVELACE [10] in order to get the connection between the dual amplitude and PCAC. It is may be more than a coincidence that we are led to understand the presence of the Lovelace zero on the basis of a Ward-like identity!

$N=2$. As N increases the complication due to the appearance of new vectors starts to show up. In particular the residue R_2 at this pole can be written as

$$R_2 = S_2 + cS_1 + \frac{c(c+1)}{2}S_0\,, \tag{5.5}$$

or

$$R_2 = \frac{1}{2}\langle \bar{P}^{(1)}_{\mu\nu}\rangle\langle Q^{(1)}_{\mu\nu}\rangle + \frac{1}{2}\langle \bar{P}^{(2)}_{\mu}\rangle\langle Q^{(2)}_{\mu}\rangle + c\langle \bar{P}^{(1)}_{\mu}\rangle\langle Q^{(1)}_{\mu}\rangle + \frac{c(c+1)}{2}\langle 1\rangle\langle 1\rangle\,. \tag{5.6}$$

In terms of $O_{3,1}$ representations R_2 contains:

a) A second-rank tensor $\langle \bar{P}^{(0)}_{\mu\nu}\rangle$.

b) Two four-vectors $\langle \bar{P}^{(1)}_{\mu}\rangle\langle \bar{P}^{(2)}_{\mu}\rangle$.

c) One Lorentz scalar $\langle 1\rangle$.

For the space components of these tensors all couplings are real (since $c>0$).

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(9) In the case of more than 3 incoming particles the « direct inspection method » to prove eq. (4.21) is indeed pretty hard!

(10) C. LOVELACE: *Phys. Lett.*, **28** B, 265 (1968).

The Ward-like identities at our disposal are the following:

$$\Pi_\mu \langle P_\mu^{(1)} \rangle = \frac{\Pi^2}{2} \langle 1 \rangle \, , \tag{5.7}$$

$$\Pi_\mu \langle P_{\mu\nu}^{(1)} \rangle = \langle D_\nu \rangle = \left\langle P_\mu^{(1)} \left(1 + \frac{\Pi^2}{2}\right) - P_\nu^{(2)} \right\rangle \, . \tag{5.8}$$

This last Ward identity is easily obtained from (4.22). At this point it is convenient to classify our states in O_3, namely to separate their angular momenta J. Instead of using $P_\mu^{(1)}$ and $P_\mu^{(2)}$ as independent vectors we shall use $P_\mu^{(1)}$ and D_μ. The resulting spin structure is shown in Table I where we have put

TABLE I. – *Tensors appearing in the residue R_2 and their O_3 content.* States with imaginary coupling are in parenthesis.

$\langle P_{\mu\nu}^{(1)} \rangle$	$\langle D_\nu \rangle$	$\langle P_\nu^{(1)} \rangle$	$\langle 1 \rangle$
$\langle P_{rs}^{(1)} \rangle$: $J=2$, $J=0$			
$\langle P_{r0}^{(1)} \rangle$, $\langle P_{0r}^{(1)} \rangle$: $(J=1)$	$\langle D_r \rangle$: $J=1$	$\langle P_r^{(1)} \rangle$: $J=1$	
$\langle P_{00}^{(1)} \rangle$: $J=0$	$\langle D_0 \rangle$: $(J=0)$	$\langle P_0^{(1)} \rangle$: $(J=0)$	$\langle 1 \rangle$: $J=0$

in parenthesis ghostlike states (time components). In terms of c.m. components the Ward identities (which are the only linear relations among our tensors) read

$$\left\{\begin{aligned} &\langle P_{0k}^{(1)} \rangle = \langle P_{k0}^{(1)} \rangle \sim \langle D_k \rangle \, , \qquad k = 1, 2, 3 \, , \\ &\langle P_{00}^{(1)} \rangle \sim \langle D_0 \rangle \, , \\ &\langle P_0^{(1)} \rangle \sim \langle 1 \rangle \, . \end{aligned}\right. \tag{5.9}$$

The relative weight of the various components in eq. (5.6) is such that:

1) The $P_{0k}^{(1)}$, $P_{k0}^{(1)}$ ghost like states are completely compensated by D_k. No spin-1 ghost appears.

