ISRAEL JOURNAL OF MATHEMATICS 85 (1994), 351–390

VIVE LA DIFFÉRENCE II. THE AX–KOCHEN ISOMORPHISM THEOREM

BY

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ABSTRACT

We show in §1 that the Ax–Kochen isomorphism theorem [AK] requires the continuum hypothesis. Most of the applications of this theorem are insensitive to set theoretic considerations. (A probable exception is the work of Moloney [Mo].) In §2 we give an unrelated result on cuts in models of Peano arithmetic which answers a question on the ideal structure of countable ultraproducts of $\mathbb{Z}$ posed in [LLS]. In §1 we also answer a question of Keisler regarding Scott complete ultrapowers of $\mathbb{R}$ (see 1.18).

The author thanks the Basic Research Fund of the Israeli Academy of Sciences, and the NSF for partial support of this research.

§1 of this paper owes its existence to Annalisa Marcja's hospitality in Trento, July 1987; Van den Dries' curiousity about Kim's conjecture; and the willingness of Hrushovski and Cherlin to look at §3 of [326] through a glass darkly and more directly the Cherlin interest in continuation during our stay in MSRI, fall 89. §2 of this paper owes its existence to a question of G. Cherlin concerning [LLS]. This paper was prepared with the assistance of the group in Arithmetic of Fields at the Institute for Advanced Studies, Hebrew University, during the special year on Arithmetic of Fields, 1991-92. Publ. 405.

Received May 27, 1992 and in revised form June 3, 1993

Introduction

In a previous paper [Sh326] we gave two constructions of models of set theory in which the following isomorphism principle fails in various strong respects:

(Iso 1) If $\mathcal{M}$, $\mathcal{N}$ are countable elementarily equivalent structures and $\mathcal{F}$ is a nonprincipal ultrafilter on ω, then the ultrapowers $\mathcal{M}^*$, $\mathcal{N}^*$ of $\mathcal{M}$, $\mathcal{N}$ with respect to $\mathcal{F}$ are isomorphic.

As is well known, this principle is a consequence of the continuum hypothesis. Here we will give a related example in connection with the well-known isomorphism theorem of Ax and Kochen. In its general formulation, that result states that a fairly broad class of Henselian fields of characteristic zero satisfying a completeness (or saturation) condition are classified up to isomorphism by the structure of their residue fields and their value groups. The case that interests us here is:

(Iso 2) If $\mathcal{F}$ is a nonprincipal ultrafilter on ω, then the ultraproducts $\prod_p \mathbb{Z}_p/\mathcal{F}$ and $\prod \mathbb{F}_p[[t]]/\mathcal{F}$ are isomorphic.

Here $\mathbb{Z}_p$ is the ring of p-adic integers and $\mathbb{F}_p$ is the finite field of order p. It makes no difference whether we work in the fraction fields of these rings as fields, in the rings themselves as rings, or in the rings as valued rings, as these structures are mutually interpretable in one another. In particular, the valuation is definable in the field structure (for example, if the residual characteristic p is greater than 2 consider the property: "$1+px^2$ has a square root"). We show that such an isomorphism cannot be obtained from the axioms of set theory (ZFC). As an application we may mention that certain papers purporting to prove the contrary need not be refereed.

Of course, the Ax–Kochen isomorphism theorem is normally applied as a step toward results which cannot be affected by set-theoretic independence results. One exception is found in the work of Moloney [Mo] which shows that the ring of convergent real-valued sequences on a countable discrete set has exactly 10 residue domains modulo prime ideals, assuming the continuum hypothesis. This result depends on the general theorem of Ax and Kochen which lies behind the isomorphism theorem for ultraproducts, and also on an explicit construction of a new class of ultrafilters based on the continuum hypothesis. It is very much an open question to produce a model of set theory in which Moloney's result no longer holds.

Our result can of course be stated more generally; what we actually show here may be formulated as follows.

PROPOSITION A: *It is consistent with the axioms of set theory that there is an ultrafilter $\mathcal{F}$ on ω such that for any two sequences of discrete rank 1 valuation rings $(R^i_n)_{n=1,2,\ldots}$ $(i = 1, 2)$ having countable residue fields, any isomorphism $F : \prod_n R^1_n/\mathcal{F} \longrightarrow \prod_n R^2_n/\mathcal{F}$ is an ultraproduct of isomorphisms F_n: $R^1_n \longrightarrow R^2_n$ (for a set of n's contained in $\mathcal{F}$). In particular most of the pairs R^1_n, R^2_n are isomorphic.*

In the case of the rings $\mathbb{F}_p[[t]]$ and $\mathbb{Z}_p$, we see that (Iso 2) fails.

From a model theoretic point of view this is not the right level of generality for a problem of this type. There are two natural ways to pose the problem:

(1) Characterize the pairs of countable models $\mathcal{M}$, $\mathcal{N}$ such that for some ultrafilter $\mathcal{F}$ in some forcing extension $\prod \mathcal{M}^\omega/\mathcal{F} \not\cong \prod \mathcal{N}/\mathcal{F}$;

(2) Characterize the pairs of countable models $\mathcal{M}$, $\mathcal{N}$ with no isomorphic ultrapowers in some forcing extension;

(there there are two variants: the ultrapowers may be formed either using one ultrafilter twice, or using any two ultrafilters).

(3) Write $\mathcal{M} \leq \mathcal{N}$ if in every forcing extension, whenever $\mathcal{F}$ is an ultrafilter on ω such that $\mathcal{N}^\omega/\mathcal{F}$ is saturated, then $\mathcal{M}^\omega/\mathcal{F}$ is also saturated. Characterize this relation.

This is somewhat like the Keisler order [Ke, Sh-a or Sh-c Chapter VI] but does not depend on the fact that the ultrafilter is regular. We can replace $\aleph_0$ here by any cardinal κ satisfying $\kappa^{<\kappa} = \kappa$.*

However the set theoretic aspects of the Ax–Kochen theorem appear to have attracted more interest than the two general problems posed here. We believe that the methods used here are appropriate also in the general case, but we have not attempted to go beyond what is presented here.

With the methods used here, we could try to show that for every $\mathcal{M}$ with countable universe (and language), if $\mathbf{P}_3$ is the partial order for adding $\aleph_3$-Cohen reals then we can build a $\mathbf{P}_3$-name for a non principal ultrafilter $\mathcal{F}$ on ω, such that in V^P $\mathcal{M}^\omega/\mathcal{F}$ resembles the models constructed in [Sh107]; we can choose the relevant bigness properties in advance (cf. Definition 1.5, clause (5.3)). This would be helpful in connection with problems (1,2) above.

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* We shall return to those problems in [Sh507], answering in particular a question of Jarden: if F^l_n is a finite field for $l < 2$, $n < w$, $F^1_n \not\cong F^2_n$ then $\prod_{n<w} F^1_n/\mathcal{F} \not\cong \prod_{n<w} F^2_n/\mathcal{F}$.

In §2 of this paper we give a result on cuts in models of Peano Arithmetic which has previously been overlooked. Applied to $\aleph_1$-saturated models, our result states that some cut does not have countable cofinality from either side; and, in general, the two cofinalities are equal. As we explain in §2, this answers a question on ideals in ultrapowers of $\mathbb{Z}$ which was raised in [LLS]. The result has nothing to do with the material in §1, beyond the bare fact that it also gives some information about ultraproducts of rings over ω.

The model of set theory used for the consistency result in §1 is obtained by adding $\aleph_3$ Cohen reals to a suitable ground model. There are two ways to get a "suitable" ground model. The first way involves taking any ground model which satisfies a portion of the GCH, and extending it by an appropriate preliminary forcing, which generically adds the *name* for an ultrafilter which will appear after addition of the Cohen reals. The alternative approach, which we prefer and called model-theoretic is to start with an L-like ground model and use instances of diamond (or related weaker principles) to prove that a sufficiently generic name already exists in the ground model (a complete proof of the case used here is included in the appendix). The theme is that forcing an object is a transparent way to build an object with no undesirable subsets (or expansions), so it is nice to prove that if some forcing argument proves that such an object exists then it really exists, provided that the original universe satisfies suitable principles. For 1-morasses this is the point of Shelah Stanley [ShSt112]. For $\diamondsuit_\lambda$ (with $\lambda = \aleph_2$) this is the principle we use: it is weaker than the one corresponding to a 1-morass, but its assumption is weaker: e.g GCH + $\lambda = \mathrm{cf}\lambda > \aleph_1$, or $\lambda = \lambda^{<\lambda} > \beth_\omega$ or $\lambda = \lambda^{<\lambda} > \beth_\omega$, implies (by [Sh460]) $(Dl)_\lambda$ which is a weakening of $\diamondsuit_\lambda$, and suffices for the principle. That was the method used in §3 of [Sh326], which is based in turn on [ShHL162] (see also the earlier [Sh82] [Sh107]) which has still not appeared as of this writing. Also the formalism as presented in [ShHL162], or [Sh326], though adequate for certain applications, turns out to be slightly too limited for our present use. More specifically, there are continuity assumptions built into that formalism which are not valid here and cannot easily be recovered. The difficulty, in a nutshell, is that a union of ultrafilters in successively larger universes is not necessarily an ultrafilter in the universe arising at the corresponding limit stages, and it can be completed to one in various ways.

After a complaint on the earlier version we give in the appendix a proof of the case where $\diamondsuit_\lambda$ is assumed (rather than the weaker $(Dl)_\lambda$) which is the case used.

So the family *App* defined below can be reused as an actual forcing notion for the most part so a reader can read the paper with this in mind omitting the parts speaking on the second approach. However we will then take note of matters relevant to the more refined argument based on a variant of the model-theoretic method, spesifically based on A6-10 here (fitting the theme above). In addition the exposition in [Sh326, §3] includes a very explicit discussion of the way such a result may be used to formalize arguments of the type given here, in a suitable ground model (in the second sense).

Notation: Note that we used trees with linearly ordered levels, necessarily well ordered.

1. Obstructing the Ax–Kochen isomorphism

1.1 DISCUSSION. We will prove Proposition A as formulated in the introduction. We begin with a few words about our general point of view. In practice we do not deal directly with valuation rings, but with trees. If one has a structure with a countable sequence of refining equivalence relations E_n (so that E_{n+1} refines E_n) then the equivalence classes carry a natural tree structure in which the successors of an E_n-class are the E_{n+1}-classes contained in it. Each element of the structure gives rise to a path in this tree, and if the equivalence relations separate points then distinct elements give rise to distinct paths. This is the situation in the valuation ring of of a valued field with value group $\mathbb{Z}$, where we have the basic family of equivalence relations: $E_n(x,y) \leftrightarrow v(x-y) \geq n$. (Or better: $E(x,y;z) =:$ "$v(x-y) \geq v(z)$".) Of course an isomorphism of structures would induce an isomorphism of trees, and our approach is to limit the isomorphisms of such trees which are available.

1.2 THE MAIN RESULT FOR TREES. We consider trees as structures equipped with a partial ordering and the relation of lying at the same level of the tree. We will also consider expansions to much richer languages. We use the method of [Sh326, §3] to prove:

PROPOSITION B: *It is consistent with the axioms of set theory that there is a nonprincipal ultrafilter $\mathcal{F}$ on ω such that for any two sequences of countable trees $(T^i_n)_{n=1,2,\ldots}$ for $i = 1,2$, with each tree T^i_n countable with ω levels, and with each node having at least two immediate successors, if $\mathcal{T}^i = \prod_n T^i_n/\mathcal{F}$, then for any* *isomorphism $F\colon T^1 \simeq T^2$ there is an element $a \in T^1$ such that the restriction of F to the cone above a is the restriction of an ultraproduct of maps $F_n\colon T^1_n \longrightarrow T^2_n$.*

1.3 Proposition B implies Proposition A: Given an isomorphism F between ultraproducts R^1, R^2 modulo $\mathcal{F}$ of discrete valuation rings R^i_n, we may consider the induced map F_+ on the tree structures T^1, T^2 associated with these rings, as indicated above. We then find by Proposition B that on a cone of T^1, F_+ agrees with an ultraproduct of maps $F_{+,n}$ between the trees T^i_n associated with the R^i_n. On this cone F is definable from F_+, in the following sense: $F(x) = y$ iff for all n, $F_+(x \bmod (\pi_1)^n) \equiv y \bmod (\pi_2)^n$, where π_i generates the maximal ideal of R^i and we identify $R^i/(\pi_i)^n$ with the n-th level of T^i. (This is expressed rather loosely; in the notation we are using at the moment, one would have to take n to vary on integers in the sense of the ultraproduct, so including nonstandard integers. After formalization in an appropriate first order language it will look somewhat different.) Furthermore F is definable in (R^1, R^2) from its restriction to this cone: the cone corresponds to a coset of some principal ideal (a) of R^1 and $F(x) = F(ax + b)/F(a)$ for any fixed choice of b from the cone. Summing up, then, there is a first order sentence valid in $(R^1, R^2; F_+)$ (with F_+ suitably interpreted as a parametrized family of maps $R^1/\pi_1^n \longrightarrow R^2/\pi_2^n$) stating that an isomorphism $F\colon R^1 \longrightarrow R^2$ is definable in a particular way from F_+; so the same must hold in most of the pairs (R^1_n, R^2_n), that is, for a set of indices n which lies in $\mathcal{F}$. In particular in such pairs we get an isomorphism of R^1 and R^2. ∎

1.4 CONTEXT. We concern ourselves solely with Proposition B in the remainder of this section. For notational convenience we fix two sequences $(T^i_n)_{n<\omega}$ of trees ($i = 1$ or 2) in advance, where each tree T^i_n is countable with ω levels, no maximal point, and no isolated branches. The tree T^i_n is considered initially as a model with two relations: the tree order and equality of level. Although we fix the two sequences of trees, we can equally well deal simultaneously with all possible pairs of such sequences, at the cost of a little more notation.

As explained in the introduction, we work in a Cohen generic extension of a suitable ground model. This ground model is assumed to satisfy $2^{\aleph_n} = \aleph_{n+1}$ for $n = 0, 1, 2$. If we use the partial order App defined below as a preliminary forcing, prior to the addition of the Cohen reals, then this is enough. If we wish to avoid any additional forcing then we assume that the ground model satisfies $\diamondsuit_S$ for $S = \{\delta < \aleph_3\colon \operatorname{cof}\delta = \aleph_2\}$, and we work with App directly in the ground model using the A6-10 in the Appendix. The model-theoretical proof requires more active participation by the reader.

Let $\mathbf{P}$ be the Cohen forcing adding $\aleph_3$ Cohen reals. An element $\mathbf{p}$ of $\mathbf{P}$ is a finite partial function from $\aleph_3 \times \omega$ to ω. For $\mathcal{A} \subseteq \aleph_3$, and $\mathbf{p} \in \mathbf{P}$, let $\mathbf{p}\restriction\mathcal{A}$ denote the restriction of $\mathbf{p}$ to $\mathcal{A} \times \omega$ and $\mathbf{P}\restriction\mathcal{A} = \{\mathbf{p}\restriction\mathcal{A} : \mathbf{p} \in \mathbf{P}\}$. Let $\underset{\sim}{x}_\beta$ be the β^{th} cohen real. The partial order App is defined below.

We deal with a number of expansions of the basic language of pairs of trees. For a forcing notion Q and G being Q-generic over V, we write ${}^G(T_n^1, T_n^2)$ for the expanded structure in which for every k, every sequence $(r_n)_{n<\omega}$ of k-place relations r_n on (T_n^1, T_n^2) is represented by a k-place relation symbol R (i.e., $R_{(r_n : n<\omega)}$); that is, R is interpreted in (T_n^1, T_n^2) by the relation r_n. This definition takes place in $V[G]$. In V we will have names for these relations and relation symbols. We write ${}^Q(T_n^1, T_n^2)$ for the corresponding collection of names. In practice Q will be $\mathbf{P}\restriction\mathcal{A}$ for some $\mathcal{A} \subseteq \omega_3$ and in this case we write ${}^{\mathcal{A}}(T_n^1, T_n^2)$.

Typically we will have certain subsets of each T_n^i singled out, and we will want to study the ultraproduct of these sets, so we will make use of the predicate whose interpretation in each T_n^i is the desired set. Really we want to deal with ${}^{\mathbf{P}}(T_n^1, T_n^2)$, but this is rather large, and so we have to pay some attention to matters of timing.

1.5 Definition: As in [Sh326], we will set up a class App of approximations to the name of an ultrafilter in the generic extension $V[\mathbf{P}]$. In [Sh326] we emphasized the use of general method of [ShHL162] to construct the name $\underset{\sim}{\mathcal{F}}$ of a suitable ultrafilter in the ground model.

The elements of App are triples $q = (\mathcal{A}, \underset{\sim}{\mathcal{F}}, \varepsilon)$ such that:

(1) $\mathcal{A}$ is a subset of $\aleph_3$ of cardinality $\aleph_1$;

(2) $\underset{\sim}{\mathcal{F}}$ is a $\mathbf{P}\restriction\mathcal{A}$-name of a nonprincipal ultrafilter on ω, called $\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}$;

(3) $\varepsilon = (\varepsilon_\alpha \colon \alpha \in \mathcal{A})$, with each $\varepsilon_\alpha \in \{0,1\}$, and $\varepsilon_\alpha = 0$ whenever $\mathrm{cf}\alpha < \aleph_2$;

(4) For $\beta \in \mathcal{A}$ we have: $[\underset{\sim}{\mathcal{F}} \cap \{\underset{\sim}{a} : \underset{\sim}{a}$ a $\mathbf{P}\restriction(\mathcal{A}\cap\beta)$-name of a subset of $\omega\}]$ is a $\mathbf{P}\restriction(\mathcal{A}\cap\beta)$-name;

(5) If $\mathrm{cf}\beta = \aleph_2$, $\beta \in \mathcal{A}, \varepsilon_\beta = 1$ then $\mathbf{P}\restriction\mathcal{A}$ forces the following:

(5.1) $\underset{\sim}{x}_\beta / \underset{\sim}{\mathcal{F}}$ is an element of $(\prod_{n<\omega} T_n^1 / \underset{\sim}{\mathcal{F}}\restriction\mathcal{A})^{V[\mathbf{P}\restriction\mathcal{A}]}$ whose level is above all levels of elements of the form $\underset{\sim}{x} / \underset{\sim}{\mathcal{F}}$ for $\underset{\sim}{x}$ a $\mathbf{P}\restriction(\mathcal{A}\cap\beta)$-name;

(5.2) $\underset{\sim}{x}_\beta$ induces a branch $\underset{\sim}{B}$ on $(\prod_{n<\omega} T_n^1 / \underset{\sim}{\mathcal{F}} \restriction (\mathcal{A}\cap\beta))^{V[\mathbf{P}\restriction(\mathcal{A}\cap\beta)]}$ which has elements in every level of that tree (such a branch will be called *full*) and which is a $\mathbf{P}\restriction(\mathcal{A}\cap\beta)$- name (and not just forced to be equal to one); so $\underset{\sim}{B}$ is a $\mathbf{P}\restriction(\mathcal{A}\cap\beta)$-name of a member of $V[G]$ and if $G\subseteq\mathbf{P}\restriction\mathcal{A}\cap\beta$ is generic over V, then $\underset{\sim}{B}[G]=\{y\colon y\in V[G\cap(\mathbf{P}\restriction(A\cap\beta))],\ y\in\prod_{n<\omega}T^1_n,\ y/\underset{\sim}{\mathcal{F}}[G]<\underset{\sim}{x}_\beta/\underset{\sim}{\mathcal{F}}\}$ is full.

