A PROOF OF THE GROTHENDIECK INEQUALITY

BY

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ABSTRACT

The fundamental constant of Grothendieck's inequality, defined below, was shown by Grothendieck to be less than $\sinh \pi/2 = 2.301+$. We improve the bound slightly, and show that for the positive definite case $\pi/2$ suffices.

In 1956, A. Grothendieck proved that the supremum of the injective tensor norms of the identity map on $\mathbf{R}^n$ is uniformly bounded for all n, with least upper bound γ between $\pi/2$ and $\sinh \pi/2$. This remarkable result leads directly to a number of important theorems involving $\mathcal{L}_p$-spaces. These are given in a paper of Lindenstrauss and Pełczyński [2], together with the following matrix formulation of γ: let $A = (a_{ij})$ be a real $n \times n$ matrix, n any finite positive integer, and $\mathcal{H}$ an arbitrary real Hilbert space with inner product $(\cdot,\cdot)$. Define $|A|_{\mathcal{H}}$ and $|A|$ by

$$(1) \qquad |A|_{\mathcal{H}} = \sup\{|\Sigma\Sigma\, a_{ij}(x_i, y_j)| : x_i, y_j \text{ in } \mathcal{H} \text{ of norm} \leqq 1\}$$

and

$$(2) \qquad |A| = \sup\{|\Sigma\Sigma\, a_{ij}t_iu_j| : -1 \leqq t_i, u_j \leqq 1\}.$$

Then $|A|_{\mathcal{H}} \leqq \gamma |A|$.

The proof given by Grothendieck [1, p. 62] and reformulated by Lindenstrauss and Pełczyński [2, p. 279] obtains $\gamma \leqq \sinh \pi/2 = 2.301+$ by averaging (2), with $t_i = \operatorname{sgn}(x_i, \omega)$ and $u_j = \operatorname{sgn}(y_j, \omega)$, over the unit sphere Ω in $\mathbf{R}^n$ with normalized surface measure $d\omega$. We obtain a slightly smaller bound, $\gamma < 2.261$,

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by averaging over $\mathbf{R}^n$ with normalized Gaussian measure and using a variational argument to determine an optimal (in this context) scalar map corresponding to the signum function. Our argument also shows that for positive definite matrices A, we have $|A|_{\mathscr{K}} \leqq \pi/2|A|$.

Sten Kaijser (private correspondence) has pointed out that our proof generalizes directly to the complex case, and yields an upper bound of 1.607 for the complex constant.

For a fixed positive integer n, let $dG(x) = (2\pi)^{-n/2}\exp(-|x|^2/2)dx$ be normalized Gaussian measure of mean zero and unit variance on $\mathbf{R}^n$. Write L^2 for $L^2(\mathbf{R}^n, dG)$ and let $\|\cdot\|$ and $(\cdot\,,\cdot)$ be the Hilbert norm and inner product in L^2.

If $x = (x_i)$ and $y = (y_i)$ are vectors in $\mathbf{R}^n$, let $|x|$ be the Euclidean norm of x, and $x \cdot y = \Sigma\, x_i y_i$ be the inner product of x and y. For $t \geqq 0$, write $dm(t) = (2/\pi)^{\frac{1}{2}}\exp(-t^2/2)dt$, where dt is Lebesgue measure. All unmarked sums Σ are taken with i and j ranging independently over the integers from 1 through n.

Let $\mathscr{U}$ be the set of measurable functions $f(t) \geqq 0$, defined on $t \geqq 0$, with $\operatorname{ess\,sup} f(t) \leqq 1$. Given $f \in \mathscr{U}$, let ψ be the odd extension of f to domain $\mathbf{R}$, i.e. $\psi(t) = f(t)$ if $t \geqq 0$ and $\psi(t) = -f(-t)$ if $t < 0$. Then given $x \in \mathbf{R}^n$, we define the real valued functions φ_x and ψ_x on $\mathbf{R}^n$ by

$$\varphi_x(z) = x \cdot z \quad \text{and} \quad \psi_x(z) = \psi(x \cdot z) \quad (z \in \mathbf{R}^n). \tag{3}$$

Clearly φ_x and ψ_x are in L^2 for each $x \in \mathbf{R}^n$.