2) The above compensation unfortunately is such that the ghost-like state D_0 dominates over $P_{00}^{(1)}$ while $P_0^{(1)}$ is accomodated by the good scalar $\langle 1 \rangle$. It is easy to get convinced that there is no way to get rid of this difficulty with our set of tensors only. If we have to compensate for the ghost like $P_{0k}^{(1)}$, we have necessarily the trouble for D_0 and *vice versa* [11].

The previous example has shown that our program of ghost killing is only partly successful. Therefore the many-particle representation in the simplified form presented here is not free of negative norm states.

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[11] The trace $P_{kk}^{(1)}$ is just an independent, positive norm scalar which, unfortunately, cannot save the situation.

The reason for this difficulty is not hard to understand from an intuitive standpoint; indeed we see that we have an infinite number of independent vectors $P_\mu^{(n)}$ whose time components $\Pi_\mu P_\mu^{(n)}$ are negative-norm scalars. One would therefore need an infinite number of independent positive-norm scalars in order to obtain a full compensation. However, from the explicit form of Sect. **3**:

$$F = \exp\left[\sum_{n=1}^{\infty} \frac{z^n}{n} (\bar{P}_\mu^{(n)} Q_\mu^{(n)} + c)\right], \tag{5.10}$$

we see that a single scalar $\langle 1 \rangle$ exists for each different vector $P^{(n)}$ and $Q^{(n)}$. The reason is that, in this simplified model with all trajectories being equal (see Sect. **3**), great compensations occur in the expressions for γ and δ given in (2.7) and eliminate all possible new scalars. On the other hand the vector part $\bar{P}^{(n)} Q^{(n)}$ of (5.10) is extremely stable against modifications of the trajectories [12]. It is therefore not unconceivable that less restrictive and more realistic models with the same vector part, could provide the new scalars which are needed. In particular one could take the realistic situation in which the external lines are pions, where trajectories coupled with even and odd numbers of lines are not the same. This would be enough in order to obtain a scalar $S^{(n)}$ for each vector $P_\mu^{(n)}$.

Another possibility [13] is to take the model itself less seriously and add phenomenologically to it a smooth background so that the sum of the pole plus background corresponds to a positive term. In this case the ghosts will not be so ghastly but would lead to observable dips (of course factorizable!) in the amplitude [14].

At this point we want to discuss higher values of N. We shall limit our attention to the two leading trajectories $J \geqslant \alpha(s_N) - 1$. At $\alpha(s_n) = n$ we have a spin n on the leading trajectory and a spin $n-1$ on the first daughter. They can both be extracted from eq. (3.15) and they read

$$R_n = \frac{1}{n!} \langle \bar{P}^{(1)}_{\mu_1 \dots \mu_n} \rangle \langle Q^{(1)}_{\mu_1 \dots \mu_n} \rangle + \frac{c}{(n-1)!} \langle \bar{P}^{(1)}_{\mu_1 \dots \mu_{n-1}} \rangle \langle Q^{(1)}_{\mu_1 \dots \mu_{n-1}} \rangle + \\ + \frac{1}{2(n-2)!} \langle \bar{P}^{(1)}_{\mu \dots \mu_{n-2}} \bar{P}^{(2)}_\nu \rangle \langle Q^{(1)}_{\mu_1 \dots \mu_{n-2}} Q^{(2)}_\nu \rangle + (\text{contributions to } J < n-1)\,. \tag{5.11}$$

([12]) The only important condition that has to be maintained in order to have the simplest factorization is the universality of the slopes.

([13]) We do not consider as particularly promising the alternative of adding more « satellites » terms to eq. (2.1) since this, unless we find an appropriate way of adding similar terms to the multiparticle amplitudes, will spoil our simple factorization.