(5.3) The branch $\underset{\sim}{B}$ intersects every dense subset of

$$[\prod_n{}^{\mathcal{A}\cap\beta}T^1_n/[\mathcal{F}\restriction(\mathcal{A}\cap\beta)]^{V[\mathbf{P}\restriction(\mathcal{A}\cap\beta)]}$$

which is definable in $(\prod_n{}^{\mathcal{A}\cap\beta}(T^1_n,T^2_n)/[\underset{\sim}{\mathcal{F}}\restriction(\mathcal{A}\cap\beta)])^{V[\mathbf{P}\restriction(\mathcal{A}\cap\beta)]}$.

Note in (5.3) that the dense subset under consideration will have a $\mathbf{P}\restriction(\mathcal{A}\cap\beta)$-name, and also that by Łoś' theorem a dense subset of the type described extends canonically to a dense subset in any larger model. The notion of "bigness" alluded to in the introduction is given by (5.3).

We write $q_1\le q_2$ if q_2 extends q_1 in the natural sense. We say that $q_2\in\text{App}$ is an *end* extension of q_1, and we write $q_1\le_{\text{end}}q_2$, if $q_1\le q_2$ and $\mathcal{A}^{q_2}\setminus\mathcal{A}^{q_1}$ follows $\mathcal{A}^{q_1}$. Here we have used the notation: $q=(\mathcal{A}^q,\mathcal{F}^q,\varepsilon^q)$.

1.6 Remark: The following comments bear on the version based on the model theoretic method. In a previous version of this method, rather than examining each $\underset{\sim}{x}_\beta$ separately, we would really group them into short blocks $X_\beta=(\underset{\sim}{x}_{\beta+\zeta}\colon\zeta<\aleph_2)$, for β divisible by $\aleph_2$. Then our assumptions on the ground model V allow us to use the method to construct the name $\underset{\sim}{\mathcal{F}}$ in V. One of the ways $\diamondsuit_S$ will be used is to "predict" certain elements $\mathbf{p}_\delta\in\mathbf{P}\restriction\delta$ and certain $\mathbf{P}\restriction\delta$-names of functions $\underset{\sim}{F}_\delta$ which amount to guesses as to the restriction to a part of $\prod_n T^1_n$ of (the name of) a function representing some isomorphism $\underset{\sim}{F}$ modulo $\underset{\sim}{\mathcal{F}}$. However in A6 this is already taken into account.

1.7 LEMMA: *If $(q_\zeta)_{\zeta<\xi}$ is an increasing sequence of at most $\aleph_1$ members of* App *such that $q_{\zeta_1}\le_{\text{end}}q_{\zeta_2}$ for $\zeta_1<\zeta_2$, then we can find $q\in\text{App}$ such that $\mathcal{A}^q=\bigcup_\zeta\mathcal{A}^{q_\zeta}$ and $q_\zeta\le_{\text{end}}q$ for $\zeta<\xi$.*

Proof: We may suppose $\xi>0$ is a limit ordinal. If $\text{cf}(\xi)>\aleph_0$ then $\bigcup_{\zeta<\xi}q_\zeta$ will do, while if $\text{cf}(\xi)=\aleph_0$ then we just have to extend $\bigcup_\zeta\underset{\sim}{\mathcal{F}}^{q_\zeta}$ to a $\mathbf{P}\restriction(\bigcup_\zeta\mathcal{A}^{q_\zeta})$-name of an ultrafilter on ω, which is no problem. (cf. [Sh326, 3.10]). $\blacksquare_{1.7}$

1.8 LEMMA: *Suppose $q\in\text{App}$, $\gamma>\sup\mathcal{A}^q$, and $\underset{\sim}{B}$ is a $\mathbf{P}\restriction\mathcal{A}^q$-name of a branch of $(\prod_n T^\varepsilon_n/\underset{\sim}{\mathcal{F}}^q)^{V[\mathbf{P}\restriction\mathcal{A}^q]}$. Then:*

1. *We can find an* r, $q \leq r \in \mathrm{App}$ *with* $\mathcal{A}^r = \mathcal{A}^q \cup \{\gamma\}$, *and a* $(\mathbf{P}{\restriction}\mathcal{A}^r)$*-name* $\underset{\sim}{x}$ *of a member of* $\prod_n T^\varepsilon_n / \underset{\sim}{\mathcal{F}}^r$ *which is above* $\underset{\sim}{B}$.
2. *We can find an* $r \in \mathrm{App}$ *with* $q \leq_{\mathrm{end}} r$ *and* $\mathcal{A}^r = \mathcal{A}^q \cup [\gamma, \gamma + \omega_1)$, *and a* $(\mathbf{P}{\restriction}\mathcal{A}^r)$ *-name* $\underset{\sim}{B}'$ *of a full branch extending* $\underset{\sim}{B}$, *which intersects every definable dense subset of* $(\prod_n {}^{\mathcal{A}^r} T^\varepsilon_n)^{V[\mathbf{P}{\restriction}\mathcal{A}^r]} / \underset{\sim}{\mathcal{F}}^r$.
3. *In (2) we can also ask that any particular type* p *over* $\prod {}^{\mathcal{A}^q}(T^1_n, T^2_n) / \underset{\sim}{\mathcal{F}}^q$ *(in* $V[\mathbf{P}{\restriction}\mathcal{A}^q]$*) is realized in* $(\prod_n {}^{\mathcal{A}^r} T^\varepsilon_n)^{V[\mathbf{P}{\restriction}\mathcal{A}^r]} / \underset{\sim}{\mathcal{F}}^r$.

Proof: 1. Make $\underset{\sim}{x}_\gamma$ realize the required type, and let $\varepsilon_\gamma = 0$.

2. We define $r_\zeta = r{\restriction}(\mathcal{A}^q \cup [\gamma, \gamma+\zeta))$ by induction on $\zeta \leq \omega_1$. For limit ζ use 1.7 and for successor ζ use part (1). One also takes care, via appropriate bookkeeping, that $\underset{\sim}{B}'$ should intersect every dense definable subset of $(\prod_n {}^{\mathcal{A}^r} T^\varepsilon_n / \underset{\sim}{\mathcal{F}}^r)^{V[\mathbf{P}{\restriction}\mathcal{A}^r]}$ by arranging for each such set to be met in some specific $(\prod_n {}^{\mathcal{A}^{r_\zeta}} T^\varepsilon_n / \underset{\sim}{\mathcal{F}}^{r_\zeta})^{V[\mathbf{P}{\restriction}\mathcal{A}^{r_\zeta}]}$ with $\zeta < \aleph_1$.

3. We can take $\alpha \in [\gamma, \gamma + \omega_1)$ with $\mathrm{cof}\,\alpha \neq \aleph_2$ and use x_α to realize the type.
$\blacksquare_{1.8}$

1.9 Lemma: *Suppose* $q_0, q_1, q_2 \in \mathrm{App}$, $q_0 = q_2{\restriction}\beta$, $q_0 \leq q_1$, $\mathcal{A}^{q_1} \subseteq \beta$.

1. *If* $\mathcal{A}^{q_2} \setminus \mathcal{A}^{q_0} = \{\beta\}$ *and* $\varepsilon^{q_2}_\beta = 0$, *then there is* $q_3 \in \mathrm{App}$, $q_3 \geq q_1, q_2$ *with* $\mathcal{A}^{q_3} = \mathcal{A}^{q_1} \cup \mathcal{A}^{q_2}$.
2. *Suppose* $\mathcal{A}^{q_2} \setminus \mathcal{A}^{q_0} = \{\beta\}$, $\mathrm{cf}(\beta) = \aleph_2$, $\varepsilon^{q_2}_\beta = 1$, *and in particular* $\sup \mathcal{A}^{q_1} < \beta$. *Assume that* $\underset{\sim}{B}_1$ *is a* $\mathbf{P}{\restriction}\mathcal{A}^{q_1}$*-name of a full branch of*
$$(\prod T^1_n / \underset{\sim}{\mathcal{F}}^{q_1})^{V[\mathbf{P}{\restriction}\mathcal{A}^{q_1}]}$$
intersecting every dense subset of this tree which is definable in $(\prod_n {}^{\mathcal{A}^{q_1}}(T^1_n, T^2_n) / \underset{\sim}{\mathcal{F}}^{q_1})^{V[\mathbf{P}{\restriction}\mathcal{A}^{q_1}]}$, *such that* $\underset{\sim}{B}_1$ *contains the branch* $\underset{\sim}{B}_0$ *which* $\underset{\sim}{x}_\beta$ *induces according to* q_2. *Then there is* $q_3 \geq q_1, q_2$ *with* $\mathcal{A}^{q_3} = \mathcal{A}^{q_1} \cup \{\beta\}$, *such that according to* q_3, $\underset{\sim}{x}_\beta$ *induces* $\underset{\sim}{B}_1$ *on* $(\prod T^1_n / \underset{\sim}{\mathcal{F}}{\restriction}\mathcal{A}^{q_1})^{V[\mathbf{P}{\restriction}\mathcal{A}^{q_1}]}$.
3. *If* $\mathcal{A}^{q_2} \setminus \mathcal{A}^{q_0} = \{\beta\}$, $\mathrm{cf}(\beta) = \aleph_2$, $\varepsilon^{q_2}_\beta = 1$, *and* $\sup \mathcal{A}^{q_1} < \gamma < \beta$ *with* $\mathrm{cf}\gamma \neq \aleph_2$, *then there is* $q_3 \in \mathrm{App}$ *with* $q_1 \leq q_3$, $q_2 \leq q_3$, $\mathcal{A}^{q_3} = \mathcal{A}^{q_1} \cup \mathcal{A}^{q_2} \cup [\gamma, \gamma + \omega_1)$.
4. *There are* $q_3 \in \mathrm{App}$, $q_1, q_2 \leq q_3$, *so that* $\mathcal{A}^{q_3} \setminus (\mathcal{A}^{q_1} \cup \mathcal{A}^{q_2})$ *has the form* $\bigcup\{[\gamma_\zeta, \gamma_\zeta + \omega_1) \colon \zeta \in \mathcal{A}^{q_2} \setminus \mathcal{A}^{q_0}, \mathrm{cf}(\zeta) = \aleph_2\}$ *where* γ_ζ *is arbitrary subject to* $\sup(\mathcal{A}^{q_2}{\restriction}\zeta) < \gamma_\zeta < \zeta$.

5. *Assume* $\delta_1 < \aleph_2$, $\beta < \aleph_3$, *that* $(p_i)_{i<\delta_1}$ *is an increasing sequence from* App, *and that* $q \in \mathrm{App}\restriction\beta$ *satisfies:*

$$\text{For } i < \delta_1\text{: } p_i\restriction\beta \leq q.$$

Then there is an $r \in \mathrm{App}$ *with* $q \leq_{\mathrm{end}} r$ *and* $p_i \leq r$ *for all* $i < \delta_1$.

6. *Assume* δ_1, $\delta_2 < \aleph_2$, $(\beta_j)_{j<\delta_2}$ *is and increasing sequence with all* $\beta_j < \aleph_3$, *that* $(p_i)_{i<\delta_1}$ *is an increasing sequence from* App, *and that* $q_j \in \mathrm{App}\restriction\beta_j$, *for* $j < \delta_2$ *where* $\delta_2 < \aleph_3$ *satisfy:*

$$\text{For } i < \delta_1,\ j < \delta_2\text{: } p_i\restriction\beta_j \leq q_j;\ \text{For } j < j' < \delta_2\text{: } q_j \leq_{\mathrm{end}} q_{j'}.$$

Then there is an $r \in \mathrm{App}$ *with* $p_i \leq r$ *and* $q_j \leq_{\mathrm{end}} r$ *for all* $i < \delta_1$ *and* $j < \delta_2$.

Proof: 1. The proof is easy and is essentially contained in the proofs following. (One verifies that $\underset{\sim}{\mathcal{F}}^{q_1} \cup \underset{\sim}{\mathcal{F}}^{q_2}$ generates a proper filter in $V[\mathbf{P}\restriction(\mathcal{A}^{q_1} \cup \mathcal{A}^{q_2})]$.)

2. Let $\mathcal{A}_i = \mathcal{A}^{q_i}$ and let $\underset{\sim}{\mathcal{F}}_i = \underset{\sim}{\mathcal{F}}^{q_i}$ for $i = 1, 2$, and $\mathcal{A}_3 = \mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{A}_1 \cup \{\beta\}$. The only non obvious part is to show that in $V[\mathbf{P}\restriction\mathcal{A}_3]$ there is an ultrafilter extending $\underset{\sim}{\mathcal{F}}_1 \cup \underset{\sim}{\mathcal{F}}_2$ which contains the sets:

$$\{n : T^1_n \models \underset{\sim}{x}(n) \leq \underset{\sim}{x}_\beta(n)\} \ \text{ for } \underset{\sim}{x} \in \underset{\sim}{B}_1,\ \underset{\sim}{x} \text{ a } \mathbf{P}\restriction\mathcal{A}_1\text{-name.}$$

If this fails, then there is some $\mathbf{p} \in \mathbf{P}\restriction\mathcal{A}_3$, a $\mathbf{P}\restriction\mathcal{A}_1$-name $\underset{\sim}{a}$ of a member of $\underset{\sim}{\mathcal{F}}_1$, a $\mathbf{P}\restriction\mathcal{A}_2$-name $\underset{\sim}{b}$ of a member of $\underset{\sim}{\mathcal{F}}_2$, and some $\underset{\sim}{x} \in \underset{\sim}{B}_1$ such that $\mathbf{p} \Vdash$"$\underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} = \emptyset$" where $\underset{\sim}{c} = \{n : \underset{\sim}{x}(n) \leq \underset{\sim}{x}_\beta(n)\}$ for some $(\mathbf{P}\restriction\mathcal{A}^{q_1})$-name $\underset{\sim}{x}$, such that $\mathbf{p}\restriction\mathcal{A}_1| \Vdash$ "$\underset{\sim}{x} \in \underset{\sim}{B}_1$" and w.l.o.g $\Vdash$ "$\underset{\sim}{x} \in \prod_m T^1_m$". Why only those three sets? by the amount of closure under intersection we have. Let $\mathbf{p}_i = \mathbf{p}\restriction\mathcal{A}_i$ for $i = 0, 1, 2$, and let $\mathbf{H}^0 \subseteq \mathbf{P}\restriction\mathcal{A}_0$ be generic over V, with $\mathbf{p}_0 \in \mathbf{H}^0$.

Let:

$$\underset{\sim}{A}^1_n[\mathbf{H}^0] = \{y \in T^1_n\text{: For some } \mathbf{p}'_1, \mathbf{p}_1 \leq \mathbf{p}'_1 \in \mathbf{P}\restriction\mathcal{A}_1,\ \mathbf{p}'_1\restriction\mathcal{A}_0 \in \mathbf{H}^0 \text{ and } \mathbf{p}'_1 \Vdash \text{“}\underset{\sim}{x}(n) \leq y, \text{ and } n \in \underset{\sim}{a}\text{”}\ \}$$

Then $\underset{\sim}{A}^1_n$ is a $\mathbf{P}\restriction\mathcal{A}_0$-name. Let $\underset{\sim}{A}^1 = (\prod_n \underset{\sim}{A}^1_n/\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}_0)^{V[\mathbf{P}\restriction\mathcal{A}_0]}$. Now $\underset{\sim}{A}^1$ is not necessarily dense in $(\prod_n T^1_n/\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}_0)^{V[\mathbf{P}\restriction\mathcal{A}_0]}$, but the set

$$\underset{\sim}{A}^* =: \{\underset{\sim}{y} \in (\prod_n T^1_n/\underset{\sim}{\mathcal{F}}^{q_0})^{V[\mathbf{P}\restriction\mathcal{A}_0]} : \underset{\sim}{y} \in \underset{\sim}{A}^1, \text{ or } \underset{\sim}{y} \text{ is incompatible in the tree with all } \underset{\sim}{y}' \in \underset{\sim}{A}^1\}$$

is dense, and it is definable (in $\prod_m {}^{(\mathbf{P}\restriction A_0)} T^1_m / \underset{\sim}{\mathcal{F}}^{q_0}$), hence not disjoint from $\underset{\sim}{B}_0$. Fix $\underset{\sim}{y} \in \underset{\sim}{A}^* \cap \underset{\sim}{B}_0$. As $\mathbf{P}\restriction\mathcal{A}_1$ forces ($\Vdash_{\mathbf{P}\restriction\mathcal{A}_1}$) that $\underset{\sim}{x} \in \underset{\sim}{B}_1$, clearly $\underset{\sim}{x}$ and $\underset{\sim}{y}$ cannot be be incompatible (in $\prod_m {}^{(\mathbf{P}\restriction A_1)} T^1_m$), so, clearly $\mathbf{p}_1\restriction\mathcal{A}_1$ forces ($\Vdash_{\mathbf{P}\restriction\mathcal{A}_1}$) that $\underset{\sim}{x}$, $\underset{\sim}{y}$ are compatible in $\prod_m {}^{\mathbf{P}\restriction A_0} T^1_m$, and thus $\underset{\sim}{y} \in \underset{\sim}{A}^1$.