LEMMA 1. *Let $f \in \mathscr{U}$ and ψ the odd extension of f. If $|x| = |y| = 1$, then*

(i) $(\varphi_x, \varphi_y) = x \cdot y$

(ii) $(\varphi_x, \psi_y) = K(x \cdot y)$ *where*

$$K = K_f = \int_0^\infty tf(t)dm(t)$$

(iii) $\|\varphi_x - \psi_x\|^2 = 1 - 2K + L$, *where*

$$L = L_f = \int_0^\infty f^2(t)dm(t).$$

PROOF. Since x and y have norm 1, we have $x = (x \cdot y)y + y'$, where y' is orthogonal to y. Then

$$\int (x \cdot z)(y \cdot z)dG(z) = (x \cdot y) \cdot \int |y \cdot z|^2 dG(z) + \int (y' \cdot z)(y \cdot z)dG(z).$$

The measure dG is characterized by assigning the function $z \to y \cdot z$ a Gaussian distribution of mean zero and variance $|y|^2$, so $\int |y \cdot z|^2 dG(z) = |y|^2 =$

1. Since the Gaussian distributions are basis-invariant, we may choose coordinates so that y is one of the basis vectors, to get $\int (y' \cdot z)(y \cdot z) dG(z) = 0$. This completes (i).

For (ii), with $x = (x \cdot y)y + y'$ as before,

$$\int (x \cdot z)\psi(y \cdot z)dG(z) = (x \cdot y) \cdot \int (y \cdot z)\psi(y \cdot z)dG(z) + \int (y' \cdot z)\psi(y \cdot z)dG(z).$$

Again choose coordinates so that y is a basis vector, and integrate first along y. We obtain

$$\int (y \cdot z)\psi(y \cdot z)dG(z) = \int_0^\infty tf(t)dm(t),$$

and

$$\int (y' \cdot z)\psi(y \cdot z)dG(z) = 0.$$

For (iii), expand $\|\varphi_x - \psi_x\|^2$, note that

$$(\psi_x, \psi_x) = \int \psi^2(x \cdot z)dG(z) = \int_0^\infty f^2(t)dm(t),$$

and apply (i) and (ii).

THEOREM 1 (Grothendieck). *There is a finite number γ, independent of n, such that $|A|_{\mathscr{H}} \leqq \gamma |A|$ for all $n \times n$ matrices A.*

PROOF. Let $f \in \mathscr{U}$. Write $K = K_f$, $L = L_f$, and let ψ be the odd extension of f. For any fixed n, let $A = (a_{ij})$ and $\mathscr{H}$ be given, with $|A| \leqq 1$.

Note that $|A|_{\mathscr{H}}$ is the supremum of the sums $\Sigma_i \|\Sigma_j a_{ij} x_j\|$, taken over x_j in $\mathscr{H}$ of norm $\leqq 1$. Since any such set $\{x_j\}$ of n vectors spans a subspace of $\mathscr{H}$ of dimension at most n, we may take $\mathscr{H} = \mathbf{R}^n$. Further, the convexity of the unit sphere in Euclidean space $\mathbf{R}^n$ allows us to choose vectors x_i, y_j of norm 1 in $\mathbf{R}^n$ such that $|A|_{\mathscr{H}} = \Sigma a_{ij}(x_i \cdot y_j)$.

By Lemma 1, the identity

$$(\psi_x, \psi_y) = (\varphi_x, \psi_y) + (\psi_x, \varphi_y) - (\varphi_x, \varphi_y) - (\varphi_x - \psi_x, \psi_y - \varphi_y)$$

yields:

$$\sum a_{ij}(\psi_{x_i}, \psi_{y_j}) = (2K - 1) \sum a_{ij}(x_i \cdot y_j) - \sum a_{ij}(\varphi_{x_i} - \psi_{x_i}, \psi_{y_j} - \varphi_{y_j}). \tag{4}$$

From (iii), the last sum above is bounded in absolute value by $|A|_{\varkappa}(1-2K+L)$, and since $|A|\leqq 1$, we also have $|\Sigma a_{ij}(\psi_{x_i},\psi_{y_j})|\leqq 1$. Hence the triangle inequality applied to (4) yields:

$$\text{(5)}\qquad 1\geqq\left|\sum a_{ij}(\psi_{x_i},\psi_{y_j})\right|\geqq|A|_{\varkappa}(|2K-1|+2K-L-1).$$