([14]) A more optimistic point of view is to say that this background will come out automatically from unitarizing the narrow resonance approximation: see, for instance, G. VENEZIANO: *Proceedings of the Coral Gables Conference, 1969.*

We can now write the «Ward» identity given in eq. (4.19) as

$$\Pi_\mu \langle V^{(1)}_{\mu\mu_1\ldots\mu_n}\rangle = \sum_{i=1}^{n-1}\left(\frac{\Pi}{4} - V^{(2)}\right)_{\mu_i} V^{(1)}_{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_n}\,. \tag{5.12}$$

Replacing now the P, s and Q, s in terms of V, s and U, s one can verify that terms odd under reflection drop as a consequence of (5.12). What is left is the expression

$$\begin{aligned}R_n = \frac{1}{n!}\langle \overline{V}^{(1)}_{\mu_1\ldots\mu_n}\rangle\langle U^{(1)}_{\mu_1\ldots\mu_n}\rangle - \frac{1}{(n-1)!}\left(\frac{\Pi^2}{4} + c + \frac{n-1}{2}\right)\cdot \\ \cdot\langle \overline{V}^{(1)}_{\mu_1\ldots\mu_{n-1}}\rangle\langle U^{(1)}_{\mu_1\ldots\mu_{n-1}}\rangle + \frac{1}{2(n-2)!}\langle \overline{V}^{(1)}_{\mu_1\ldots\mu_{n-2}}\overline{V}^{(2)}_\nu\rangle\langle U^{(1)}_{\mu_1\ldots\mu_{n-2}}U^{(2)}_\nu\rangle + (J<n-1)\,.\end{aligned} \tag{5.13}$$

The tensor $\langle \overline{V}^{(1)}_{\mu_1\ldots\mu_{n-2}}\overline{V}^{(2)}_\nu\rangle$ is not irreducible and the only spin $n-1$ that is contained in it comes from the complete symmetrical component

$$\overline{V}^{(1)}_{\mu_1\ldots\mu_{n-2}}\overline{V}^{(2)}_\nu \sim \frac{1}{n-1}\sum \overline{V}^{(2)}_{\mu_i}\overline{V}^{(1)}_{\mu_1\ldots\mu_{i-1}\mu_{i+1}\ldots\mu_{i-1}} = \overline{V}^{(2,2)}_{J=n-1}\,. \tag{5.14}$$

Using eqs. (5.12)-(5.14) we finally get $(\Pi^2 = -2(n-a))$

$$\begin{aligned}R_n = \frac{1}{n!}\langle \overline{V}^{(1)}_{J=n}\rangle\langle U^{(1)}_{J=n}\rangle + \frac{c}{2(n-1)!}\langle \overline{V}^{(1)}_{J=n-1}\rangle\langle U^{(1)}_{J=n-1}\rangle + \\ + \frac{c}{2(n-1)!\,(n-1)(n-a)}\langle \overline{V}^{(1,2)}_{J=n-1}\rangle\langle U^{(1,2)}_{J=n-1}\rangle + (J<n-1)\,.\end{aligned} \tag{5.15}$$

Equation (5.15) tells us that the second trajectory is actually twice degenerate (with the only exception of $N=1$ as we have seen) and both its constituents have the right coupling. Once more the ghostlike $J=n-1$ component of $V^{(2)}_{\mu_1\mu_2\ldots\mu_n}$ has been compensated by other tensors of rank $n-1$ through Ward-like identities.

A further interesting feature that appears here is the occurrence of parity doublets ($M\neq 0$ Toller poles). This happens the first time at $N=3$ where the tensor $\overline{V}^{(1)}_\mu\overline{V}^{(1)}_1$ appears. As we noticed already such an object is not irreducible and we can write:

$$\begin{aligned}\langle \overline{V}^{(1)}_\mu\overline{V}^{(2)}_\nu\rangle\langle U^{(1)}_\mu U^{(2)}_\nu\rangle = \tfrac{1}{4}\langle \overline{V}^{(1)}_\mu\overline{V}^{(2)}_\nu + \overline{V}^{(1)}_\nu\overline{V}^{(2)}_\mu\rangle\langle U^{(1)}_\mu U^{(2)}_\nu + U^{(1)}_\nu U^{(2)}_\mu\rangle + \\ + \tfrac{1}{4}\langle \overline{V}^{(1)}_\mu\overline{V}^{(2)}_\nu - \overline{V}^{(1)}_\nu\overline{V}^{(2)}_\mu\rangle\langle U^{(1)}_\mu U^{(2)}_\nu - U^{(1)}_\nu U^{(2)}_\mu\rangle\,.\end{aligned} \tag{5.16}$$

The first term on the r.h.s. of (5.16) is the $\overline{V}^{(1,2)}_{\mu\nu}$ term considered in eq. (5.14). The second one is a new object which, like the $F_{\mu\nu}$ electromagnetic field, is composed of one vector and one axial vector particle (parity doublet of spin 1). $N = 3$ is the lowest-energy level at which $M \neq 0$ quantum numbers begin to occur in these models. Notice that the first two highest trajectories are not parity doublets.