The following sets are in $\underset{\sim}{\mathcal{F}}^{V[\mathbf{H}^0]}$:

$$\underset{\sim}{A} = \{n\colon \text{ for some } \mathbf{p}_1', \mathbf{p}_1 \le \mathbf{p}_1' \in \mathbf{P}\restriction\mathcal{A}_1, \mathbf{p}_1'\restriction\mathcal{A}_0 \in \mathbf{H}^0 \text{ and } \mathbf{p}_1' \Vdash \text{“}\underset{\sim}{x}(n) \le \underset{\sim}{y}(n), \text{ and } n \in \underset{\sim}{a}\text{”}\}.$$

$$\underset{\sim}{B} = \{n\colon \text{ for some } \mathbf{p}_2', \mathbf{p}_2 \le \mathbf{p}_2' \in \mathbf{P}\restriction\mathcal{A}_2, \mathbf{p}_2'\restriction\mathcal{A}_0 \in \mathbf{H}^0 \text{ and } \mathbf{p}_2' \Vdash \text{“}\underset{\sim}{y}(n) \le \underset{\sim}{x}_\beta(n), \text{ and } n \in \underset{\sim}{b}\text{”}\}.$$

For example, $\underset{\sim}{A}$ is a subset of ω in $V[\mathbf{H}^0]$ which is in $\underset{\sim}{\mathcal{F}}^{q_1}$. As the complement of $\underset{\sim}{A}$ cannot be in $\underset{\sim}{\mathcal{F}}^{q_0}[H^0]$, $\underset{\sim}{A}$ must be in $\underset{\sim}{\mathcal{F}}^{q_0}[H^0]$.

Now for any $n \in \underset{\sim}{A} \cap \underset{\sim}{B}$ we can force $n \in \underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c}$ by amalgamating the corresponding conditions $\mathbf{p}_1', \mathbf{p}_2'$; as said above this finishes the proof of the existence of q_3.

3. Let $\underset{\sim}{B}_0$ be the $\mathbf{P}\restriction\mathcal{A}^{q_0}$-name of the branch which $\underset{\sim}{x}_\beta$ induces. By 1.8 (2) there is q_1^*, $\mathcal{A}^{q_1^*} = \mathcal{A}^{q_1} \cup [\gamma, \gamma + \omega_1)$, $q_1 \le q_1^* \in$ App and there is a $\mathbf{P}\restriction\mathcal{A}^{q_1^*}$-name $\underset{\sim}{B}_1 \supseteq \underset{\sim}{B}_0$ of an appropriate branch for q_1^*. Now apply part (2) to q_0, q_1^*, q_2.

4. As in [Sh326, 3.9(2)], by induction on the order type γ of $(\mathcal{A}^{q_2} \setminus \mathcal{A}^{q_1})$: If $\gamma = 0$ trivial; If $\gamma = \gamma' + 1$, β last member of $\mathcal{A}^{q_2}$, $\varepsilon_\beta^{q_2} = 0$ use part (1); If $\gamma = \gamma' + 1$ and β last member of $\mathcal{A}^{q_2}$, $\varepsilon_\beta^{q_2} = 1$ use part (3). If γ is a limit ordinal, use part (6) below.

5, 6. Since (6) includes (5), it suffices to prove (6); but as we go through the details we will treat the cases corresponding to (5) first. We point out at the outset that if δ_2 is a successor ordinal or a limit of uncountable cofinality, then we can replace the q_j by their union, which we call q, setting $\beta = \sup_j \beta_j$, so all these cases can be treated using the notation of (5).

We will prove by induction on $\gamma < \omega_2$ that if all $\beta_j \le \gamma$ and all p_i belong to App$\restriction\gamma$, then the claim (6) holds for some r in App$\restriction\gamma$.

We first dispose of most of the special cases which fall under clause (5) (so for the present, q is well defined). If $\delta_1 = \delta_0 + 1$ is a successor ordinal it suffices to apply (1) on (3) to p_{δ_0} and q. So we assume for the present that δ_1 is a limit ordinal. In addition if $\gamma = \beta$ we take $r = q$, so we will assume $\beta < \gamma$ throughout.

THE CASE $\gamma = \gamma_0 + 1$, A SUCCESSOR: In this case our induction hypotheses applies to the $p_i \restriction \gamma_0$, q, β, and γ_0, yielding r_0 in $\mathrm{App} \restriction \gamma_0$ with $p_i \restriction \gamma_0 \leq r_0$ and $q \leq_{\mathrm{end}} r_0$. What remains to be done is an amalgamation of r_0 with all of the p_i, where $\mathrm{dom}\, p_i \subseteq \mathrm{dom}\, r_0 \cup \{\gamma_0\}$, and where one may as well suppose that γ_0 is in $\mathrm{dom}\, p_i$ for all i. This is a slight variation on 1.9 (1) or (3) (depending on the value of $\varepsilon_\gamma^{p_i}$, which is independent of i).

THE CASE γ A LIMIT OF COFINALITY GREATER THAN $\aleph_1$: Since $\delta_1 < \aleph_2$ there is some $\gamma_0 < \gamma$ such that all p_i lie in $\mathrm{App} \restriction \gamma_0$ and $\beta < \gamma_0$, and the induction hypothesis then yields the claim.

THE CASE γ A LIMIT OF COFINALITY $\aleph_1$: Choose γ_j a strictly increasing and continuous sequence of length at most ω_1 with supremum γ, starting with $\gamma_0 = \beta$. By induction choose $r_j \in \mathrm{App} \restriction \gamma_j$ for $i < \omega_1$ such that:

(0) $$r_0 = q;$$

(1) $$r_j \leq_{\mathrm{end}} r_{j'} \text{ for } j < j' < \omega_1;$$

(2) $$p_i \restriction \gamma_j \leq r_j \text{ for } i < \delta_i \text{ and } j < \omega_1.$$

At successor stages the inductive hypothesis is applied to $p_i \restriction \gamma_{j+1}$, r_j, γ_j, and γ_{j+1}. At limit stages j we apply the inductive hypothesis to $p_i \restriction \gamma_j$, $r_{j'}$ for $j' < j$, $\gamma_{j'}$ for $j' < j$, and γ_j; and here (6) is used, inductively.

Finally let $r = \bigcup r_j$.

We now make an observation about the case of (5) that we have not yer treated, in which γ has cofinality ω. In this case we can use the same construction used when γ has cofinality $\aleph_1$, except for the last step (where we set $r = \bigcup r_j$, above). What is needed at this stage would be an instance of (6), with the r_j in the role of the q_j and $\delta_2 = \omega$.

This completes the induction for the cases that fall under the notation of (5), apart from the case in which γ has cofinality ω, which we reduced to an instance of (6) with the same value of γ and with $\mathrm{cf}\,\delta_2 = \omega$. Accordingly as we deal with the remaining case we may assume $\mathrm{cf}\,\delta_2 = \omega$. In this case $q = \bigcup q_j$ is a well-defined object, but not necessarily in App, as the filter $\underset{\sim}{\mathcal{F}}^q$ is not an ultrafilter (there are reals generated by $\mathbf{P}\restriction(\operatorname{dom} q)$ which do not come from any $\mathbf{P}\restriction(\operatorname{dom} q_j)$).

Now we prove part (6), so by the cases already treated, δ_2 is a limit ordinal. We distinguish two cases. If $\beta =: \sup \beta_j$ is less than γ (remember $q_j \in \mathrm{App}\restriction\beta_j$), then the induction applies, delivering an element $r_0 \in \mathrm{App}\restriction\beta$ with $p_i\restriction\beta \leq r_0$ and all $q_j \leq_{\mathrm{end}} r_0$. This r_0 may then play the role of q in an application of 1.9 (5) for the same γ, and either it has already been proved or it is the last case above which was reduced to a case of 1.9 (5) in which $\beta = \gamma$, a case treated below.

In some sense the main case (at least as far as the failure of continuity is concerned) is the remaining one in which $\beta = \gamma$. Notice in this case that although $p_i\restriction\beta_j \leqq q_j$ it does not follow that $p_i\restriction\beta \leq q$ (for the reason mentioned above: $p_i\restriction\beta$ includes an ultrafilter on part of the universe, while the filter associated with q need not be an ultrafilter). All that is needed at this stage is an ultrafilter containing all $\underset{\sim}{\mathcal{F}}^{p_i} \cup \underset{\sim}{\mathcal{F}}^{q_j}$. As this is a directed system of filters, it sufficed to check the compatibility of each such pair, as was done in 1.9(2). $\blacksquare_{1.9}$

1.10 Construction, first version. We force with App and the generic object G gives us a $\mathbf{P}$-name of an ultrafilter in $V[\mathrm{App}][\mathbf{P}] = V[G][\mathbf{P}]$. The forcing is $\aleph_2$-complete by 1.9 (6). We also claim that it satisfies the $\aleph_3$-chain condition (see below), and hence does not collapse cardinals and does not affect our assumptions on cardinal arithmetic. (Subsetes of $\aleph_2$ are added, but not very many). Let $\underset{\sim}{\mathcal{F}}^G = \bigcup\{\underset{\sim}{\mathcal{F}}^r : r \in G\}$, it is a $\mathbf{P}$-name of an ultrafilter on ω, it belongs to $V[G]$. In particular for each member r of the generic subset of App we have $\underset{\sim}{\mathcal{F}}[G] \cap \mathcal{P}(\omega)^{V[G]^{\mathbf{P}\restriction\mathcal{A}^r}} = \underset{\sim}{\mathcal{F}}[G] \cap \mathcal{P}(\omega)^{V[\mathbf{P}\restriction\mathcal{A}^r]}$ and $(\prod^{\mathcal{A}^r}(T_n^1, T_n^2)/\underset{\sim}{\mathcal{F}}^r)^{V[\mathbf{P}\restriction\mathcal{A}^r]}$ both are $\mathbf{P}\restriction\mathcal{A}^r$-names, not depending on forcing with App, i.e. on $\underset{\sim}{G}$.

We now check the chain condition. Suppose we have an antichain $\{q_\alpha\}$ of cardinality $\aleph_3$ in App, where for convenience the index α is taken to vary over ordinals of cofinality $\aleph_2$. We claim that by Fodor's lemma, we may suppose that the condition $q_\alpha\restriction\alpha$ is constant. One application of Fodor's lemma allows us to assume that $\gamma = \sup(\mathcal{A}^{q_\alpha} \cap \alpha)$ is constant. Once γ is fixed, there are only $\aleph_2$ possibilities for $q_\alpha\restriction\gamma$, by our assumptions on the ground model, and a second application of Fodor's lemma allows us to take $q_\alpha\restriction\gamma$ to be constant.

Now fix α_1 of cofinality $\aleph_2$ (or more accurately, in the set of indices which survive two applications of Fodor's lemma), and let $q'_1 =: q_{\alpha_1}$, $\beta = \sup \mathcal{A}^{q_1}$, and take $\alpha_2 > \beta$ of cofinality $\aleph_2$. We find that $q'_2 =: q_{\alpha_2}$ and q_1 are compatible, by

1.9(4), and this is a contradiction.

Let $G^\beta = G \cap (\mathrm{App}\restriction\beta)$ and $\underset{\sim}{\mathcal{F}}^\beta = \bigcup\{\underset{\sim}{\mathcal{F}}^r : r \in G_\beta\}$ (for $\beta < \aleph_3$).

1.11 Construction, second version. As we wish to apply the model theoretic method (over a suitable ground model) and build the name of our ultrafilter in the ground model, we proceed as follows. For $\alpha \le \aleph_3$ we choose $G^\alpha \subseteq \mathrm{App}\restriction\alpha$, directed under $\le$, increasing with α, inductively as in A 22 making all the commitments we can; more specifically, take $\mathcal{N}_\alpha \prec (H(\beth^+_{\omega+1}), \in)$ of cardinality $\aleph_2$ with $\alpha \in \mathcal{N}_\alpha$, $\aleph_2 \subseteq \mathcal{N}_\alpha$, $\mathcal{N}_\alpha$ includes the sequence of the first α moves and is $(< \aleph_2)$-complete, increasing with α, and the oracle associated with $\diamondsuit_S$ belongs to $\mathcal{N}_\alpha$, and in stage α if the Guelf will make all the commitments known to $\mathcal{N}_\alpha$, then G^α is in the ground model but behaves like a generic object for $\mathrm{App}\restriction\alpha$ in V, and in particular gives rise to a name $\underset{\sim}{\mathcal{F}}^\alpha$. Note that $V[G_\alpha] = V$ here.

The lengthy discussion in [Sh326] is useful for developing intuition. Here we will just note briefly that what is called a commitment here is really an isomorphism type of a commitment, in a more conventional sense: this is a device for compressing $\aleph_3$ possible commitments into a set of size $\aleph_2$.

The axioms in the appendix have been given in a form suitable to their application to the proof of the relevant combinatorial theorem, rather than in the form most convenient of verification. 1.9 above represents the sort of formulation we use when we are actually verifying the axioms.

We will now add a few details connecting 1.9 with the eight axioms of paragraph A6. The first three of these are formal and it may be expected that they will be visibly true of any situation in which this method would be applied. The fourth axiom is the so-called amalgamation axiom which has been given in a slightly more detailed form in 1.9(4). The last four axioms are various continuity axioms, which are instances of 1.9(6). We reproduce them here:

5'. If $(p_i)_{i<\delta}$ is an increasing sequence in App of length less than λ, then it has an upper bound q.

6'. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ of members of $\mathrm{App}\restriction(\beta+1)$, with $\beta < \lambda^+$ and if $q \in \mathrm{App}\restriction\beta$ satisfies $p_i\restriction\beta \le q$ for all $i < \delta$, then $\{p_i : i < \delta\} \cup \{q\}$ has an upper bound r in App with $q \le_{\mathrm{end}} r$.

7'. If $(\beta_j)_{j<\delta}$ is a striclty increasing sequence of length less than λ, with each $\beta_j < \lambda^+$, and $p \in \mathrm{App}$, $q_i \in \mathrm{App}\restriction\beta_i$, with $p\restriction\beta_j \le q_j$, and $p_{j'}\restriction\beta_j = p_j$ for $j < j' < \delta$, then $\{p\} \cup \{q_j : i < \delta\}$ has an upper bound r with all $q_j \le_{\mathrm{end}} r$.

8'. Suppose δ_1, δ_2 are limit ordinals less than λ, and $(\beta_j)_{j<\delta_2}$ is a strictly increasing continuous seqeuence of ordinals less than λ^+. Let $I(\delta_1, \delta_2) := (\delta_1 + 1) \times (\delta_2 + 1) \setminus \{(\delta_1, \delta_2)\}$. Suppose that for $(i,j) \in I(\delta_1, \delta_2)$ we have $p_{ij} \in \mathrm{App}\restriction\beta_i$ such that

$$i \leq i' \Longrightarrow p_{ij} \leq p_{i'j}$$
$$j \leq j' \Longrightarrow p_{ij} = p_{ij'}\restriction\beta_j;$$

Then $\{p_{ij}: (i,j) \in I(\delta_1, \delta_2)\}$ has an upper bound r in App with $r\restriction\beta_j = p_{\delta_1,j}$ for all $j < \delta_1$.

The first three are visibly instances of 1.9(6). In the case of axiom (8') we set $p_i = p_{i,\delta_2}$ for $i < \delta_1$ and $q_j = p_{\delta_1,j}$ for $j < \delta_2$. Then $p_i\restriction\beta_j = p_{i,j} \leq q_j$, so 1.9(6) applies and yields (8').

1.12 Lemma: *Suppose $\delta < \aleph_3$, $\mathrm{cf}(\delta) = \aleph_2$, and $\mathbf{H}^\delta \subseteq \mathbf{P}\restriction\delta$ is generic for $\mathbf{P}\restriction\delta$. Then in $V[G^\delta][\mathbf{H}^\delta]$ we have:*

$$\textstyle\prod_n {}^\delta(T_n^1, T_n^2)/\underset{\sim}{\mathcal{F}}{}^\delta[\mathbf{H}^\delta] \text{ is } \aleph_2\text{-compact.}$$

Proof: Similar to 1.8(3). We can use some $\underset{\sim}{x}_\beta$ with β of cofinality less than $\aleph_2$ to realize each type. In the forcing version, this means App forces our claim to hold since it can't force the opposite. In the alternative approach, what we are saying is that the commitments we made include commitments to make our claim true. As $2^{\aleph_1} = \aleph_2$ in $V[\mathbf{H}^\delta]$ we can "schedule" the commitments conveniently, so that each particular type of cardinality $\aleph_1$ that needs to be considered by stage δ in fact appears before stage δ. $\blacksquare_{1.12}$

1.13 Killing isomorphisms. We begin the verification that our filter $\underset{\sim}{\mathcal{F}}$ satisfies the condition of Proposition B. We suppose therefore that we have a $\tilde{\mathbf{P}}$-name $\underset{\sim}{F}$ and a condition $\mathbf{p}^* \in \mathbf{P}$ forcing:

"$\underset{\sim}{F}$ is a map from $\prod_n T_n^1$ onto $\prod_n T_n^2$ which represents an isomorphism modulo $\underset{\sim}{\mathcal{F}}$."

We then have a stationary set S of ordinals $\delta < \aleph_3$ of cofinality $\aleph_2$ which satisfy:

(a) $\mathbf{p}^* \in \mathbf{P}\restriction\delta$.
(b) For every $\mathbf{P}\restriction\delta$-name $\underset{\sim}{x}$ for an element of $\prod_n T_n^1$, $\underset{\sim}{F}(\underset{\sim}{x})$ is a $\mathbf{P}\restriction\delta$-name.
(c) Similarly for $\underset{\sim}{F}^{-1}$.

If we are using our second approach, over an L-like ground model:

(d) At stage δ of the construction of the G^α, the diamond "guessed" $\mathbf{p}^\delta = \mathbf{p}^*$ and $\underset{\sim}{F}_\delta = \underset{\sim}{F}\restriction\delta$.

(In this connection, recall that the guesses made by diamond influence the choice of "commitments" made in the construction of the G^δ.) Let $\underset{\sim}{y}^* =: \underset{\sim}{F}(\underset{\sim}{x}_\delta)$. Then:

$$(*)_{\underset{\sim}{y}^*} \qquad \mathbf{p}^* \Vdash \begin{cases} \text{"}\underset{\sim}{y}^* \text{ induces a branch in } (\prod_n T_n^2/\underset{\sim}{\mathcal{F}})^{V[\mathbf{P}\restriction\delta]} \\ \text{which is the image under } \underset{\sim}{F}_\delta \\ \text{of the branch which } \underset{\sim}{x}_\delta \text{ induces on } (\prod_n T_n^1/\underset{\sim}{\mathcal{F}})^{V[\mathbf{P}\restriction\delta]}\text{."} \end{cases}$$

Now we come to one of the main points. We claim that there is some $q^* \in G$ such that: $q^*\restriction\delta \in G^\delta$, $\underset{\sim}{x}_\delta$, y^* are $(\mathbf{P}\restriction\mathcal{A}^{q^*})$-names, and with the following property:

$(\dagger)_\delta$ Given $q_1 \in G^\delta$ with $q^*\restriction\delta \le q_1$ and $\mathbf{P}\restriction\mathcal{A}^{q_1}$-names $(\underset{\sim}{x}, \underset{\sim}{y})$ with $\underset{\sim}{x} \in \prod T_n^1$, $\underset{\sim}{y} \in \prod T_n^2$, then for any $q_3' \in \text{App}$ with $q_1, q^* \le q_3'$ and $q_3'\restriction\delta \in G^\delta$, we have: $\underset{\sim}{x}$, $\underset{\sim}{y}$ are $(\mathbf{P}\restriction\mathcal{A}^{q_3'\restriction\delta})$-names and $p^*\restriction\mathcal{A}^{q_3'}$ forces (i.e. $\Vdash_{\mathbf{P}\restriction\mathcal{A}^{q_3'}}$.) the following:

(a) "If $\underset{\sim}{y} = \underset{\sim}{F}_\delta(\underset{\sim}{x})$ then: $\underset{\sim}{x} \le \underset{\sim}{x}_\delta$ iff $\underset{\sim}{y} \le \underset{\sim}{y}^*$, and

(b) if $\underset{\sim}{y}$ and $\underset{\sim}{F}(\underset{\sim}{x})$ are incomparable, then $\underset{\sim}{x} \le \underset{\sim}{x}_\delta$ implies $\underset{\sim}{y} \not\le \underset{\sim}{y}^*$."