This inequality holds for any function f in $\mathcal{U}$. In particular, let f be identically 1. Then $K=(2/\pi)^{\frac{1}{2}}$ and $L=1$, so (5) reads

$$1\geqq|A|_{\varkappa}(|2(2/\pi)^{\frac{1}{2}}-1|+2(2/\pi)^{\frac{1}{2}}-2).$$

The coefficient of $|A|_{\varkappa}$ is easily seen to be greater than .19, hence $|A|_{\varkappa}<(.19)^{-1}=5.2+$.

Corollary 1. *If $f\in\mathcal{U}$ is such that $2K_f^2>L_f$, then $\gamma\leqq(2K_f^2-L_f)^{-1}$.*

Proof. From the proof of Theorem 1, for any $f\in\mathcal{U}$ we have:

$$\text{(5)}\qquad 1\geqq|A|_{\varkappa}(|2K-1|+2K-L-1).$$

When f is replaced by $cf, c>0$, we get $K_{cf}=cK_f$ and $L_{cf}=c^2L_f$, so (5) becomes:

$$\text{(6)}\qquad c^2\geqq|A|_{\varkappa}(|2cK-1|+2cK-c^2L-1).$$

If c is such that $2cK-1\leqq 0$, then (6) is trivial, so we only consider constants $c>0$ for which $4cK-c^2L-2>0$. By (6), $|A|_{\varkappa}\leqq c^2(4cK-c^2L-2)^{-1}$ for all such c. For fixed K and L, the minimum value of the right side of this inequality occurs when $c=1/K$. Since $2K^2-L>0$, this value of c yields $|A|_{\varkappa}\leqq(2K^2-L)^{-1}$. This completes the proof.

We observe that the real-valued mapping $f\to(2K_f^2-L_f)$ is lower, semi-continuous in the weak*-topology on $L^\infty(\mathbf{R}^+,dm(t))$ and $\mathcal{U}$ is a weak*-closed subset of the unit sphere of this L^∞ space. Hence $2K_f^2-L_f$ attains its maximum on $\mathcal{U}$, say at $\mu(t)$. We now determine $\mu(t)$ explicitly.

Let h be a bounded measurable function on $[0,\infty)$ such that $h(t)\geqq 0$ on $[\mu=0]$, $h(t)\leqq 1$ on $[\mu=1]$ and $\operatorname{ess\,sup}_{t\geqq 0}|\mu(t)+\epsilon h(t)|=1$ for $\epsilon>0$ sufficiently small. Fix such an ϵ, write K' and L' for the K and L corresponding to the function $\mu+\epsilon h$, and delete the μ-subscript on K_μ and L_μ. Then

$$2K'^2-L'=2K^2-L+2\epsilon\int_0^\infty[2Kh(t)t-\mu(t)h(t)]dm(t)+O(\epsilon^2).$$

Since $2K^2-L\geqq 2K'^2-L'$, and ϵ can be taken arbitrarily small,

$$0 \geqq \int_0^\infty [2Kh(t)t - \mu(t)h(t)]dm(t). \tag{7}$$

Three cases need to be considered.

Case 1. Let $h(t) \geqq 0$ and supported on $[\mu = 0]$. By (7), $0 \geqq \int_0^\infty 2tKh(t)dm(t)$, which is impossible unless $[\mu = 0]$ has measure zero. Hence $\mu(t) > 0$ a.e.

Case 2. Let $h(t) \leqq 0$ and supported on $[\mu = 1]$. Then (7) implies $0 \geqq \int_0^\infty h(t)(2Kt - 1)dm(t)$, so $2Kt - 1 \geqq 0$ for all t such that $\mu(t) = 1$.

Case 3. Let $h(t)$ be supported on $[0 < \mu < 1]$. Note that, for any $\epsilon \leqq 1$, there are functions h, satisfying the given properties, which are strictly positive or strictly negative on $[0 < \mu < 1]$. For any such functions $h, 0 \geqq \int_0^\infty h(t)[2Kt - \mu(t)]dm(t)$, and this is possible only if $2Kt = \mu(t)$ a.e. on $[0 < \mu < 1]$.