It is clear and easy to verify that all discrete representations of the Lorentz group are indeed present or, conversely, all higher Toller trajectories should be seen. This again confirms that simple two-body considerations are very misleading in the estimate of the level structure.

6. – Conclusions.

We first summarize briefly the main results of our work:

a) The residue of each pole at $s_n = A + Bn$ factorizes into a finite number of terms, independently on the number of initial and final external lines.

b) The number of such terms increases with an $\exp[a\sqrt{n}]$ law as a function of n. This very fast increase, characteristic of a system with an infinite number of degrees of freedom, could not be expected on simple intuitive counting of trajectories in the elastic problem.

c) The different terms appearing in the factorized residue are four dimensional tensors. As it is expected, the c.m. time components of these tensors give rise to unphysical states (« ghosts ») with negative metric.

d) Fortunately one can find Ward identities connecting the divergences of higher tensors (*i.e.* c.m. time components) with tensors of lower rank. Those relations give rise to a compensation of entire families of ghosts (*e.g.* for the second trajectory).

The previous results suggest that the future outlook on the structure of elementary particles could be rather different from what one could have expected a few years ago. First of all the somewhat unexpected result of the enormous multiplicity of the levels shows that models with straight-line trajectories and duality exhibit new peculiar features. The essential many-body nature of the problem can also be understood from other, less direct, arguments.

Indeed two-body problems with short-range potential do necessarily lead to trajectories which after having reached a certain value of J, start to decrease. This is because the centrifugal barrier does not allow the existence of resonances with large J.

It was also pointed out by MANDELSTAM [15] that infinitely rising trajectories could follow from the opening of more and more channels with increasing

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[15] S. MANDELSTAM: *Proceedings of the 1966 Tokyo Summer Lectures on Theoretical Physics* (New York, *1966*).

energy so that the appearance of new thresholds compensates the decrease (due to centrifugal barrier of the two-body part of $\alpha(t)$. It can also be seen that the strict duality present in the models discussed here, requires the presence of Feynman graphs which correspond to a many-body problem both in the s- and t-channel. The situation is simply illustrated in Fig. 4.