Notice here that q_3' need not be in G.

The reason for this depends slightly on which of the two approaches to the construction of G we have taken. In a straight forcing approach, we may say that some $q^* \in G$ forces $(*)_{\underset{\sim}{y}^*}$, and this yields $(\dagger)_\delta$. In the second, pseudo-forcing, approach we find that our "commitments" include a commitment to falsify $(*)_{\underset{\sim}{y}^*}$ if possible; as we did not do so, at a certain point it must have been impossible to falsify it, which again translates into $(\dagger)_\delta$.

We now fix q^* satisfying $(\dagger)_\delta$, and we set $q_0 = q^*\restriction\delta$. At this stage, $(\dagger)_\delta$ gives some sort of local definition of $\underset{\sim}{F}\restriction\delta$, on a cone in $(\prod{}^\delta(T_n^1/\underset{\sim}{\mathcal{F}}^\delta)^{V[\mathbf{P}\restriction\delta]}$, (the cone is determined by q_0). The next result allows us to put this definition in a more useful form (and this is nailed down in 1.15). One may think of this as an elimination of quantifiers.

1.14 LEMMA: *Suppose that:*

(1) q_0, q_1, q_2, q_3 *are in* App *with* $q_0 = q_2\restriction\beta_0 \le q_1 \le_{\text{end}} q_3$, *and* $q_2 \le q_3$.

(2) $q_0 \le r_0 \in \text{App}$ *with* $\mathcal{A}^{q_1} \subseteq \mathcal{A}^{r_0} \subseteq \beta_0$.

Let $\mathcal{A}_i = \mathcal{A}^{q_i}$ *for* $i = 0, 1, 2, 3$, *and suppose that*

(3) $\underset{\sim}{f}_0$ *is a* $\mathbf{P}\restriction\mathcal{A}^{r_0}$*-name of a partial map from* $(\prod_n(T_n^1, T_n^2))^{V[\mathbf{P}\restriction\mathcal{A}_1]}$ *into* $(\prod_n(T_n^1, T_n^2))^{V[\mathbf{P}\restriction\mathcal{A}^{r_0}]}$ *representing a partial elementary embedding of*

$(\prod_n {}^{\mathcal{A}_0}(T_n^1,T_n^2)/\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}_1)^{V[\mathbf{P}\restriction\mathcal{A}_1]}$ *into* $(\prod_n {}^{\mathcal{A}_0}(T_n^1,T_n^2)/\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}^{r_0})^{V[\mathbf{P}\restriction\mathcal{A}^{r_0}]}$ *which is equal to the identity on* $(\prod_n(T_n^1,T_n^2)/\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}_0)^{V[\mathbf{P}\restriction\mathcal{A}_0]}$.

Then there is an $r\in$ App *with:* $q_2\le r$; $r_0\le_{\mathrm{end}} r$, $\mathcal{A}_3\subseteq\mathcal{A}^r$; $\mathcal{A}^r\cap\beta_0=\mathcal{A}^{r_0}$; *and there is a* **P***-name* $\underset{\sim}{f}$ *of a function from* $(\prod_n(T_n^1,T_n^2))^{V[\mathbf{P}\restriction\mathcal{A}_3]}$ *into*

$$(\prod_n(T_n^1,T_n^2))^{V[\mathbf{P}\restriction\mathcal{A}^r]}$$

representing an elementary embedding of $(\prod_n^{\mathcal{A}_2}(T_n^1,T_n^2)/\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}_3)^{V[\mathbf{P}\restriction\mathcal{A}_3]}$ *into*

$$(\prod_n {}^{\mathcal{A}_2}(T_n^1,T_n^2)/\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}^r)^{V[\mathbf{P}\restriction\mathcal{A}^r]}$$

which is the identity on $(\prod_n(T_n^1,T_n^2)/\underset{\sim}{\mathcal{F}}\restriction\mathcal{A}_2)^{V[\mathbf{P}\restriction\mathcal{A}_2]}$.

Proof: It will be enough to get $\underset{\sim}{f}$ as a partial elementary embedding, as one may then iterate 1.8(3) $\aleph_1$ times.

We may suppose $\beta_0=\inf(\mathcal{A}_3\setminus\mathcal{A}^{r_0})$. Let $\mathcal{A}_3\setminus\beta_0=(\beta_i)_{i<\xi}$ be enumerated in increasing order. We will construct two increasing sequences, one of names $\underset{\sim}{f}_i$ and and one of elements $r_i\in$ App, indexed by $i\le\xi$, such that our claim holds for $\underset{\sim}{f}_i, q_2\restriction\beta_i, q_3\restriction\beta_i, r_i$, and in addition $\mathcal{A}^{r_i}\subseteq\beta_i$ and r_i is $\le_{\mathrm{end}}$-increasing. At the end we take $r=r_\xi$ and $\underset{\sim}{f}=\underset{\sim}{f}_\xi$.

THE CASE $i=0$: Initially r_0 and $\underset{\sim}{f}_0$ are given.

THE LIMIT CASE: Suppose first that i is a limit ordinal of cofinality $\aleph_0$, and let $\mathcal{A}=\bigcup_{j<i}\mathcal{A}^{r_j}$. In this case $\bigcup_{j<i}\underset{\sim}{\mathcal{F}}^{r_j}$ is not an ultrafilter in $V[\mathbf{P}\restriction\mathcal{A}]$ and the main point will be to prove that there is a $\mathbf{P}\restriction\mathcal{A}$-name for an ultrafilter $\underset{\sim}{\mathcal{F}}_i$ extending $\underset{\sim}{\mathcal{F}}^{q_2\restriction\beta_i}$ and $\bigcup_{j<i}\underset{\sim}{\mathcal{F}}^{r_i}$, such that

(*) The map $\underset{\sim}{f}_i$ defined as the identity on $(\prod_n(T_n^1,T_n^2))^{V[\mathbf{P}\restriction(\mathcal{A}_2\cap\beta_i)]}$ and as $\bigcup_{j<i}\underset{\sim}{f}_j$ on the latter's domain is a partial elementary map from

$$(\prod_n {}^{\mathcal{A}_2\cap\beta_i}(T_n^1,T_n^2)/\underset{\sim}{\mathcal{F}}\restriction(\mathcal{A}_3\cap\beta_i))^{V[\mathbf{P}\restriction(\mathcal{A}_3\cap\beta_i)]}$$

into

$$(\prod_n {}^{\mathcal{A}_2\cap\beta_i}(T_n^1,T_n^2)/\underset{\sim}{\mathcal{F}}^{r_i})^{V[\mathbf{P}\restriction\mathcal{A}]}.$$

So it will suffice to find $\underset{\sim}{\mathcal{F}}_i$ making (*) true. This means we must check the finite intersection property for a certain family of (names of) sets. Suppose toward a contradiction that we have a condition $\mathbf{p} \in \mathbf{P}\restriction\mathcal{A}$ forcing "$\underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} = \emptyset$," where for some $j < i$:

(A) $\underset{\sim}{a}$ is a $\mathbf{P}\restriction\mathcal{A}^{r_j}$-name for a member of $\underset{\sim}{\mathcal{F}}^{r_j}$

(B) $\underset{\sim}{b}$ is a $\mathbf{P}\restriction\mathcal{A}^{q_2\restriction\beta_i}$-name for a member of $\underset{\sim}{\mathcal{F}}^{q_2\restriction\beta_i}$

(C) $\underset{\sim}{c}$ is the name of a set of the form:

$$\{n:\ {}^{\mathcal{A}^{q_2\restriction\beta_i}}(T_n^1, T_n^2) \models \underset{\sim}{\varphi}(\underset{\sim}{\mathbf{x}}(n), \underset{\sim}{f}_j(\underset{\sim}{\mathbf{y}})(n))\}$$

(note: $\underset{\sim}{f}_j(\underset{\sim}{y})$ is a $(\mathbf{P}\restriction\mathcal{A}^{r_1})$-name).

(C1) $\underset{\sim}{\mathbf{x}}, \underset{\sim}{\mathbf{y}}$ are finite sequences from $(\prod_n(T_n^1, T_n^2))^{V[\mathbf{P}\restriction\mathcal{A}^{q_2\restriction\beta_i}]}$ and $\operatorname{Dom}\underset{\sim}{f}_j \subseteq (\prod_n(T_n^1, T_n^2))^{V[\mathbf{P}\restriction(\mathcal{A}_3\cap\beta_j)]}$ respectively.

(C2) $\underset{\sim}{\varphi}$ is a $\mathbf{P}\restriction\mathcal{A}^{q_2\restriction\beta_i}$-name for a formula in the language of $\prod_n {}^{\mathcal{A}^{q_2\restriction\beta_i}}(T_n^1, T_n^2)$

(C3) $\underset{\sim}{\varphi}(\underset{\sim}{\mathbf{x}}, \underset{\sim}{\mathbf{y}})$ holds in $(\prod_n {}^{\mathcal{A}_2\cap\beta_i}(T_n^1, T_n^2)/\underset{\sim}{\mathcal{F}}\restriction(\mathcal{A}_3\cap\beta_i))^{V[\mathbf{P}\restriction\mathcal{A}_3\cap\beta_i]}$.

Here $j < i$ arises as the supremum of finitely many values below i. Note that as i is a limit ordinal, we have no "bigness" condition. As $\underset{\sim}{\mathbf{x}}$ can be absorbed into the language, we will drop it.

Now let $\mathbf{H}$ be generic for $\mathbf{P}\restriction(\mathcal{A}_2\cap\beta_j)$ with $\mathbf{p}\restriction(\mathcal{A}_2\cap\beta_j) \in \mathbf{H}$, and define:

$$\underset{\sim}{A}_n =: \{\mathbf{u} : \text{for some } \mathbf{p}_2 \geq \mathbf{p}\restriction(\mathcal{A}_2\cap\beta_i) \text{ in } \mathcal{P}^{\mathcal{A}_2\cap\beta_j} \text{ with } \mathbf{p}_2\restriction(\mathcal{A}_2\cap\beta_j) \in \mathbf{H},$$
$$\mathbf{p}_2 \Vdash \text{"} n \in \underset{\sim}{b} \text{ and } {}^{\mathcal{A}^{q_2\restriction\beta_i}}(T_n^1, T_n^2) \models \underset{\sim}{\varphi}(\mathbf{u})\text{."}\}$$

$\underset{\sim}{A}_n$ is a $\mathbf{P}\restriction(\mathcal{A}_2\cap\beta_j)$-name of a subset of T_n^2. Note $(\underset{\sim}{A}_n)_{n<\omega}$ is a relation in $\prod {}^{\mathcal{A}^{q_2\restriction\beta_j}}(T_n^1, T_n^2)$. By hypothesis $\{n:\ {}^{\mathcal{A}^{q_2\restriction\beta_i}}(T_n^1, T_n^2) \models \underset{\sim}{\varphi}(\underset{\sim}{\mathbf{y}}(n))\} \in \underset{\sim}{\mathcal{F}}^{q_3\restriction\beta_i}$, and this set is contained in the set $\underset{\sim}{c}' =: \{n:\ \underset{\sim}{\mathbf{y}}(n) \in \underset{\sim}{A}_n\}$, hence $\mathbf{p}$ forces $\underset{\sim}{c}'$ to be in $\underset{\sim}{\mathcal{F}}^{q_3\restriction\beta_i}$. But $\underset{\sim}{c}'$ is a $(\mathbf{P}\restriction\mathcal{A}^{q_3\restriction\beta_j})$ -name. Therefore $\underset{\sim}{c}' \in \underset{\sim}{\mathcal{F}}^{q_3\restriction\beta_j}$ and applying $\underset{\sim}{f}_j$, we find:

$$\{n:\ \underset{\sim}{f}_j(\underset{\sim}{\mathbf{y}})(n) \in \underset{\sim}{A}_n\} \in \underset{\sim}{\mathcal{F}}^{r_j}.$$

Hence we may suppose that $\mathbf{p}$ forces: for $n \in \underset{\sim}{a}$, $\underset{\sim}{f}_j(\underset{\sim}{\mathbf{y}})(n) \in \underset{\sim}{A}_n$. But then any element of $\underset{\sim}{a}$ can be forced by some extension of $\mathbf{p}$ to lie in $\underset{\sim}{b} \cap \underset{\sim}{c}$, by amalgamating appropriate conditions over $\mathcal{A}_2 \cap \beta_j$.

Limits of larger cofinality are easier.

THE SUCCESSOR CASE: Suppose now that $i = j+1$. We may suppose that $\beta_j \in \mathcal{A}_2$ as otherwise there is nothing to prove. If $\varepsilon^{q_2}_{\beta_j} = 0$ we argue as in the previous case. So suppose that $\varepsilon^{q_2}_{\beta_j} = 1$. In particular β_j has cofinality $\aleph_2$.

Using 1.8(3) repeatedly, and the limit case, we can find $\underset{\sim}{B}$, $q'_1, r', \underset{\sim}{f}'$ such that (remember that by the first sentence in the proof we look for f' with domain $(\mathrm{Dom}\, \underset{\sim}{f}_j) \cup \prod_m (T^1_m, T^2_m)^{V[\mathbf{P}\restriction \mathcal{A}^{q_2\restriction\beta_i}]}$):

(1) $q_3\restriction\beta_j \leq_{\mathrm{end}} q'_1$; $\mathcal{A}^{q'_1} \subseteq \beta_j$;

(2) $r_j \leq_{\mathrm{end}} r'$; $\mathcal{A}^{r'} \subseteq \beta_j$;

(3) $\underset{\sim}{f}'$ is a map from $\prod_n (T^1_n, T^2_n)^{V[\mathbf{P}\restriction\mathcal{A}^{q'_1}]}$ onto $\prod_n (T^1_n, T^2_n)^{V[\mathbf{P}\restriction\mathcal{A}^{r'}]}$ representing an elementary embedding of $(\prod_n {}^{\mathcal{A}^{q_2\restriction\beta_j}}(T^1_n, T^2_n)/\underset{\sim}{\mathcal{F}}^{q'_1})^{V[\mathbf{P}\restriction\mathcal{A}^{q'_1}]}$ into $(\prod_n {}^{\mathcal{A}^{q_2\restriction\beta_j}}(T^1_n, T^2_n)/\underset{\sim}{\mathcal{F}}^{r'})^{V[\mathbf{P}\restriction\mathcal{A}^{r'}]}$ extending f_j;

(4) $\underset{\sim}{B}$ is a $\mathbf{P}\restriction\mathcal{A}^{q'_1}$-name of a branch of $(\prod_n T^1_n/\underset{\sim}{\mathcal{F}}^{q'_1})^{V[\mathbf{P}\restriction\mathcal{A}^{q'_1}]}$ which is sufficiently generic;

(5) $\underset{\sim}{f}'[\underset{\sim}{B}]$ is a $\mathbf{P}\restriction\mathcal{A}^{r'}$ -name of a branch of $(\prod_n T^1_n/\underset{\sim}{\mathcal{F}}^{r'})^{V[\mathbf{P}\restriction\mathcal{A}^{r'}]}$ which is sufficiently generic.

(6) $\underset{\sim}{B}$ includes $\{\underset{\sim}{x}\colon \underset{\sim}{x}$ is a $\mathbf{P}\restriction\mathcal{A}^{q_2\restriction\beta_j}$-name of a member of $\prod_n T^1_n$ which is below $\underset{\sim}{x}_{\beta_j}$ according to $q_2\restriction\beta_j\}$, (remember that this is a $\mathbf{P}\restriction\mathcal{A}^{q_2\restriction\beta_j}$-name by the definition of App).

Let q'_3 satisfy $q_3\restriction\beta_i \leq q'_3$, $q'_1 \leq_{\mathrm{end}} q'_3$, with $\mathcal{A}^{q'_3} \subseteq \beta_i$ such that according to q'_3 the vertex $\underset{\sim}{x}_{\beta_j}$ lies above $\underset{\sim}{B}$ (using 1.9(2)). We intend to have r_i put $\underset{\sim}{x}_{\beta_j}$ above $\underset{\sim}{f}'[\underset{\sim}{B}]$ (to meet conditions (5.2, 5.3) in the definition of App), while meeting our other responsibilities. As usual the problem is to verify the finite intersection property for a certain family of names of sets. Suppose therefore toward a contradiction that we have a condition $\mathbf{p} \in \mathbf{P}$ forcing "$\underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} \cap \underset{\sim}{d} = \emptyset$," where:

$\underset{\sim}{a}$ is a $\mathbf{P}\restriction\mathcal{A}^{r'}$-name of a member of $\underset{\sim}{\mathcal{F}}^{r'}$;

$\underset{\sim}{b}$ is a $\mathbf{P}\restriction\mathcal{A}^{q_2\restriction\beta_i}$-name of a member of $\underset{\sim}{\mathcal{F}}^{q_2\restriction\beta_i}$;

$\underset{\sim}{c}$ is the name of a set of the form

$$\{n\colon {}^{\mathcal{A}^{q_2\restriction\beta_j}}(T^1_n, T^2_n) \models \underset{\sim}{\varphi}(\underset{\sim}{x}_{\beta_j}(n), \underset{\sim}{\mathbf{z}}, \underset{\sim}{f}'(\underset{\sim}{\mathbf{y}})(n))\}$$

$\underset{\sim}{d}$ is $\{n\colon T^1_n \models \underset{\sim}{x}(n) < \underset{\sim}{x}_{\beta_j}(n)\}$

where in connection with $\underset{\sim}{c}$ and $\underset{\sim}{d}$ we have:

$\underset{\sim}{\mathbf{y}}$ is a finite sequence from $(\prod_n (T_n^1, T_n^2))^{V[\mathbf{P}\restriction \mathcal{A}^{q_1'}]}$,

$\underset{\sim}{\mathbf{z}}$ is a finite sequence from $\prod_n (T_n^1, T_n^2)^{V[\mathbf{P}\restriction \mathcal{A}^{q_2 \restriction \beta_i}]}$

$\underset{\sim}{\varphi}(\underset{\sim}{x}_{\beta_j}, \underset{\sim}{\mathbf{z}}\ \underset{\sim}{\mathbf{y}})$ is defined and holds in $(\prod_n{}^{\mathcal{A}_2 \cap \beta_i}(T_n^1, T_n^2)/\underset{\sim}{\mathcal{F}}^{q_3'})^{V[\mathbf{P}\restriction \mathcal{A}^{q_3'}]}$,

$\underset{\sim}{x}$ is a $\mathbf{P}\restriction \mathcal{A}^{q_1'}$-name for a member of $\underset{\sim}{f}'[\underset{\sim}{B}]$ (in connection with $\underset{\sim}{d}$).