The information obtained from these cases determines a.e. the maximal function $\mu(t) = 2Kt$ if $0 \leqq t \leqq 1/2K$ and $\mu(t) = 1$ if $t \geqq 1/2K$.

Calculating $2K^2 - L$ for this μ, we immediately obtain the equations $1 = 2 \cdot \int_0^{1/2k} dm(t)$ and $2K^2 - L = 2K(2/\pi)^{\frac{1}{2}} \exp(-1/8K^2) - \frac{1}{2}$. From tables in [3], $2K^2 - L > .4423$, so by Theorem 2, $\gamma < 2.261$.

If the given matrix A is positive definite, the corresponding value of γ is at most $\pi/2$. Grothendieck showed [1, p. 51] that $\pi/2$ is a lower bound for γ (over all matrices) by calculating, for each n, the exact value of a particular tensor norm of the identity on $\mathbf{R}^n$, showing that this norm is dominated by the supremum of the injective tensor norms, and letting $n \to \infty$. We show the sufficiency of $\pi/2$ for positive definite A by exploring the fact that the last sum in inequality (4) is non-positive for such matrices.

Let A be a positive definite $n \times n$ matrix with $|A| \leqq 1$. Consider the vector space of n-tuples of vectors in $\mathbf{R}^n$. This is a Hilbert space when endowed with the inner product $(\boldsymbol{x}, \boldsymbol{y}) = \Sigma x_i \cdot y_i$. Here $\boldsymbol{x} = (x_1, \cdots, x_n)$ and $\boldsymbol{y} = (y_1, \cdots, y_n)$, each x_i and y_i in $\mathbf{R}^n$. This inner product induces the norm $\|\boldsymbol{x}\|^2 = \Sigma |x_i|^2$. Now $\Sigma a_{ij}(x_i \cdot y_j) = (A\boldsymbol{x}, \boldsymbol{y}) = (A^{\frac{1}{2}}\boldsymbol{x}, A^{\frac{1}{2}}\boldsymbol{y})$, since A is positive definite: hence $|\Sigma a_{ij}(x_i \cdot y_j)| \leqq \|A^{\frac{1}{2}}\boldsymbol{x}\| \cdot \|A^{\frac{1}{2}}\boldsymbol{y}\|$, with equality, in particular, whenever $\boldsymbol{x} = \boldsymbol{y}$. This establishes:

LEMMA 3. *If A is positive definite, there exist vectors $x_1, \cdots, x_n$, each of unit norm in $\mathbf{R}^n$, such that $|A| = \Sigma a_{ij}(x_i \cdot x_j)$.*

THEOREM 4. *For any positive definite $n \times n$ matrix A, $|A|_{\varkappa} \leqq \pi/2$.*

PROOF. We follow the proof of Theorem 2. Again $f \in \mathcal{U}$ and ψ is the odd extension of f.

Assume $|A| \leqq 1$ and choose $x_1, \cdots, x_n$ of norm 1 in $\mathbf{R}^n$ such that $|A|_{\varkappa} = \Sigma a_{ij}(x_i \cdot x_j)$. Then inequality (4) of the proof of Theorem 2 holds with y_j replaced by x_j. But $\Sigma a_{ij}(\varphi_{x_i} - \psi_{x_i}, \varphi_{x_j} - \psi_{x_j})$ is non-negative, so $\Sigma a_{ij}(\psi_{x_i}, \psi_{x_j}) \geqq |A|_{\varkappa}(2K-1)$ and $1 \leqq |A|_{\varkappa}(2K-1)$. Replace f by $cf, c > 0$. Then $c^2 \geqq |A|_{\varkappa}(2cK-1)$, so $|A|_{\varkappa} \leqq c^2(2cK-1)^{-1}$ whenever $c \geqq 1/2K$. The minimum value of this bound for $|A|_{\varkappa}$, over all c, occurs when $c = 1/K$, so $|A|_{\varkappa} \leqq 1/K^2$. K is clearly maximized when f is identically 1, and then $K = (2/\pi)^{\frac{1}{2}}$. Thus $|A|_{\varkappa} \leqq \pi/2$.

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