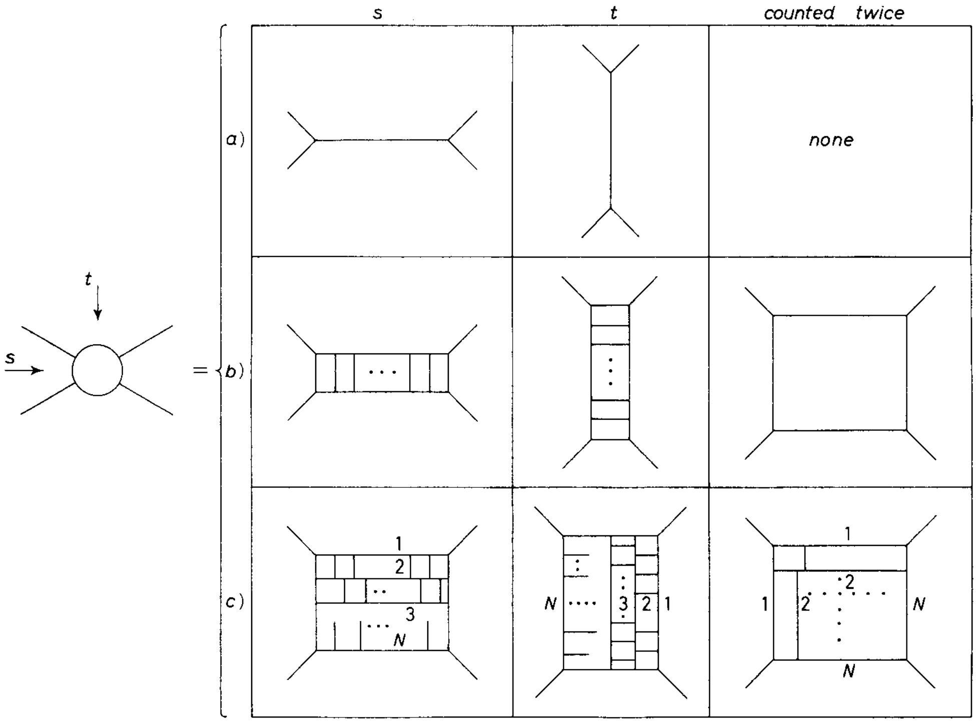

Fig. 4. – Perturbation theory diagrams for a two-body process. Row *a*) is the old Born approximation; Row *b*) is a twice composite ladder (strip) approximation; Row *c*) is a N-times composite ladder approximation. The first and second column represent s and t channel exchange respectively. The third column shows the diagrams responsible for double counting.

Fig. 4*a*) shows the old fashioned case of the exchange of an elementary particle. In this case the two poles $1/(s-M^2)$ and $1/(t-M^2)$ should be summed coherently *interference model for resonances*). Figure 4*b*) exhibits the graph which are taken into account in the strip approximation which was suggested by CHEW and FRAUTSCHI on the basis of the nearest singularities dominance phylosophy. Again (apart from the square graph shown in Fig. 4*b*)) the interference model is almost valid. The situation is completely different with the graphs of Fig. 4*c*) in which a large number N of particles is exchanged in both the s- and t-channel. In this case adding coherently singularities in s and t will make us guilty of a bad crime of double counting. Finally in the limit

$N \to \infty$ the same graph represents at the same time poles in s and in t. So we finally see that the evidence we find for the many-body structure of our levels is not inconsistent with other features of dual-resonance models.

In a simple quark language one could say that the present models point in the direction of $q\bar{q}$ excitation more than that of l-excitation of higher levels.

It is amusing that this phenomenon of compound levels is not new in nuclear physics. Indeed such an effect has been found in isobar analog resonances, the giant dipole resonance and in fission. The possibility that in analogy with the nuclear case, also elementary particle levels should have a fine structure has been raised by FESHBACH [16]. Our theoretical analysis shows that this may very well be the case.

Another important, although somewhat less pleasant feature is the appearance of ghosts with negative metric. This phenomenon too is not unexpected: if one classifies Regge trajectories in parallel Lorentz families, the particles related to the integer intersection of those trajectories can be classified as Lorentz particles. They will correspond to finite nonunitary representations of the Lorentz group and some of the particles will have a negative metric. A similar situation is present in the program of saturation of superconvergence relations, where the simplest schemes do indeed correspond to fake solutions with Lorentz particles of indefinite metric. The surprising fact is that the model does provide a mechanism for the compensation of effects due to ghost exchange. What is helping us is the great richness of levels. We indeed find well-behaved levels which are coupled to all channels in the same way as ghosts and compensate their effect. This mechanism shows striking analogies which the well-known compensation between longitudinal and timelike photons in electrodynamics.

Unfortunately in the schematized model with all equal trajectories we are only able to obtain compensation for the low-lying states. In spite of that, we have not lost the hope that a generalized, more realistic model can be found where no unphysical state appears at least for bosonic reactions [17].

If one considers the general properties of the dual resonance models one sees that many of the good properties of a reasonable S-matrix are indeed present. The important exception is of course unitarity, so that one might feel that the dual-resonance formulae are the lowest-order Born approximation of a new theory. From this point of view one is led to construct corrections to the simple models by means of rules for constructing the equivalent of higher-order Feynman graphs.

The rules for constructing higher-order dual graphs has been given, at least

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(16) H. FESHBACH: *Comm. Nucl. Part. Phys.*, **1**, 40 (1967).

(17) When half-integer spins are present, the parity degeneracy due to MacDowell symmetry is likely to increase the difficulty of this problem.

for a large class of graphs, by KIKKAWA, SAKITA and VIRASORO ([18]). It is amusing to note that those rules could be obtained in an almost unambiguous way from duality only. The main question is whether these graphs take into account at least in a perturbative way unitarity. This can be directly verified if one is able to recognize that the vertex functions appearing in those graphs are the same as those appearing in the « Born-like graphs » studied in this paper.