We can absorb the parameters $\underset{\sim}{\mathbf{z}}$ occurring in $\underset{\sim}{\varphi}$ into the expanded language which is associated with $\prod (T_n^1, T_n^2)^{\mathcal{A}^{q_2 \restriction \beta_i}}$ as individual constants so w.l.o.g $\underset{\sim}{\mathbf{z}}$ disappears.

Let $\mathbf{H}^* \subseteq \mathbf{P}$ be generic over V with $\mathbf{H} \subseteq \mathbf{H}^*$ and $\mathbf{p} \in \mathbf{H}^*$. So $\mathbf{H} = \mathbf{H}^* \restriction \mathcal{A}^{q_2 \restriction \beta_j}$ and set $\mathbf{H}_1 = \mathbf{H}^* \restriction \mathcal{A}^{q_1'}$, and $\mathbf{H}_3 = \mathbf{H}^* \restriction \mathcal{A}^{q_3'}$. In $V[\mathbf{H}]$ we define:

$$\underset{\sim}{A}{}_n^1 =: \{(x, \mathbf{u})\colon \text{ For some } \mathbf{p}_1 \in \mathbf{P}\restriction \mathcal{A}^{r'}, \text{ with } \mathbf{p}_1 \geq \mathbf{p}\restriction \mathcal{A}^{r'} \text{ and } \mathbf{p}_1 \restriction \mathcal{A}^{q_2 \restriction \beta_j} \in \mathbf{H},$$
$$\mathbf{p}_1 \text{ forces: } ``n \in \underset{\sim}{a},\ \underset{\sim}{x}(n) = x,\ \underset{\sim}{f}'(\underset{\sim}{\mathbf{y}})(n) = \mathbf{u}.\}$$

$$\underset{\sim}{A}{}_n^2 =: \{(x^*, \mathbf{u})\colon \text{ For some } \mathbf{p}_2 \in \mathbf{P}\restriction \mathcal{A}^{q_2 \restriction \beta_2} \text{ with } \mathbf{p}_2 \geq \mathbf{p}\restriction (\mathcal{A}_2 \cap \beta_i)$$
$$\text{and } \mathbf{p}_2 \restriction (\mathcal{A}_2 \cap \beta_j) \in \mathbf{H},$$
$$\mathbf{p}_2 \text{ forces: } ``n \in \underset{\sim}{b},\ \underset{\sim}{x}_{\beta_j}(n) = x^*, \text{ and } \underset{\sim}{\varphi}(x^*, \mathbf{u}).\text{''}\}$$

In $V[\mathbf{H}]$ there is no n satisfying:

$$(*_n) \qquad (\exists x, x^*, \mathbf{u})[(x, \mathbf{u}) \in \underset{\sim}{A}{}_n^1 \ \&\ (x^*, \mathbf{u}) \in \underset{\sim}{A}{}_n^2 \ \&\ x < x^*.]$$

Otherwise we could extend $\mathbf{p}$ by amalgamating suitable conditions $\mathbf{p}_1$, $\mathbf{p}_2$, to force such an n into $\underset{\sim}{a} \cap \underset{\sim}{b} \cap \underset{\sim}{c} \cap \underset{\sim}{d}$.

For $n < \omega$ and $\mathbf{u} \in T_n^1$ let

$$\underset{\sim}{A}{}_n^2(\mathbf{u}) =: \{x \in T_n^1\colon (x, \mathbf{u}) \in A_n^2\}$$
$$\underset{\sim}{A}{}_n^3(\mathbf{u}) =: \{x \in T_n^1\colon \text{Either } (x, \mathbf{u}) \in \underset{\sim}{A}{}_n^2 \text{ or there is no } x' \text{ above } x$$
$$\text{in } T_n^1 \text{ for which } (x', \mathbf{u}) \in \underset{\sim}{A}{}_n^2\}.$$

Then $\underset{\sim}{A}{}_n^3(\mathbf{u})$ is dense in T_n^1, and hence $\underset{\sim}{A}{}^3 =: \prod \underset{\sim}{A}{}_n^3/\underset{\sim}{\mathcal{F}}^{q_2 \restriction \beta_j}[\mathbf{H}]$ is a dense subset of $(\prod_n T_n^1/\underset{\sim}{\mathcal{F}}^{q_2 \restriction \beta_i})^{V[\mathbf{P}\restriction \mathcal{A}^{q_2 \restriction \beta_j}]}$.

Let $\mathcal{T} = (T^1, T^2; A^2, A^3)$ be the ultraproduct

$$(\prod_n (T^1_n, T^2_n; \underset{\sim}{A}{}^2_n, \underset{\sim}{A}{}^3_n)/\underset{\sim}{\mathcal{F}}{}^{q'_3})^{V[\mathbf{H}_3]}.$$

Now $\underset{\sim}{\varphi}[\underset{\sim}{x}_{\beta_j}, \underset{\sim}{\mathbf{y}}]$ holds in $\prod_n {}^{\mathcal{A}_2 \cap \beta_i}(T^1_n, T^2_n)/\mathcal{F}^{q'_3}[\mathbf{H}_3]$, so $\underset{\sim}{x}_{\beta_j}[\mathbf{H}_3] \in A^2(\underset{\sim}{\mathbf{y}}[\mathbf{H}_3])$ (using Łoś' theorem to keep track of the meaning of A^2 in this model). By the choice of $\underset{\sim}{B}$, $\underset{\sim}{B}[\mathbf{H}_1]$ meets $A^3(\underset{\sim}{\mathbf{y}}[\mathbf{H}_1])$ (as the later is dense) and indeed:

(1) $$A^3(\underset{\sim}{\mathbf{y}}[\mathbf{H}_1]) \cap \underset{\sim}{B}[\mathbf{H}_1] \text{ is unbounded in } \underset{\sim}{B}[\mathbf{H}_1].$$

For $\underset{\sim}{z} \in A^3(\underset{\sim}{\mathbf{y}}[\mathbf{H}_1]) \cap \underset{\sim}{B}[\mathbf{H}_1]$, as $\underset{\sim}{z} < \underset{\sim}{x}_{\beta_j}$ we have also $\underset{\sim}{z} \in A^2(\underset{\sim}{\mathbf{y}}[\mathbf{H}_1]) \cap \underset{\sim}{B}[\mathbf{H}_1]$. Hence in $V[\mathbf{H}_1]$ we have:

(2) $$A^2[\underset{\sim}{\mathbf{y}}] \cap \underset{\sim}{B}[\mathbf{H}_1] \text{ is unbounded in } \underset{\sim}{B}[\mathbf{H}_1].$$

Hence $A^2(\underset{\sim}{f}'(\underset{\sim}{\mathbf{y}})) \cap \underset{\sim}{f}'[\underset{\sim}{B}][\mathbf{H}^* \restriction \mathcal{A}^{r'}]$ is unbounded in $\underset{\sim}{f}'[\underset{\sim}{B}][\mathbf{H}^* \restriction \mathcal{A}^{r'}]$, and we can find $\underset{\sim}{z} \in A^2(\underset{\sim}{f}'(\underset{\sim}{\mathbf{y}}[\mathbf{H}_3])) \cap \underset{\sim}{f}'[\underset{\sim}{B}][\mathbf{H}^* \restriction \mathcal{A}^{r'}]$ with $\underset{\sim}{x} < \underset{\sim}{z}$ (all in the ultraproduct $\prod_n T^1_n/\mathcal{F}^{r'}[\mathbf{H}^* \restriction \mathcal{A}^{r'}]$ as f' is an elementary embedding). In particular for some $n \in \underset{\sim}{a}[\mathbf{H}^*]$, we have $\underset{\sim}{x}(n)[\mathbf{H}^*] < \underset{\sim}{z}(n)[\mathbf{H}^*]$ in T^1_n and $\underset{\sim}{z}(n) \in A^2(\underset{\sim}{\mathbf{y}}(n))$. Letting $x = \underset{\sim}{x}(n)[\mathbf{H}_1]$, $x^* = \underset{\sim}{z}(n)[\mathbf{H}_1]$, and $u = \underset{\sim}{f}'(\underset{\sim}{\mathbf{y}})(n)[\mathbf{H} \restriction \mathcal{A}^{r'}]$, we find that $(*_n)$ holds in $V[\mathbf{H}]$, a contradiction. $\blacksquare_{1.14}$

1.15 Weak definability.

Proposition: *Let $\delta < \aleph_3$ be an ordinal of cofinality $\aleph_2$ satisfying conditions 1.13 (a-d). Suppose $q_1, q_2 \in G$, $q_2 \restriction \delta = q_0 \le q_1$, $\mathcal{A}^{q_1} \subseteq \delta$, $\delta \in \mathcal{A}^{q_2}$, $\underset{\sim}{y}^*$ is a $\mathbf{P} \restriction \mathcal{A}^{q_2}$-name of an element of $\prod_n T^2_n$, and $\varepsilon^{q_2}_\delta = 1$. Suppose further that $\underset{\sim}{x}'$, $\underset{\sim}{x}''$ and $\underset{\sim}{y}'$, $\underset{\sim}{y}''$ are $\mathbf{P} \restriction \mathcal{A}^{q_1}$-names, $\mathbf{p} \in \mathbf{P}$, $\mathbf{p}_i = \mathbf{p} \restriction \mathcal{A}^{q_i}$ $(i = 1, 2)$, and:*

$\mathbf{p}_2 \Vdash$ "$\underset{\sim}{F}(\underset{\sim}{x}_\delta) = \underset{\sim}{y}^*$"

$\mathbf{p}_1 \Vdash$ "$\underset{\sim}{x}', \underset{\sim}{x}'' \in \prod_n T^1_n$, *and* $\underset{\sim}{y}', \underset{\sim}{y}'' \in \prod_n T^2_n$";

$\mathbf{p}_1 \Vdash$ "*The types of* $(\underset{\sim}{x}', \underset{\sim}{y}')$ *and of* $(\underset{\sim}{x}'', \underset{\sim}{y}'')$ *over* $\{\underset{\sim}{x}/\underset{\sim}{\mathcal{F}} : \underset{\sim}{x}$ *a* $\mathbf{P} \restriction \mathcal{A}^{q_0}$-*name of a member of* $\prod_n {}^{\mathcal{A}^{q_0}}(T^1_n, T^2_n)\}$ *in the model* $(\prod_n {}^{\mathcal{A}^{q_0}}(T^0_n, T^1_n)/\underset{\sim}{\mathcal{F}}{}^{q_1})^{V[\mathbf{P} \restriction \mathcal{A}^{q_1}]}$ *are equal.*"

Then the following are equivalent.

1. *There is* $r^0 \in \mathrm{App}$ *such that* $q_1, q_2 \le r^0$, $r^0 \restriction \delta \in G^\delta$, *and*

$$\mathbf{p} \Vdash \text{“}\prod_n T_n^1/\underset{\sim}{\mathcal{F}}^{r^0} \models (\underset{\sim}{x}'/\underset{\sim}{\mathcal{F}}^{r^0} < \underset{\sim}{x}_\delta/\underset{\sim}{\mathcal{F}}^{r^0}) \text{ and}$$
$$\prod_n T_n^2/\underset{\sim}{\mathcal{F}}^{r^0} \models (\underset{\sim}{y}'/\underset{\sim}{\mathcal{F}}^{r^0} < \underset{\sim}{y}^*/\underset{\sim}{\mathcal{F}}^{r^0})\text{”};$$

2. *There is* $r^1 \in \mathrm{App}$ *such that* $q_1, q_2 \le r^1$, $r^1 \restriction \delta \in G^\delta$ *and*

$$\mathbf{p} \Vdash \text{“}\prod_n T_n^1/\underset{\sim}{\mathcal{F}}^{r^1} \models (\underset{\sim}{x}''/\underset{\sim}{\mathcal{F}}^{r^1} < \underset{\sim}{x}_\delta/\underset{\sim}{\mathcal{F}}^{r^1}) \text{ and}$$
$$\prod_n T_n^2/\underset{\sim}{\mathcal{F}}^{r^1} \models (\underset{\sim}{y}''/\underset{\sim}{\mathcal{F}}^{r^1} < \underset{\sim}{y}^*/\underset{\sim}{\mathcal{F}}^{r^1}).\text{”}$$

Proof: By symmetry it suffices to show that (1) implies (2). Take $\mathbf{H}^\delta \subseteq \mathbf{P}\restriction\delta$ generic over V with $\mathbf{p}_1 \in \mathbf{H}^\delta$, and suppose that r^0 is as in (1). Let $r_0 = r^0\restriction\delta$ and let $\underset{\sim}{f}_0$ be the extension of the identity map on $(\prod T_n^1)^{V[\mathbf{P}\restriction\mathcal{A}]^{q_0}}$ by: $\underset{\sim}{f}_0(\underset{\sim}{x}') = \underset{\sim}{x}''$, $\underset{\sim}{f}_0(\underset{\sim}{y}') = \underset{\sim}{y}''$. Writing $\beta_0 = \delta$ and taking q_3 provided by 1.9(4), we recover the assumptions of 1.14, which produces a certain r in App, an end extension of $r^0\restriction\delta$; here we may easily keep $r\restriction\delta \in G^\delta$ (cf. 1.11). It suffices to take $r^1 = r$. $\blacksquare_{1.15}$

1.16 Definability. We claim now that $\underset{\sim}{F}$ is definable on a cone by a first order formula. For a stationary set S_0 of $\delta < \aleph_3$ of cofinality $\aleph_2$, we will have conditions (a-d) of 1.13 which may be expressed as follows:

Both $\underset{\sim}{F}\restriction(\mathbf{P}\restriction\delta - \text{names})$ and $\underset{\sim}{F}^{-1}\restriction(\mathbf{P}\restriction\delta - \text{names})$ are $\mathbf{P}\restriction\delta$-names;

When working with $\diamondsuit_S$:

$\diamondsuit_S$ guessed the names of these two restrictions and also guessed $\mathbf{p}^*$ correctly;

and hence for suitable $\underset{\sim}{y}_\delta$ and q_δ^* we have the corresponding conditions $(*)_{\underset{\sim}{y}_\delta}$ and $(\dagger)_\delta$ (with q_δ^* in place of q^*). By Fodor's lemma, on a stationary set $S_1 \subseteq S_0$ we have $q_0 = q_\delta^*\restriction\delta$ is constant, and also the isomorphism type of the pair $(q_\delta^*, \underset{\sim}{y}_\delta)$ over $\mathcal{A}^{q_0}$ is constant.

So for δ in S_1, we have the following two properties, holding for $\underset{\sim}{x}'$ in $V[\mathbf{P}\restriction\delta]$ and $\underset{\sim}{y}' = \underset{\sim}{F}(\underset{\sim}{x}')$), by $(\dagger)_\delta$ and 1.15 respectively:

1. The decision to put $\underset{\sim}{x}'$ below $\underset{\sim}{x}_\delta$ implies that $\underset{\sim}{y}'$ must be put below $\underset{\sim}{y}^*$; and

2. This decision is determined by the type of the pair $(\underset{\sim}{x}', \underset{\sim}{y}')$ in $\prod {}^{\mathcal{A}^{q_0}}(T^1_n, T^2_n)/\underset{\sim}{\mathcal{F}}^{V[\mathbf{H}][\mathbf{P}\restriction\delta/\mathbf{H}]}$.

As S_1 is unbounded below $\aleph_3$ this holds generally.

This gives a definition by types of the isomorphism $\underset{\sim}{F}$ above the branch in $\prod T^1_n/\underset{\sim}{\mathcal{F}}^{V[\mathbf{P}\restriction\mathcal{A}^{q_0}]}$ which the condition q^*_δ says that the vertex $\underset{\sim}{x}_\delta$ induces there (using 1.9(2)), and this branch does not depend on δ. Note that this set contains a cone, and the image of this cone is a cone in the image. Now by $\aleph_2$-compactness equivalently $\aleph_2$-saturation, of $\prod_n {}^{\mathcal{A}^{q_0}}(T^1_n, T^2_n)/\underset{\sim}{\mathcal{F}}^{V[\mathbf{P}\restriction\mathcal{A}^{q_0}]}$ we get a first order definition on a smaller cone; this last step is written out in detail in Lemma 1.17 below. This proves Proposition B.

1.17 Lemma (true definability): *Let M be a λ-saturated structure, and $A \subseteq M$ with $|A| < \lambda$. Let $(D_1; <_1)$, $(D_2; <_2)$ be A-definable trees in M; that is, the partial orderings $<_i$ are linear below each node. Assume that every node of D_1 or D_2 has at least two immediate successors. Let F: $D_1 \longrightarrow D_2$ be a tree isomorphism which is A-type-definable in the following sense:*

$$[f(x) = y \,\&\, \operatorname{tp}((x,y),A) = \operatorname{tp}((x',y'),A)] \Longrightarrow f(x') = y'.$$

Then f is A-definable (i.e. by a first order formula with parameters from A), on some cone of D_1.

Note: We do not require a relation eq meaning equality of level exists. Before entering into the proof, we note that we use somewhat less information about F (and its domain and range) than is actually assumed; and this would be useful in working out the most general form of results of this type (which will apply to some extent in any unsuperstable situation). We intend to develop this further elsewhere†, as it would be too cumbersome for our present purpose. Note that this fits well with the framework of [Sh72], [Sh107]—also there there is a lemma saying every type definable object of a specific kind in a quite saturated model is definable. See more in [Sh384].

The proof may be summarized as follows. If a function F is definable by types in a somewhat saturated model, then on the locus of each 1-type, it agrees with the restriction of a definable function. If F is an automorphism and the locus of some 1-type separates the points in a definable set C in an appropriate

† See [Sh384], [Sh482]

sense, then F can be recovered, definably, on C. Finally, in sufficiently saturated trees of the type under consideration, some 1-type separates the points of a cone. Details follow.

Proof: If we replace M by a λ-saturated elementary extension, the definition of F by types continues to work (and the extension is an elementary extension for the expansion by F). In particular, replacing $|M|$ by a more saturated structure, if necessary, but keeping A fixed, we may suppose that $\lambda > |T|, |A|, \aleph_0$.

We show first:

(1) There is a 1-type p defined over A such that its set of realizations $p[D_1]$ is dense in a cone of D_1,

i.e., for some a in D_1 we have that any element above a lies below a realization of p. Why? For any 1-type p over A, if $p[D_1]$ does not contain a cone of D_1 then by saturation there is some $\varphi \in p$ with:

$$\forall a \exists b > a \ \neg \exists x > b \ \varphi(x)$$

So if (1) fails we may choose one such formula φ_p for each 1-type p over A, and then it is consistent (hence true) that we have a wellordered increasing sequence a_p (in the tree ordering) such that for each 1-type p, above a_p we have:

$$\neg \exists x > a_p \, \varphi_p(x)$$

By saturation there is a further element a above all a_p (either by further increasing λ or by paying attention to what we are actually doing) and we have arranged that there is no 1-type left for it to realize. As this is impossible, (1) holds. We fix a 1-type p (which is a complete type over A) and an element a_0 in D_1 so that the realizations of p are dense in the cone above a_0. It is important to note at this point that the density implies that any two distinct vertices above a_0 are separated by the realizations of p in the sense that there is a realization of p lying above one but not the other (here we use the immediate splitting condition we have assumed in the tree D_1).

Let a realize the type p, and let q be the type of the pair $(a, F(a))$ over A. If b is any other realization of p, then there is an element c with the pair (b, c) realizing q, and hence $F(b) = c$ so in particular $F(a)$ is definable over $A \cup \{a\}$ and by the assumption of the lemma p determines q uniquely. So each realization a of p determines a unique element d such that the pair (a, d) realizes q, and by saturation there is a formula $\varphi(x,y) \in q$ so that $\varphi(x,y) \Longrightarrow \exists! z\, \varphi(x,z)$. Hence $p \cup \{\varphi\} \vdash q$.

Now the following holds in M:

$$p(x) \cup p(x') \cup \{\varphi(x,y), \varphi(x',y')\} \Longrightarrow (x < x' \leftrightarrow y < y')$$

and hence for some formula $\psi(x) \in p$ the same holds with p replaced by ψ. Increasing φ we may suppose $\varphi(x,y) \Longrightarrow \psi(x)$ and conclude that $\varphi(x,y)$ defines a partial isomorphism f. Let B be $\{a > a_0 : \exists y \varphi(a,y)\}$. Now f coincides with F on the set of realizations of p above a, and the action of F on this set determines its action on the cone above a by density (or really by the separation condition mentioned above), so f coincides with F on B. Furthermore the action of F on B determines its action on the cone above a_0 definably, so F is definable above a.

The definition $\varphi^*(x,y)$ of F on the cone above a obtained in this manner may easily be written down explicitly:

$$\text{“}\forall x', y' \ [\varphi(x',y') \Longrightarrow (x < x' \leftrightarrow y < y')]\text{”} \qquad \blacksquare_{1.17}$$

For the application in 1.15 we take $\lambda = \aleph_2$. $\blacksquare_{1.16}$

Here we have finished proving the main theorem 1.2 and proposition A from the Introduction. $\blacksquare_{1.2}$

1.18 Proposition: **P** *forces: In* $\prod_n T_n^1/\underset{\sim}{\mathcal{F}}$ $(\underset{\sim}{\mathcal{F}} = \underset{\sim}{\mathcal{F}}[G^{\aleph_3}])$, *every full branch is an ultraproduct of branches in the original trees* T_n^1.

First Proof (in brief): Following the line of the previous argument we argue as follows: If $\underset{\sim}{B}$ is a **P**-name for such a branch, then for a stationary set of ordinals $\delta < \aleph_3$ of cofinality $\aleph_2$, $\underset{\sim}{B} \cap (\prod_n T_n^1/\underset{\sim}{\mathcal{F}})^{V[\mathbf{P}\restriction\delta]}$ will be a full branch and a $\mathbf{P}\restriction\delta$-name, guessed correctly by $\diamondsuit_S$. We tried to make a commitment to terminate this branch, but failed, and hence for some q^* and y^*, witnessing to the failure, we were unable to omit having $q^*\restriction\delta \in G^\delta$ where q^* is essentially the support of "y^* is a bound". Using 1.13 one shows that the branch was definable at this point by types in $\aleph_1$ parameters, and by $\aleph_2$-compactness we get a first order definition, which by Fodor's lemma can be made independent of δ. $\blacksquare_{1.18}$

Filling in the details in the foregoing argument constitutes an excellent, morally uplifting exercise for the reader. However the more pragmatic reader may prefer the following dull derivation of the proposition from Proposition B.

Second Proof: We can derive the result from Proposition B. In the first place, we may replace the trees T_n^1 in the proposition above by the universal tree of this type, which we take to be $T = \mathbb{Z}^{<\omega}$ (writing $\mathbb{Z}$ rather than ω for the sake of the notation used below). Now apply Proposition B to the pair of sequences (T_n^1), (T_n^2) in which $T_n^i = T$ for all i, n. Using the model of ZFC and the ultrafilter referred to in Proposition B, suppose B is a full branch of $T^* = \prod T_n^2/\mathcal{F}$ (in $V[G^{\aleph_3}]$), and let $\mathbb{Z}^* = \mathbb{Z}^\omega/\mathcal{F}$, $\mathbb{N}^* = \mathbb{N}^\omega/\mathcal{F}$. For each $i \in \mathbb{N}^*$ let B_i be the i-th node of B; this is a sequence in $(\mathbb{Z}^*)^{[0,i)}$ which is coded in $\mathbb{N}^*$. Define an automorphism f_B of T^* whose action on the i-th level is via addition of B_i (pointwise addition of sequences). Applying Proposition B and Łoś' theorem to this automorphism, we see that f_B is the ultraproduct of addition maps corresponding to various branches of T, and that B is the ultraproduct of these branches. $\blacksquare_{1.18}$

1.19 Corollary: *It is consistent with ZFC that $\mathbb{R}^\omega/\mathcal{F}$ is Scott-complete for some ultrafilter $\mathcal{F}$.*

Here $\mathbb{R}^\omega/\mathcal{F}$ is called **Scott-complete** if it has no proper dedekind cut (A, B) in which $\inf\{b - a : a \in A, b \in B\}$ is 0 in $\mathbb{R}^\omega/\mathcal{F}$. Now 1.18 is sufficient for this by Keisler Schmere [KeSc, Prop. 1.3]. This corollary shows that a positive answer to Question 4.3 of [KeSc, p. 1024] is relatively consistent with ZFC.

1.20 Remark: In the proof of 1.2 the predicate "at the same level" may be omitted from the language of the trees T_n^i throughout as the condition on $\underset{\sim}{x}_\delta$ that uses this (the "full branch" condition) follows from the "bigness" condition: meeting every suitable dense subset.

2. Cuts in models of Peano arithmetic

2.1 Introduction. We refer to a proper Dedekind cut (A, B) in a linear order as a **gap**. We refer to the cofinality of A and the coinitiality of B as the left and right cofinalities of the gap, respectively. For results provable in ZFC see [Sh-a, Sh-c, VI 3.12 p. 357]; for example, in $\mathbb{N}^\omega/\mathcal{F}$, $\mathcal{F}$ in ultrafilter on ω, if we take $A = \mathbb{N}$ ($\subseteq \mathbb{N}^\omega/\mathcal{F}$) and B its complement then any regular cardinal in the interval $(\aleph_0, 2^{\aleph_0}]$ can be the right cofinality of this cut. In general the possible values of these cofinalities in ultrapowers of the linearly ordered set $\mathbb{N}$, or other reduced products, depend heavily on the set-theoretic background. (See [DW] for background information.) However we show here by a simple argument:

2.2 THEOREM: *Let $\mathcal{N}$ be a nonstandard model of Peano arithmetic. Then there is a gap in $\mathcal{N}$ whose left and right cofinalities are equal.*

As a corollary, any $\aleph_1$-saturated elementary extension of $\mathbf{N}$, and in particular any ultrapower $\mathcal{N}^I/\mathcal{F}$ with respect to an ω-incomplete ultrafilter, has a gap whose left and right cofinalities are both uncountable. This answers a question posed in a slightly different formulation in [LLS] (and, as we have lately learnd, by Renling Jim), which we review in 2.5 below.

2.3 CONSTRUCTION. We will write $\exp x$ for x^x.

We will construct elements $a_{\alpha,n}$, $b_{\alpha,n}$ in $\mathcal{N}$ for $n < \omega$ and $\alpha < \gamma_0$ for some limit ordinal γ_0, such that for all n and for all $\alpha < \beta < \gamma_0$:

$$a_{\alpha,n} < a_{\beta,n} < b_{\beta,n} < b_{\alpha,n}; \tag{1}$$

$$\exp(b_{\alpha,n+1}) < a_{\alpha+1,n} - a_{\alpha,n}. \tag{2}$$

The construction is by induction on limit ordinals γ. At each stage we construct all of the elements $a_{\alpha,n}$ and $b_{\alpha,n}$ for $\alpha < \gamma$, as long as this is possible.

To initiate the construction, with $\gamma = \omega$, we first choose infinite elements $d_n \in \mathcal{N}$ for n finite such that for all n we have $\exp d_{n+1} << d_n$, where we write $x << y$ if $kx < y$ for all finite k. We let $a_{i,n} = d_{n+1} + i \cdot \exp(d_{n+1})$ and $b_{i,n} = d_n - i - 1$. In particular $a_{i,n} < [d_n/2] < b_{i,n}$ for i, n finite.

2.4 THE INDUCTIVE STEP. Now suppose the elements $a_{\alpha,n}$ and $b_{\alpha,n}$ have been chosen for $\alpha < \gamma$ with γ a limit ordinal. Let A_n, B_n be the ranges of the sequences $a_{\alpha,n}$, $b_{\alpha,n}$ (for $\alpha < \gamma$) respectively. If one of the pairs (A_n, B_n) determines a gap in $\mathcal{N}$, then it is the desired gap (i.e. the gap $(\{x\colon (\exists y \in A_n)(x < y)\}, \{x\colon (\exists y \in B_n)(y < x)\})$. Assume therefore:

For all n there is an element c_n with $A_n < c_n < B_n$

i. e. $(\forall x \in A_n)(\forall y \in B_n)[x < c_n < y]$.

Under this assumption we will continue the construction by defining $a_{\gamma+i,n}$ and $b_{\gamma+i,n}$ for all finite i, n.

We set $c'_n = c_n - \exp c_{n+1}$ and we observe that $A_n < c'_n$ (i. e. $(\forall x \in A_n)(x < c'_n)$) since:

$$a_{\alpha,n} < a_{\alpha+1,n} - \exp(b_{\alpha,n+1}) < a_{\alpha+1,n} - \exp c_{n+1} < c_n - \exp c_{n+1} = c'_n \text{ for } \alpha < \gamma.$$

We set (for $i, n < \omega$):

$$a_{\gamma+i,n} =: c'_n + i \cdot \exp(c_{n+1} - 1);$$

$$b_{\gamma+i,n} =: c'_n + c_{n+1} \cdot \exp(c_{n+1} - 1) - i.$$

Condition (1) clearly remains valid: $a_{\gamma+i,n}$ increase with i [by its definition], for $\alpha < \gamma$, $a_{\alpha,n} < a_{\gamma+i,n}$ [as $a_{\alpha,n} \in A_n$ hence by a previous statement $a_{\alpha,n} < c'_n$ and trivially $c'_n < a_{\gamma+i,n}$], also $a_{\gamma+i,n} < b_{\gamma+i,n}$ [as c_{n+1} is nonstandart], $b_{\gamma+i,n}$ decrease with i [check definition] and for $\alpha < \gamma$, $b_{\gamma+i,n} < b_{\alpha,n}$ [as $c_{n+1} \cdot \exp(c_{n+1} - 1) < \exp c_{n+1}$ (by the definition of exp) so by the definition of c'_n, $b_{\gamma+i,n}$ we have $b_{\gamma+i,n} < c'_n + \exp c_{n+1} = c_n$ but by the choice of c_n, we have $c_n < b_{\alpha,n}$]. Furthermore since $c_{n+1} \cdot \exp(c_{n+1} - 1) < \exp c_{n+1} - 1$ we have $b_{\alpha,n} < c'_n + \exp c_{n+1} - 1 = c_n - 1$ (for $\alpha = \gamma + i$ for $i < \omega$), hence $\exp(b_{\alpha,n+1}) < \exp(c_{n+1} - 1) = a_{\alpha+1,n} - a_{\alpha,n}$ and this yields condition (2). $\blacksquare_{2.2}$

2.5 Discussion: by G. Cherlin. We recall briefly the way the question was posed in [LLS]. Let $\mathcal{Z} = \mathbb{Z}^\omega/\mathcal{F}$ be an ultrapower of the ring of integers. Each prime ideal lies below a unique maximal ideal in this ring, and the set of prime ideals below a given maximal ideal is linearly ordered under inclusion. In [LLS] the question is posed, whether in such a ring the following holds for every maximal ideal $\mathfrak{m}$:

> There is a prime ideal below $\mathfrak{m}$ which is neither a union nor an intersection of countably many principal ideals.

It was shown above that this is true, and now we want to make this more explicit. This requires two steps. The analysis is simplest in the case in which $\mathfrak{m}$ is principal, and the general case will reduce to this one. The background for what follows is given in [Ch].

Suppose first that $\mathfrak{m}$ is principal. Then each prime ideal $\mathfrak{p}$ below $\mathfrak{m}$ has a representation as $\mathfrak{p} = \mathfrak{m}^{\mathfrak{I}}$ where $\mathfrak{I}$ is an initial segment of $\mathcal{N} =: \mathbf{N}^\omega/\mathcal{F}$ and $\mathfrak{m}^{\mathfrak{I}} = \bigcap_{n \in \mathfrak{I}} \mathfrak{m}^n$. Here $\mathfrak{I}$ must be closed under addition, or equivalently under multiplication by 2, and conversely for $\mathfrak{I}$ additively closed, $\mathfrak{m}^{\mathfrak{I}}$ is prime. We associate to $\mathfrak{I}$ the initial segment $\log \mathfrak{I} =: \{n \in \mathcal{N} : 2^n \in \mathfrak{I}\}$ and we find that $\mathfrak{I}$ is additively closed if and only if $\log \mathfrak{I}$ is closed under addition of 1, or in other words $\log \mathfrak{I}$ is the left half of a gap in $\mathcal{N}$. Conversely a gap $(\bar{J}_0, \bar{J}_1)$ in $\mathcal{N}$ corresponds to an additively closed initial segment $\bar{\mathfrak{I}} = \{n\colon (\exists m \in I_0)[n \leq 2^m]\}$

and hence to a prime ideal below $\mathfrak{m}$. Furthermore this correspondence (is one to one into and) preserves left and right cofinalities. So the result proved above shows that in an ω_1-saturated model, our claim holds below a principal maximal ideal.

If $\mathfrak{m}$ is nonprincipal it is necessary to use more machinery. The details of this machinery, which involves a reduction of general ideals to principal ideals by passage to a definable ultrapower of $\mathcal{N}$, are given in [Ch, §4]. What interests us here is the following: the prime ideals below $\mathfrak{m}$ are again classified by gaps in an order, but the order is not the order on $\mathcal{N}$; rather it is the order on a definable ultrapower $\mathcal{N}^*$ of $\mathcal{N}$ taken with respect to a *bounded* ultrafilter on the definable sets of $\mathcal{N}$ [Ch, Theorems 4.5 and 4.8]. By "bounded" we simply mean that the ultrafilter contains some bounded set.

To conclude, it suffices to prove that the model $\mathcal{N}^*$ is again ω_1-dense. This follows from Lemma 2.1.1 of [Ri]; in [Ri] it is also shown that the ω_1-density condition implies ω_1-saturation in models of Peano arithmetic. For the reader's convenience we give a self-contained proof of the density condition.

2.6 Proposition: *Let $\mathcal{N}$ be an ω_1-saturated model of PA, and let $\mathcal{F}$ be an ultrafilter on the (Boolean algebra of the) definable subsets of $\mathcal{N}$ that contains the bounded definable set A. Then the definable ultrapower $\mathcal{N}^* := \mathrm{Def}(\mathcal{N})/\mathcal{F}$ is ω_1-dense.*

Proof: We take elements m_i, n_i in $\mathcal{N}^*$ with $m_1 \leq m_2 \leq \cdots \leq n_2 \leq n_1$. These elements are represented by definable functions f_i, g_i in $\mathcal{N}$, and actually it suffices to take the restrictions of f_i, g_i to A, which are coded by elements of $\mathcal{N}$. By the saturation hypothesis, there are sequences of functions of length K with $K \in \mathcal{N}$ infinite, which extend the given two sequences and are again coded in $\mathcal{N}$. (We have now verified the hypothesis of [Ri, Lemma 2.1.1], and could therefore stop at this point.) So we may speak of f_i and g_i for $i \leq K$, as functions defined on A.

For $x \in A$ let $i(x)$ be the largest $i \leq K$ such that:

$$f_1(x) \leq f_2(x) \leq \cdots \leq f_i(x) \leq g_i(x) \leq \cdots \leq g_2(x) \leq g_1(x).$$

Observe that for i finite, $\{x : i(x) \geq i\} \in \mathcal{F}$. We may also suppose that $i(x) \geq 1$ on A.

Set $f(x) = f_{i(x)}(x)$ for $x \in A$ and observe that this definition makes sense in $\mathcal{N}$. Accordingly f represents an element m of $\mathcal{N}^*$, and by the construction $m_i \leq m \leq n_i$ for all finite i. $\blacksquare_{2.6}$

$\blacksquare_{2.5}$

Appendix

Omitting types. This appendix bears only on the version of §1 that depends on the ideas of [ShHL162]. On the one hand, we wish to recall explicitly what those ideas are. On the other hand, we will propose a variant of the formalism of [ShHL162] more suitable for the present purpose. All in all we consider three variants for framework in A1, A6 and inside A11.