We see therefore that the study of factorization is a necessary « easy » preliminary step in the « hard » program of enforcing unitarity in a general dual formula. In a more elementary way we can say that, since unitarity involves a completeness sum on all states, the first step is to count how many states we have to deal with.

We have seen in this paper that even this level counting is not so easy and involves questions, like that of ghosts, which are far from being solved. Only future investigation will tell whether the present stage of developments of dual models is the first step for the construction of a reasonable self consistent theory of elementary particles.

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APPENDIX

In this Appendix we discuss some technical details related to the reflection operation R introduced in Sect. **2**. We have seen that the factorization properties of the amplitude involves the fundamental vectors

$$P^{(n)} = \sum_{i=0}^{r=1} \varrho_i^n p_i \qquad (n = 1, 2 \ldots) , \tag{A.1}$$

where $\varrho_i = x_i x_{i-1} \ldots x_2 x_1$ $(i = 1, 2, \ldots r)$ and, by definition,

$$\varrho_0 = 1 , \qquad \varrho_{r+1} = 0 . \tag{A.2}$$

In particular it will be useful to define $P^{(0)}$ from eq. (A.1) and the prescription $\varrho_{r+1}^0 = \lim_{\varepsilon \to 0} (\varepsilon)^0 = 1$, namely

$$P^{(0)} = \sum_{i=0}^{r+1} p_i = \Pi = Q^{(0)} = \sum_{i=0}^{s+1} q_i . \tag{A.3}$$

([18]) K. KIKKAWA, B. SAKITA and M. A. VIRASORO: University of Wisconsin preprint coo-224 (1969). A dual formula for the box diagram was also independently obtained by the present authors (unpublished).

The reflection operation R acts on the vectors $P^{(n)}$ as

$$P^{(n)} \underset{R}{\rightarrow} \bar{P}^{(n)} = \sum_{i=0}^{r-1} \bar{\varrho}_i^n \bar{p}_i\,, \tag{A.4}$$

where (see eq. (2.14))

$$\bar{P}_i = P_{r+1-i}\,, \qquad \bar{\varrho}_i = 1 - \varrho_{r+1-i}\,, \tag{A.5}$$

the linear transformation from $P^{(n)}$ to $\bar{P}^{(n)}$ turns out to be simply

$$\bar{P}^{(n)} = \sum_{i=0}^{n} d_i^{(n)} P^{(i)}\,, \qquad d_i^{(n)} = (-1)^i \binom{n}{i}\,, \tag{A.6}$$

the coefficients $d_i^{(n)}$ are given also by

$$(1-x)^n = \sum_i d_i^{(n)} x^i\,. \tag{A.7}$$

In order to show that (A.6) is true we simply observe that

$$\bar{P}^{(n)} = \sum_i \bar{\varrho}_i^n \bar{p}_i = \sum_i (1-\varrho_{r+1-i})^n p_{r+1-i} = \\ = \sum_j (1-\varrho_j)^n p_j = \sum_j \sum_k d_k^{(n)} \varrho_j^n p_j = \sum_k d_k^{(n)} P^{(k)}\,. \tag{A.8}$$

It will also be useful to define new vectors

$$V^{(n)} = \sum_{i=0}^{n} (\tfrac{1}{2} - \varrho_i)^n p_i\,, \qquad V^{(0)} = \Pi\,. \tag{A.9}$$

These vectors have the simple reflection property:

$$V^{(n)} \underset{R}{\rightarrow} \bar{V}^{(n)} = (-1)^n V^{(n)}\,, \tag{A.10}$$

which is easily verified.