In the context of this paper, the formalism of [ShHl162] is intended to provide a combinatorial refinement of forcing with App, which gives a $\mathcal{P}_3$-name $\underset{\sim}{\mathcal{F}}$ in suitalbe ground model for an ultrafilter which will have the desired properties in a $\mathcal{P}_3$-generic extension. We now review this material. Our discussion complements the discussion in [Sh326], which focussed more on filling the gap between the intuitive notion of "sufficiently generic" and the formalism given in [ShHL162]. Here the focus of our discussion is more technical: we discuss the replacement of the continuity axiom of [ShHL162] by a more flexible setup. For the reader who wants to understand how to apply the method and is not familiar with [ShHL162] the discussion in the appendix to [Sh326] should be more useful than the present discussion.

In sections A1-A5 we are presenting the material of [ShHL162] as it was summarized in [Sh326]. An alternative setup is presented in sections A6-A10. The axioms given in section A6 below should supercede the axioms given in section A1, and one would check that the proofs of [ShHL162] work with these new axioms. For completeness we give a proof under somewhat weaker set theoretic condition which applies in the case of §1.

A1 Uniform partial orders. We review the formalism of [ShHL162].

With the cardinal λ fixed, a partially ordered set $(\mathcal{P}, <)$ is said to be *standard λ^+-uniform* if $\mathcal{P} \subseteq \lambda \times P_\lambda(\lambda^+)$ (we refer here to subsets of λ^+ of size strictly less than λ), has the following properties (if $p = (\alpha, u)$ we write $\operatorname{dom} p$ for u, and we write $\mathcal{P}_\beta$ for $\{p \in \mathcal{P}\colon \operatorname{dom} p \subseteq \beta\}$):

1. If $p \leq q$ then $\operatorname{dom} p \subseteq \operatorname{dom} q$.

2. For all $p \in \mathcal{P}$ and $\alpha < \lambda^+$ there exists a $q \in \mathcal{P}$ with $q \leq p$ and $\operatorname{dom} q = \operatorname{dom} p \cap \alpha$; furthermore, there is a unique maximal such q, for which we write $q = p \restriction \alpha$.
3. (*Indiscernibility*) If $p = (\alpha, v) \in \mathcal{P}$ and $h : v \to v' \subseteq \lambda^+$ is an order-isomorphism onto v' then $(\alpha, v') \in \mathcal{P}$. We write $h[p] = (\alpha, h[v])$. Moreover, if $q \leq p$ then $h[q] \leq h[p]$.
4. (*Amalgamation*)† For every $p, q \in \mathcal{P}$ and $\alpha < \lambda^+$, if $p \restriction \alpha \leq q$ and $\operatorname{dom} q \subseteq \alpha$, then there exists $r \in \mathcal{P}$ so that $p, q \leq r$.
5. For all $p, q, r \in \mathcal{P}$ with $p, q \leq r$ there is $r' \in \mathcal{P}$ so that $p, q \leq r'$ and $\operatorname{dom} r' = \operatorname{dom} p \cup \operatorname{dom} q$.
6. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ, then it has a least upper bound q, with domain $\bigcup_{i<\delta} \operatorname{dom} p_i$; we will write $q = \bigcup_{i<\delta} p_i$, or more succinctly: $q = p_{<\delta}$.
7. For limit ordinals δ, $p \restriction \delta = \bigcup_{\alpha<\delta} p \restriction \alpha$.
8. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ, then $(\bigcup_{i<\delta} p_i) \restriction \alpha = \bigcup_{i<\delta} (p_i \restriction \alpha)$.

It is shown in [ShHL162] that under a diamond-like hypothesis, such partial orders admit reasonably generic objects. The precise formulation is given in A5 below.

A2 Density systems. Let $\mathcal{P}$ be a standard λ^+-uniform partial order. For $\alpha < \lambda^+$, $\mathcal{P}_\alpha$ denotes the restriction of $\mathcal{P}$ to $p \in \mathcal{P}$ with domain contained in α. A subset G of $\mathcal{P}_\alpha$ is an *admissible ideal* (of $\mathcal{P}_\alpha$) if it is closed downward, is λ-directed (i.e. has upper bounds for all small subsets), and for every p in $\mathcal{P}_\alpha \setminus G$ some $q \in G$ is incompatible with p (in $\mathcal{P}_\alpha$). For G an admissible ideal in $\mathcal{P}_\alpha$, $\mathcal{P}/G$ denotes the restriction of $\mathcal{P}$ to $\{p \in \mathcal{P} : p \restriction \alpha \in G\}$.

If G is an admissible ideal in $\mathcal{P}_\alpha$ and $\alpha < \beta < \lambda^+$, then an (α, β)-*density system* for G is a function D from pairs (u, v) in $P_\lambda(\lambda^+)$ with $u \subseteq v \in \mathcal{P}_\lambda(\lambda^+)$ into subsets of $\mathcal{P}$ with the following properties:

(i) $D(u, v)$ is an upward-closed dense subset of $\{p \in \mathcal{P}/G : \operatorname{dom} p \subseteq v \cup \beta\}$;
(ii) For pairs $(u_1, v_1), (u_2, v_2)$ in the domain of D, if $u_1 \cap \beta = u_2 \cap \beta$ and

† Actually this implies that we can weaken the demand $\operatorname{dom} q \subseteq \alpha$ to $(\operatorname{dom} p) \cap (\operatorname{dom} q) = (\operatorname{dom} p) \cap \alpha$; this holds also for the framework in A11(2) as we can find $n < w$, $\alpha_0 < \alpha_1 < \cdots < \alpha_n = \lambda^+$ from W^*_λ (see there) such that $(\operatorname{dom} p) \cap \alpha_0 \subseteq \operatorname{dom}(q) \cap \alpha_1$, and for $l \in (1, n-1)$, $(\operatorname{dom} p) \cap [\alpha_l, \alpha_{l+1}) \neq \emptyset \Leftrightarrow (\operatorname{dom} q) \cap [\alpha_l, \alpha_{l+1}) \neq \emptyset$. Not so in A6.

$v_1 \cap \beta = v_2 \cap \beta$, and there is an order isomorphism from v_1 onto v_2 carrying u_1 to u_2, then for any γ we have $(\gamma, v_1) \in D(u_1, v_1)$ iff $(\gamma, v_2) \in D(u_2, v_2)$.

An admissible ideal G' (of $\mathcal{P}_\gamma$) is said to *meet* the (α,β)-density system D for G if $\gamma \geq \alpha$, $G' \supseteq G$ and for each $u \in P_\lambda(\gamma)$ there is $v \in P_\lambda(\gamma)$ containing u such that G' meets $D(u, v)$.

A3 The genericity game. Given a standard λ^+-uniform partial order $\mathcal{P}$, the *genericity game* for $\mathcal{P}$ is a game of length λ^+ played by Guelfs and Ghibellines, with Guelfs moving first. The stages of the game are $\alpha < \lambda^+$ such that $\beta < \alpha \Rightarrow \beta' < \alpha$ (see below). The Ghibellines build an increasing sequence of admissible ideals meeting density systems set by the Guelfs. Consider stage α. If α is a successor, we write α^- for the predecessor of α; if α is a limit, we let $\alpha^- = \alpha$. Now at stage α for every $\beta < \alpha$ an admissible ideal G_β in some $\mathcal{P}_{\beta'}$ is given, and one can check that there is a unique admissible ideal G_{α^-} in $\mathcal{P}_{\alpha^-}$ containing $\bigcup_{\beta<\alpha} G_{\beta'}$ (remember A 1(5)) or [Lemma 1.3, ShHL 162]. The Guelfs now supply at most λ density systems D_i over G_{α^-} for (α, β_i) and also fix an element g_α in $\mathcal{P}/G_{\alpha^-}$. Let α' be minimal such that $g_\alpha \in \mathcal{P}_{\alpha'}$ and $\alpha' \geq \sup \beta_i$. The Ghibellines then build an admissible ideal $G_{\alpha'}$ for $\mathcal{P}_{\alpha'}$ containing G_{α^-} as well as g_α, and meeting all specified density systems, or forfeit the match; they let $G_{\alpha''} = G_{\alpha'} \cap \alpha''$ when $\alpha \leq \alpha'' < \alpha'$. The main result is that the Ghibellines can win (i.e. not forfeit at any stage) with a little combinatorial help in predicting their opponents' plans, see A4 below.

For notational simplicity, we assume that G_δ is an $\aleph_2$-generic ideal on App$\restriction\delta$, when cf$\delta = \aleph_2$, which is true on a club in any case.

A4 Dl_λ. The combinatorial principle Dl_λ states that there are subsets Q_α of the power set of α for $\alpha < \lambda$ such that $|Q_\alpha| < \lambda$, and for any $A \subseteq \lambda$ the set $\{\alpha : A \cap \alpha \in Q_\alpha\}$ is stationary. This follows from $\diamondsuit_\lambda$ or inaccessibility, obviously, and Kunen showed that for successors, Dl and $\diamondsuit$ are equivalent. In addition Dl_λ implies $\lambda^{<\lambda} = \lambda$.

A5 A general principle.

Theorem: *Assuming Dl_λ, the Ghibellines can win any standard λ^+-uniform $\mathcal{P}$-game.*

This is Theorem 1.9 of [ShHL 162].

A6 UNIFORM PARTIAL ORDERS REVISITED. We introduce a second formalism that fits the setups encountered in practice more closely. In our second version we write "quasiuniform" rather than "uniform" throughout as the axioms have been weakened slightly.

With the cardinal λ fixed, a partially ordered set $(\mathcal{P}, <)$ is said to be *standard λ^+-quasiuniform* if $\mathcal{P} \subseteq \lambda \times P_\lambda(\lambda^+)$ has the following properties (if $p = (\alpha, u)$ we write $\operatorname{dom} p$ for u, and we write $\mathcal{P}_\beta$ for $\{p \in \mathcal{P}\colon \operatorname{dom} p \subseteq \beta\}$):

1′. If $p \leq q$ then $\operatorname{dom} p \subseteq \operatorname{dom} q$.

2′. For all $p \in \mathcal{P}$ and $\alpha < \lambda^+$ there exists a $q \in \mathcal{P}$ with $q \leq p$ and $\operatorname{dom} q = \operatorname{dom} p \cap \alpha$; furthermore, there is a unique maximal such q, for which we write $q = p{\restriction}\alpha$ and then we write $q \leq_{\text{end}} p$.

3′. *(Indiscernibility)* If $p = (\alpha, v) \in \mathcal{P}$ and $h : v \to v' \subseteq \lambda^+$ is an order-isomorphism onto v' then $(\alpha, v') \in \mathcal{P}$. We write $h[p] = (\alpha, h[v])$. Moreover, if $q \leq p$ then $h[q] \leq h[p]$.

4′. *(Amalgamation)* For every $p, q \in \mathcal{P}$ and $\alpha < \lambda^+$, if $p{\restriction}\alpha \leq q$, $cf(\alpha) = \lambda$ and $\operatorname{dom} q \subseteq \alpha$, then there exists $r \in \mathcal{P}$ so that $p, q \leq r$.

5′. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ, then it has an upper bound q.

6′. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ of members of $\mathcal{P}_{\beta+1}$, with $\beta < \lambda^+$ and if $q \in \mathcal{P}_\beta$ satisfies $p_i{\restriction}\beta \leq q$ for all $i < \delta$, then $\{p_i\colon i < \delta\} \cup \{q\}$ has an upper bound r in $\mathcal{P}$ with $q = r{\restriction}\beta$.

7′. If $(\beta_i)_{i<\delta}$ is a strictly increasing sequence of length less than λ, with each $\beta_i < \lambda^+$, and $q \in \mathcal{P}$, $p_i \in \mathcal{P}_{\beta_i}$, with $q{\restriction}\beta_i \leq p_i$, then $\{p_i\colon i < \delta\} \cup \{q\}$ has an upper bound r with $p_j = r{\restriction}\beta_j$ for all $j < \delta$.

8′. Suppose δ_1, δ_2 are limit ordinals less than λ, and $(\beta_i)_{i<\zeta}$ is a strictly increasing continuous sequence of ordinals less than λ^+. Let $I(\delta_1, \delta_2) := (\delta_1 + 1) \times (\delta_2 + 1) \smallsetminus \{(\delta_1, \delta_2)\}$. Suppose that for $(i, j) \in I(\delta_1, \delta_2)$ we have $p_{ij} \in \mathcal{P}{\restriction}\beta_i$ such that

$$i \leq i' \Longrightarrow p_{ij} \leq p_{i'j}$$
$$j \leq j' \Longrightarrow p_{ij} \leq p_{ij'}{\restriction}\beta_j;$$

Then $\{p_{ij}\colon (i, j) \in I(\delta_1, \delta_2)\}$ has an upper bound r in $\mathcal{P}$ with $r{\restriction}\beta_j = p_{\delta_1, j}$ for all $j < \delta_2$.

These axioms apply in the case of the partial order App by 1.9.

A7 Remark. We can weaken the end extension requirements in the conclusions of these axioms but this does not seem useful.

A8 Density systems revisited. Let $\mathcal{P}$ be a standard λ^+-quasiuniform partial order. A subset G of $\mathcal{P}_\alpha$ is a *quasiadmissible ideal* (of $\mathcal{P}_\alpha$) if it is closed downward and is λ-directed (i.e. has upper bounds for all small subsets) and for every $p \in \mathcal{P}_\alpha \setminus G$ some $q \in G$ is incompatible with p (in $\mathcal{P}_\alpha$). For G a quasiadmissible ideal in $\mathcal{P}_\alpha$, $\mathcal{P}/G$ denotes the restriction of $\mathcal{P}$ to $\{p \in \mathcal{P}: p\restriction\alpha \in G\}$. If $\langle G_\alpha: \alpha < \beta\rangle$ is increasing, G_α quasiaddmessble ideal of $\mathbf{P}_\alpha$, then $\mathbf{P}/\bigcup_{\alpha<\beta} G_\alpha = \{p: p\restriction\alpha \in G_\alpha$ for $\alpha < \beta\}$.

If $\bar{G} = \langle G_\gamma: \gamma < \alpha\rangle$ is an increasing sequence, G_γ is a quasi-admissible ideal in $\mathcal{P}_\gamma$ and $\alpha \leq \beta < \lambda^+$, then an (α, β)-*density system* for $\bar{G}$ is a function D from sets u in $P_\lambda(\lambda^+)$ into subsets of $\mathcal{P}$ with the following properties:

(i) $D(u)$ is an upward-closed dense subset of $\mathcal{P}/\bigcup_{\gamma<\alpha} G_\gamma$;

(ii) For pairs (u_1, v_1) and (u_2, v_2) with u_1, u_2 in the domain of D, and $v_1, v_2 \in P_\lambda(\lambda^+)$ with $u_1 \subseteq v_1$, $u_2 \subseteq v_2$, if $u_1 \cap \beta = u_2 \cap \beta$ and $v_1 \cap \beta = v_2 \cap \beta$, and there is an order isomorphism from v_1 onto v_2 carrying u_1 to u_2, then for any γ we have $(\gamma, v_1) \in D(u_1)$ iff $(\gamma, v_2) \in D(u_2)$.

For $\gamma \geq \alpha$, a quasiadmissible ideal G' of $\mathcal{P}_\gamma$ is said to *meet* the (α, β)-density system D for $\bar{G}$ if ($G' \supseteq \bigcup_{\gamma<\alpha} G_\gamma$ and) for each $u \in P_\lambda(\gamma)$ G' meets $D(u)$.

A9 The genericity game revisited. Given a standard λ^+-quasiuniform partial order $\mathcal{P}$, the *genericity game* for $\mathcal{P}$ is a game of length λ^+ played by Guelfs and Ghibellines, with Guelfs moving first. The Ghibellines build an increasing sequence of quasi admissible ideals meeting density systems set by the Guelfs. Consider stage α. Now at stage α for every $\beta < \alpha$ an admissible ideal G_β in $\mathcal{P}_\beta$ is given. The Guelfs now supply at most λ density systems D_i over $\bigcup_{\beta<\alpha} G_\beta$ for (α, β_i) and also fix an element g_α in $\mathcal{P}/\bigcup_{\beta<\alpha} G_\beta$. Let α' be minimal such that $g_\alpha \in \mathcal{P}_{\alpha'}$ and $\alpha' \geq \sup \beta_i$. The Ghibellines then build an admissible ideal $G_{\alpha'}$ for $\mathcal{P}_{\alpha'}$ containing $\bigcup_{\beta<\alpha} G_\beta$ as well as g_α, and meeting all specified density systems, or forfeit the match; they let $G_{\alpha''} = G_{\alpha'} \cap \alpha''$ when $\alpha \leq \alpha'' < \alpha'$. The main result is that the Ghibellines can win with a little combinatorial help in predicting their opponents' plans.

A10 Theorem. *Assuming Dl_λ, the Ghibellines can win any standard λ^+-uniform $\mathcal{P}$-game.*

A11 CLAIM. For proving A10, (for a given λ) it is enough:

(1) to prove it for a framework as in A6 reduced to closed $u \in \mathcal{P}_\lambda(\lambda^+)$ and β's in $W'_\lambda =: \{\delta < \lambda^+ : \mathrm{cf}(\delta) = \lambda\}$ (we call this the closed λ^+-quasi uniform setting). (We call $u \subseteq \lambda^+$ closed if $0 \in u$ and $[\delta = \sup(u \cap \delta) \& \ \delta$ is a limit ordinal $\Rightarrow \delta \in u]$. We define the closure of u, $\mathrm{cl}(u)$ naturally.

(2) to prove it when the following stronger version (i.e. with stronger requirements) holds. Let $W^*_\lambda = \{\alpha < \lambda^+ : \neg[\aleph_0 \le \mathrm{cf}(\aleph) < \lambda]\}$.

A partially ordered set $(\mathcal{P}, <)$ is said to be *standard λ^+-semiuniform* if $\mathcal{P} \subseteq \lambda \times \{u : u \subseteq \lambda^+, |u| < \lambda^+, u \text{ is closed}\}$ has the following properties (if $p = (\alpha, u)$ we write $\mathrm{dom}\, p$ for u, and we write $\mathcal{P}_\beta$ for $\{p \in \mathcal{P} : \mathrm{dom}\, p \subseteq \beta\}$):

1″. If $p \le q$ then $\mathrm{dom}\, p \subseteq \mathrm{dom}\, q$.

2″. For all $p \in \mathcal{P}$ and $\alpha \in W^*_\lambda$ there exists a $q \in \mathcal{P}$ with $q \le p$ and $\mathrm{dom}\, q = \mathrm{dom}\, p \cap \alpha$; furthermore, there is a unique maximal such q, for which we write $q = p \restriction \alpha$ and then we write $q \le_{\mathrm{end}} p$.

3″. *(Indiscernibility)* If $p = (\alpha, v) \in \mathcal{P}$ and $h : v \to v' \subseteq \lambda^+$ is an order-isomorphism onto v' and v' is closed then $(\alpha, v') \in \mathcal{P}$. We write $h[p] = (\alpha, h[v])$. Moreover, if $q \le p$ then $h[q] \le h[p]$.

4″. *(Amalgamation)* For every $p, q \in \mathcal{P}$ and $\alpha < \lambda^+$, if $p \restriction \alpha \le q$, $\alpha \in W^*_\lambda$ and $\mathrm{dom}\, q \subseteq \alpha$, then there exists $r \in \mathcal{P}$ so that $p, q \le r$ and $\mathrm{Dom}\, r = (\mathrm{Dom}\, p) \cup (\mathrm{Dom}\, q)$.

5″. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ, then it has an upper bound q and $\mathrm{Dom}(q) = \mathrm{cl}(\bigcup_{i<\delta} \mathrm{Dom}\, p_i)$.

6″. If $(p_i)_{i<\delta}$ is an increasing sequence of length less than λ of members of $\mathcal{P}_{\beta+1}$, with $\beta \in W'_\lambda$ and if $q \in \mathcal{P}_\beta$ satisfies $p_i \restriction \beta \le q$ for all $i < \delta$, then $\{p_i : i < \delta\} \cup \{q\}$ has an upper bound r in $\mathcal{P}$ with $q = r \restriction \beta$ and $\mathrm{Dom}(r) = \mathrm{cl}[(\mathrm{Dom}\, q) \cup \bigcup_{i<\delta} \mathrm{Dom}\, p_i]$.

7″. If $(\beta_i)_{i<\delta}$ is a strictly increasing sequence of length less than λ, with each $\beta_i \in W^*_\lambda$, and $q \in \mathcal{P}$, $p_i \in \mathcal{P}_{\beta_i}$, with $q \restriction \beta_i \le p_i$, then $\{p_i : i < \delta\} \cup \{q\}$ has an upper bound r with all $p_j = r \restriction \beta_j$ and $\mathrm{Dom}\, r = \mathrm{cl}[\mathrm{Dom}\, q \cup \bigcup_{i<\delta} \mathrm{Dom}(p_i)]$.

8″. Suppose δ_1, δ_2 are limit ordinals from W^*_λ, and $(\beta_i)_{i<\zeta}$ is a strictly increasing sequence of ordinals from W^*_λ. Let $I(\delta_1, \delta_2) := (\delta_1 + 1) \times (\delta_2 + 1) \setminus \{(\delta_1, \delta_2)\}$. Suppose that for $(i, j) \in I(\delta_1, \delta_2)$ we have $p_{ij} \in \mathcal{P} \restriction \beta_i$ such that

$$i \le i' \Longrightarrow p_{ij} \le p_{i'j}$$
$$j \le j' \Longrightarrow p_{ij} \le p_{ij'} \restriction \beta_j;$$

Then $\{p_{ij}\colon (i,j) \in I(\delta_1,\delta_2)\}$ has an upper bound r in $\mathcal{P}$ with $r\restriction\beta_j = p_{\delta_1,j}$ for all $j < \delta_2$ and $\mathrm{Dom}(r) = \mathrm{cl}(\cup\{\mathrm{Dom}(p_{i,j}\colon (i,j) \in I_{\delta_1,\delta_2}\})$.

These axioms apply in the case of the partial order App by 1.9.

In the paralel of A8 (density system) we use only closed u and also the game is defined as in A4 and we define admissible ideals of $\mathcal{P}_\alpha$ (for $\alpha \in W^*_\lambda$).

Proof:

(1) Easy, as this framework includes more cases.

(2) We are give a framework as in A1 and we shall "translate" to a new one. Of course instead $\mathcal{P} \subseteq \lambda^+ \times \mathcal{P}_\lambda(\lambda^+)$ we can use $\mathcal{P} \subseteq A \times \mathcal{P}_\lambda(\lambda^+)$ for any set A of cardinality λ. Let $A = \{(\alpha,\zeta,\bar{v})\colon \alpha < \lambda$ and $\bar{v}$ has the form $\langle v_\varepsilon\colon \varepsilon < \zeta\rangle$, $\zeta < \lambda$ not limit and v_ε is a subset of λ of cardinality $< \lambda\}$ (possibly empty). For $x = (\alpha,\zeta,\bar{v}) \in A$ and $u \subseteq \lambda^+$ closed of order type ζ, we let $u^{[x]} = \{\lambda\gamma + i\colon \gamma \in u, i \in v_{\mathrm{otp}\,(\gamma\cap u)}\}$. Let $\mathcal{P}' = \{(x,u)\colon x = (\alpha,\zeta,\bar{v}) \in A$, $u \subseteq \lambda^+$ has order type ζ, u is closed and $(\alpha, u^{[\bar{v}]}) \in \mathcal{P}\}$,
We define a function from $\mathcal{P}'$ onto $\mathcal{P}$:

$$f(x,u) = (\alpha, u^{[x]}) \text{ when } \quad x = (\alpha,\zeta,\bar{v}) \in A$$

We define the partial order $<$ on $\mathcal{P}'$ such that f is an isomorphism i.e.: $p \leq q$ iff $f(p) \leq f(q)$. We now show that $\mathcal{P}'$ satisfies $(1)''$–$(8)''$.

It is straightforward to check $(1)''$, $(2)''$, $(3)''$.

For $(4)''$ the point in that if for p,q,α are as there, we know that $f(p), f(q)$ has a common upper bound, r. By the indiscernibility condition w.l.o.g, if $\alpha \in \mathrm{Dom}\, r$, letting $\delta \leq \alpha < \delta + \lambda$, δ divisible by λ, we have $\delta \in cl(\mathrm{Dom}\, f(p))$ or $\delta \in cl(\mathrm{Dom}\, f(q))$ (remember $D \in (\mathrm{Dom}\, p) \cap (\mathrm{dom}\, q)$ a by the definition of a closed set). So we can find $\bar{r}$, $f(\bar{r}) = r$, with the right domain. $\blacksquare_{A11}$

A12 Notation: From now we will work toward proving A10, in the content A11(2), this suffices concentrating on $\lambda > \aleph_0$.

(1) For sets a, b of ordinals, let $OP_{a,b}$ be the function: $OP_{a,b}(\alpha) = \beta$ iff $\alpha \in a$, $\beta \in b$ and $otp(a \cap \alpha) = otp(b \cap \beta)$ so dom$\,(OP_{a,b})$ is an initial segment of a, rang$(OP_{a,b})$ is an initial segment of b, and in at least one case we have equality.

(2) $\bar{a}$ is a λ-representation of A if $\bar{a} = \langle a_i\colon i < \lambda\rangle$ is increasing contintuous, $a_i \subseteq A = \bigcup_{j<\lambda} a_j$, $|a_i| < \lambda$.

(3) $\bar{p}$ is a candidate for $\mathcal{P}_\alpha$ (or α-candidate) if $\bar{p} = \langle p_i\colon i < \lambda\rangle$ and $i < j < \lambda \Rightarrow p_i \leq p_j \in \mathcal{P}_\alpha$.

(4) If $\alpha_1 \leq \alpha_2$, $\bar{p}^l$ an α_l-candidate then $\bar{p}^1 \leq \bar{p}^2$ means $\wedge_{i<\lambda} \vee_{j<\lambda} p^1_i \leq p^2_j \restriction \alpha_1$.

(5) $\bar{p}$ represents G_α (for α) if $\bar{p}$ is a candidate for $\mathcal{P}_\alpha$ and $G_\alpha = \{q \in \mathcal{P}_\alpha: \vee_{i<\lambda} q \leq p_1\}$ and we write $G_\alpha = G_\alpha[\bar{p}]$. Let $\bar{p}^1 \approx \bar{p}^2$ iff $G_{\alpha_1}[\bar{p}^1] = G_{\alpha_2}[\bar{p}^2]$ and $\alpha_1 = \alpha_2$. See A13(2).

(6) $\bar{p}^1 \approx \bar{p}^2$ iff $\wedge_{i<\lg \bar{p}^1} \vee_{j<\lg \bar{p}^2} p^1_i \leq p^2_j$ and $\wedge_{i<\lg \bar{p}^2} \vee_{j<\lg \bar{p}^1} p^2_i \leq p^1_j$ (so $\approx$ is an equivalence relation).

(7) If $\bar{p} = \langle p_i: i < i^* \rangle$, $p_i \in \mathcal{P}$, h a partial funcion from λ^+ to λ^+ then: $h(\bar{p}) = \langle h(p_i): i < i^* \rangle$.

(8) Let S be a family of λ^+ subsets of λ, $[S_1, S_2 \in \mathcal{S} \Rightarrow |S_1 \cap S_2| < \lambda$ & $\lambda = \sup S_1]$ and $\diamondsuit_S$ (when $\lambda > \aleph_0$) for $S \in \mathcal{S}$ (see A13(4)).

A13 Claim:

(1) *If $\alpha < \lambda^+$, G_α an addimissible ideal of $\mathcal{P}_\alpha$, (or is just a λ-directed subset of $\mathcal{P}_\alpha$), then for some candidate $\bar{p}$ for $\mathcal{P}_\alpha$, $\bar{p}$ represents G_α (for α).*

(2) *If $\alpha < \lambda^+$, and $\bar{p}^1$, $\bar{p}^2$ are candidates for $\mathcal{P}_\alpha$ both representing one G then $\bar{p}^1 \approx p^2$, if in addition λ is uncountable then $\{\delta < \lambda: \bar{p}^1 \restriction \delta \approx \bar{p}^2 \restriction \delta\}$ is a club of λ.*

(3) *In A12(6) if $G[\bar{p}^1]$ is addmissible for $\mathcal{P}_{\alpha_1}$ then $\wedge_i \vee_j p^1_i \leq p^2_j$ suffices.*

(4) *If $\diamondsuit_\lambda$ holds (or $\lambda = \aleph_0$) then $\mathcal{S}$ exists.*

A14 Claim: *Assume $S \subsetneq \lambda$ is stationary, $\diamondsuit_\lambda$ and $\beta < \lambda^+$ then we can find $\mathfrak{p} = \langle \gamma_\delta, S_\delta, \bar{p}_\delta: \delta \in S \rangle$ such that:*

(α) (i) $\gamma_\delta < \lambda$

(ii) $\bar{p}_\delta = \langle p_{\delta,i}: i < \delta \rangle$

(iii) $p_{\delta,i} \in \mathcal{P}$

(iv) $\operatorname{Dom} p_{\delta,i} \subseteq \beta \cup [\beta, \beta + \gamma_\delta)$

(v) $i < j \Rightarrow p_{\delta,i} \leq p_{\delta,j}$

(vi) $\langle \bigcup_{i<\delta} \operatorname{dom}(p_{\delta,i}): \delta \in S \rangle$ *is increasing continuous*

(vii) $s_\delta \subseteq [\beta, \beta + \gamma_\delta)$

(β) *if $\gamma \in S$, $\bar{p} = \langle p_i: i < \lambda \rangle$ generates an addmissible ideal of $\mathcal{P}_\alpha$, and $\langle a_i: i < \lambda \rangle$ is a λ-representation of γ then $\{\delta \in S: \gamma_\delta = \operatorname{otp}(a_\delta)$ and $\wedge_{i<\delta} OP_{\beta \cup a_\delta, \beta \cup [\beta, \beta+\gamma_\delta]}(p_i) = p_{\delta,i}\}$ is a stationary subset of λ.*

A15 Definition: We say $\mathfrak{c}$ is an explicit β-commitement if:

(α) $\beta < \lambda^+$

(β) $\mathfrak{c}$ consists of $\beta^{\mathfrak{c}}, \gamma, \mathfrak{p}^{\mathfrak{c}} = \langle \gamma_\delta^{\mathfrak{c}}, s_\delta^{\mathfrak{c}}, \bar{p}_\delta^{\mathfrak{c}}: \delta \in S^{\mathfrak{c}} \rangle$ and $\mathfrak{q}^{\mathfrak{c}} = \langle \mathfrak{q}_\delta^{c}: \delta \in S^{\mathfrak{c}} \rangle$ of course $\mathfrak{p}_\delta^{\mathfrak{c}} = \langle p_{\delta,i}^{\mathfrak{c}}: i < \delta \rangle$

(γ) $\mathfrak{p}^{\mathfrak{c}}$ is as in A14

(δ) for $\delta \in S, \mathfrak{q}_\beta^{\mathfrak{c}} \in \mathcal{P}, \operatorname{Dom} \mathfrak{q}_\delta^{\mathfrak{c}} \subseteq \beta \cup [\beta, \beta + \gamma_\delta)$ and $\mathfrak{q}_\beta^{\mathfrak{c}}$ is an upper bound of $\{\mathfrak{p}_{\delta,i}^{\mathfrak{c}}: i < \delta\}$

A16 Definition: Let $\mathfrak{c}$ be an explicit β-commitment, $\beta < \alpha \in W_\lambda^*$, and $\bar{p}$ a candidate for $\mathcal{P}_\alpha$, $\bar{a} = \langle a_i: i < \lambda \rangle$, $a_i \supseteq \cup\{\operatorname{Dom} p_j: j < i\}$ $\bar{a}$ is a representation of α. We say $\bar{p}$ satisfies $\mathfrak{c}$ if: for some club E of λ, for every $\delta \in S^{\mathfrak{c}} \cap E$ for some $\gamma \leq \gamma_\delta^{\mathfrak{c}}$ we have

(i) $OP_{\beta \cup [\beta, \beta+\gamma_\delta^{\mathfrak{c}}), \beta \cup a_\delta}(\bar{p}_\delta^{\mathfrak{c}}) \approx \langle p_i: i < \delta \rangle \Rightarrow OP_{\beta \cup [\beta, \beta+\gamma_\delta^{\mathfrak{c}}), \beta \cup a_\delta}(q_\delta^{\mathfrak{c}}) = p_\delta$

(ii) for every $\gamma' \in [\beta, \gamma)$ we have: $\aleph_0 \leq \operatorname{cf}(\gamma') < \lambda \Leftrightarrow OP_{\beta \cup [\beta+\gamma_\delta^{\mathfrak{c}}), \beta \cup a_\delta}(\gamma') \in s_\delta^{\mathfrak{c}}$

A17 Claim: *For $S \in \mathcal{S}$ there is an expticit of commitement $\mathfrak{c} = \mathfrak{c}[S]$ with $S^{\mathfrak{c}} = S$, such that: if $\alpha < \lambda$, $\bar{p}$ a candidate for $\mathcal{P}_\alpha$, $\bar{p}$ satisfies $\mathfrak{c}$ then $G_\alpha[\bar{p}]$ is an addmissible ideal of $\mathcal{P}_\alpha$.*

A18 Notation: Γ denotes a function with domain a subset of $\mathcal{S}$ of cardinality $\leq \lambda$, each $\Gamma(S)$ an explicit β-commitement for some $\beta \leq \alpha$, $S^{\Gamma(S)} = S$, one of them is the one from A17 above. We say $\bar{p}$ (an α-candidate for some $\alpha \in W_\lambda^*$) satisfies Γ if it satisfies every $\Gamma(S)$ for $S \in \operatorname{Dom}\Gamma$.

A19 Claim: *Assume $\alpha < \alpha'$ are from W_λ^*,*

(1) *If $\bar{p}$ is an α-candidate satisfying Γ then there is an α'-candidate $\bar{p}'$ satisfying Γ with $\bar{p} \leq \bar{p}'$.*

(2) *Moreover if $r \in \mathcal{P}_{\alpha'}$, $r \restriction \alpha \in G_\alpha[\bar{p}]$ then we can demand $r \in G_{\alpha'}[\bar{p}]$.*

A20 Claim: *Assume $\delta < \lambda^+$, $\aleph_0 \leq \kappa = f(\delta) < \lambda$, $w \subseteq \delta = \sup(w)$, $\langle \bar{p}^\alpha: \alpha \in w \rangle$ is such that each $\bar{p}^\alpha$ is an α-candidate, $[\alpha^1 < \alpha^2 \in w \Rightarrow \bar{p}^{\alpha^1} \leq \bar{p}^{\alpha^2}]$, $\langle \Gamma_\alpha: \alpha \in w \rangle$ is increasing, $\bar{p}^\alpha$ satisfies $\mathcal{P}_\alpha$ (for $\alpha \in w$) then*

(1) *there is a δ-candidate $\bar{p}$ satisfying $\bigcup_{\alpha \in w} \Gamma_\alpha$, such that $\wedge_{\alpha \in w} \bar{p}^\alpha \leq \bar{p}$.*

(2) *if $\delta \leq \alpha' < \lambda^+$, $p \in \mathcal{P}_\alpha$, $\bigwedge_{\alpha \in w} p \restriction \alpha \in G[\bar{p}^\alpha]$ then we can find an α'-candidate $\bar{p}$ satisfying $\bigcup_{\alpha \in w} \Gamma_\alpha$ such that $\bigwedge_\alpha \bar{p}^\alpha \leq \bar{p}$ and $p \in G_{\alpha'}[\bar{p}]$.*

A21 CLAIM: *As in A20, when* $\mathrm{cf}(\delta) = \lambda$.

A22 CLAIM: *(1) Assume* $\alpha \in W^*_\lambda$, $\bar{p}$ *an* α*-candidate,* $G_\alpha = G_\alpha[\bar{p}]$ *is an admissible ideal of* $\mathcal{P}_\alpha$. *For any* (α, β)*-density system* D *over* G_α *and* $S \in \mathcal{S}$ *there is a* β*-explicit* β*-commitement* $\mathfrak{c}$ *satisfied by* G_β *with* $S^{\mathfrak{c}} = \mathcal{S}$, *such that:*

if $\alpha' \leq \beta$, $\bar{p}' \leq \bar{p}$ *are* α'*-candidate such that* $G_{\alpha'} = G_\alpha[\bar{p}]$ *is an addmissible ideal on* $\mathcal{P}_{\alpha'}$, *and* $\bar{p}'$ *satisfies* $\mathfrak{c}$ *then* $G_{\alpha'}[\bar{p}']$ *meets* D.

(2) We can replace in (1) G_α *by* $\langle G_{\alpha'} \colon \alpha' \in \alpha \cap W^*_\lambda \rangle$, G_α *increasing with* α

A23 Proof of Theorem A10 when $\diamondsuit_\lambda$ *holds:* The Ghibellines in addition to choosing for $\alpha \in W^*_\lambda$ an addmissible ideal G_α (increasing with α), choose on the side Γ_α, increasing with α, such that G_α satisfies Γ_α. The previous claims do the job.

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