The linear transformation between $P^{(n)}$ and $V^{(n)}$ is given by

$$\begin{cases} V^{(n)} = \sum_{i=0}^{n} C^{(n)} P^{(i)}\,, \\ P^{(n)} = \sum_{i=0}^{n} C^{(n)} V^{(n)}\,, \end{cases} \tag{A.11}$$

where

$$\left(\frac{1}{2} - x\right)^n = \sum_i C_i^{(n)} x^i\,, \qquad C_i^{(n)} = (-1)^i \binom{n}{i} \left(\frac{1}{2}\right)^{n-1}. \tag{A.12}$$

Equations (A.6) and (A.11) can be given the expression of formal power expansions

$$\bar{P}^{(n)} = (1-P)^n\,, \qquad V^{(n)} = (\tfrac{1}{2} - P)^n\,. \tag{A.13}$$

Before deriving the identities that will prove useful in Sect. 4, we have to derive the simple equality

$$\text{(A.14)} \qquad \sum_{n=1}^{\infty} C_i^{(n)}\, t^{n+1} = -\left(\frac{-t}{1-t/2}\right)^{i+1}.$$

Equation (A.14) is proven by expanding $[1-t(\frac{1}{2}-x)]^{-1}$ in power series of $t(\frac{1}{2}-x)$, using (A.12) and then equating each power in x.

We can now obtain the desired identities:

$$\text{(A.15)} \qquad \sum_{n=0}^{\infty} V^{(n)}\, y^{n+1} = \sum_{i=0}^{\infty} P^{(i)} \sum_{n=1}^{\infty} C_i^{(n)}\, y^{n+1} = -\sum_{n=0}^{\infty} P^{(n)} z^{n+1},$$

where

$$\text{(A.16)} \qquad z = \frac{-y}{1-y/2}, \qquad \text{or} \qquad \frac{1}{z} + \frac{1}{y} = \frac{1}{2}.$$

One can similarly proceed to derive the identity

$$\text{(A.17)} \qquad \sum_{n=0}^{\infty} \overline{P}^{(n)} z^{n+1} = -\sum_{n=0}^{\infty} \overline{V}^{(n)} \left(\frac{-z'}{1-z'/2}\right)^{n+1} = \\ = \sum_{n=0}^{\infty} V'^{(n)} \left(\frac{z'}{1-z'/2}\right)^{n+1} = -\sum_{n=0}^{\infty} P^{(n)} z^{n+1},$$

provided

$$\text{(A.18)} \qquad \frac{1}{z} + \frac{1}{z'} = 1.$$

Equation (A.17) can be also derived by the formal expression (A.13)

$$\text{(A.19)} \qquad z' \sum_{n=0}^{\infty} (1-P)^n z'^n = \frac{z'}{1-(1-P)z'} = -\frac{z}{1-Pz} = -z \sum_{n=0}^{\infty} P^{(n)} z^n,$$

where the second equality is just the condition (A.18). Taking the logarithm of the second equality in (A.19) we get

$$\text{(A.20)} \qquad \log z' - \log[1-(1-P)z'] = \log(-z) - \log(1-Pz).$$

From (A.18) we have also

$$\text{(A.21)} \qquad \log z' = \log(-z) - \log(1-z), \qquad \log(1-z) + \log(1-z') = 0.$$

Equation (A.20) can then be transformed into

$$\text{(A.22)} \qquad \tfrac{1}{2}\log(1-z') - \log[1'-(1-P)z'] = \tfrac{1}{2}\log(1-z) - \log[1-Pz].$$

By expanding in powers of z and z' and recalling that $P^{(0)} = \pi$ we have finally

(A.23) $$\sum_{n=1}^{\infty} \frac{\mathscr{P}^{(n)} z^n}{n} = \sum_{n=1}^{\infty} \frac{\mathscr{P}^{(n)} z'^n}{n},$$

where

(A.24) $$\mathscr{P}^{(n)} = P^{(n)} - \frac{\pi}{2}, \qquad \mathscr{P}^{(n)} = \bar{P}^{(n)} - \frac{\pi}{2}.$$

We now proceed in the same way, starting from the equality

(A.25) $$\frac{y}{1 - yV} = -\frac{z}{1 - zP} \qquad (V = \tfrac{1}{2} - P).$$

Taking the log and adding on both sides $\frac{1}{2}\log(1-z)$ we get

(A.26) $$\frac{1}{2}\log\left(\frac{(1-z)y^2}{z^2}\right) - \log(1 - yV) = \frac{1}{2}\log(1-z) - \log(1 - zP).$$

The argument of the first log is just $(1 - y^2/4)$, so that we get

(A.27) $$\sum_{n=1}^{\infty} \frac{\mathscr{V}^{(n)} y^n}{n} = \sum_{n=1}^{\infty} \frac{\mathscr{P}^{(n)} z^n}{n},$$

where $\mathscr{P}^{(n)}$ is defined in (A.24) and

(A.28) $$\mathscr{V}^{(n)} = \begin{cases} \mathscr{V}^{(n)} & \text{for } n \text{ odd}, \\ \mathscr{V}^{(n)} - 2^{-n}\Pi & \text{for } n \text{ even}. \end{cases}$$

Obviously (A.23) and (A.27) can also be proven directly without any appeal to the formal eq. (A.13).

Note. – After this work was completed, one of us (G.V.) and the authors of ref. ([18]) have been able to derive unambiguously a formula for Feynman-like dual diagrams with a single loop, using the factorization properties obtained in this paper.

In the case of the self-energy diagram we have found

$$A_{\text{S.E.}} = \int_{-\infty}^{+\infty} d^4 l \int_0^1 \int_0^1 dx_1\, dx_2\, x_1^{-\alpha(q_1^2)-1} x_2^{-\alpha(q_2^2)-1} [(1-x_1)(1-x_2)]^{-\alpha(0)-1} \cdot$$
$$\cdot \exp\left[-\sum_{i=1}^{\infty} \frac{(x_1 x_2)^i}{i(1 - x_1^i x_2^i)}\, 2b\, p_1 \cdot p_2 (x_1^i + x_2^i)\right] [f(x_1 x_2)]^{4+4b\mu^2},$$

where p_1, p_2 are the external momenta $(p_1 + p_2 = 0)$; q_1, q_2 the internal ones, and $f(x) = \prod_{k=1}^{\infty} (1 - x^k)^{-1}$ is the partition function ([8]).

This formula, which demands some slight modification of the prescription of ref. ([18]), is now being investigated.

RIASSUNTO

In questo lavoro viene studiata la struttura dei livelli nei modelli duali con risonanze strette. A questo scopo vengono usate le ampiezze per la diffusione di molte particelle proposte recentemente da vari autori, e si considera la struttura del residuo in ciascun polo. Si ottiene che per ciascun autovalore dell'energia $E_n = \sqrt{s_n}$ il residuo si fattorizza in un numero finito di temini (grado di degenerazione del livello) e che questo numero cresce con n come exp $[cE_n]$. L'interpretazione fisica di questo rapido aumento è da trovarsi nella natura essenzialmente di molti corpi dei modelli compatibili con la dualità. La presenza di stati con accoppiamento immaginario discende facilmente dal formalismo covariante quadridimensionale, che d'altra parte assicura l'assenza di singolarità cinematiche. Si riesce comunque a trovare un meccanismo di compensazione analogo a quello che esiste in elettrodinamica quantistica (identità di Ward). Anche se il problema di una sistematica compensazione di tutti gli stati non fisici non è stato risolto, tali difficoltà sono eliminate per le traiettorie più elevate.

Структура уровней для моделей двойных резонансов.

Резюме (*). — В этой статье рассматривается структура уровней для моделей узкого резонанса с двойственностью. Для этой цели мы будем использовать много-частичные двойные амплитуды, недавно предложенные несколькими авторами, и будем изучать структуру вычета для каждого полюса, в том, что касается его факторизации. Мы получаем, что для каждого собственного значения энергии $E_n = \sqrt{s_n}$ вычет, действительно, факторизуется в конечное число членов (число вырожденных уровней), и что это число увеличивается с n, как exp $[cE_n]$. Физическая интерпретация этого сильного увеличения состоит, по существу, в много-частичной природе моделей, соответствующих двойственности. Появление состояний с мнимой константой связи следует непосредственно из ковариантного четырех-мерного подхода, который мы выбрали и который обеспечивает отсутствие кинематических сингулярностей. Тем не менее, обнаружено, что здесь также появляется механизм уничтожения, аналогичный механизму уничтожения, существующему в квантовой электродинамике (тождества Уорда). Хотя проблема систематического уничтожения всех « духов » в реальном случае не решена, мы находим, что таким путем главный и причиняющий наибольшее беспокойство дух, в действительности, исключается.

(*) *Переведено редакуией.*