MIXING PROPERTIES OF A CLASS OF SKEW-PRODUCTS

BY

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ABSTRACT

Skew-products of the powers of an ergodic measure preserving transformation with a Bernoulli base are shown to be k-automorphisms.

Introduction

Let T be an invertible measure preserving transformation (m.p.t.) on a probability triple (Ω, B, P).

Let T' be a Bernoulli shift on (Ω', B', P'); and let η be a countable partition of Ω' in B' whose T'-iterates are P'-independent and span B'. See ([4], th. (10.13)). Let X be a one-to-one integer valued function on η.

Define, on the product space $(\Omega' \times \Omega, B' \times B, P' \times P)$, a m.p.t. S by

(1) $$S(w', w) = (T'w', T^{X(w')}w).$$

The present paper studies mixing properties of the transformation S.

The transformation S is a special kind of *Skew-Product*. The general case replaces the powers of the single transformation T by an arbitrary family of m.p.t.'s on (Ω, B, P), parametrized by a function on (Ω', B', P'). See ([2], p. 91).

The purpose of this note is to prove that if T is totally ergodic or if T is ergodic and X is strongly aperiodic, S is a k-automorphism.

Questions about mixing properties of skew-products are raised, among other places, in [1], [2] and [5].

It would be interesting to study which of these special skew-products are (or rather are *not*) Bernoulli shifts, once we know them to be k-automorphisms.

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Adler and Shields [1] establish Bernoullicity when T is an irrational rotation of the circle. The problem raised by B. Weiss in [6], p. 682 remains unsolved to the present.

We will assume the reader is familiar with the notions of ergodicity, k-automorphism, Bernoulli shift, as well as aperiodicity, and strong aperiodicity of integer valued random variables. Otherwise, see [4] and [5]. *Total ergodicity* of a m.p.t. means ergodicity of all of its (positive integer) powers.

Local uniformity of the distribution of sums

LEMMA. *Let $X_1, X_2, X_3, \cdots$ be independent and identically distributed, strongly aperiodic, integer valued random variables. Denote for $n \geqq 1$, $S_n = X_1 + X_2 + \cdots + X_n$. Then for every positive integer M,*

$$\lim_{n \to \infty} \sum_{m=-\infty}^{\infty} \sum_{i=0}^{M-1} \left| P(S_n = mM + i) - (1/M) \sum_{j=0}^{M-1} P(S_n = mM + j) \right| = 0. \tag{2}$$

PROOF. For the case $M = 2$, this is a rephrasing of one side of the "zero–two law", see [3]. The more general case follows then as an easy consequence.

Keeping to the notations brought forward in the introduction,

THEOREM. *Assume X is strongly aperiodic. If η is an infinite partition of Ω', assume further that the distribution of X is such that the partial sums of i.i.d. variables distributed like X form a recurrent random walk. Under these assumptions, if T is ergodic, S is a k-automorphism.*

PROOF. Let α be a finite partition of Ω in B. We will first show S to be a k-automorphism on $e_\alpha = \vee_{n=-\infty}^{\infty}(S^n(\eta \times \alpha))$ (rather than on $B' \times B$).

To obtain that, it is enough to show that for each e_α-measurable bounded real function f,

$$\operatorname*{Sup}_{g} |E(g \cdot S^n f) - E(g)E(f)| \to 0 \text{ as } n \to \infty, \tag{3}$$

where $S \cup P$ is taken over all real functions g that are bounded by 1 in absolute value, measurable with respect to $\vee_{n=-\infty}^{-1}(S^n(\eta \times \alpha))$.

For convenience, use the common statistical notation

$$\operatorname{Cov}(f, g) = E(f \cdot g) - E(f)E(g). \tag{4}$$

We will now show that to obtain (3) for any e_α-measurable bounded f, it is enough to consider f's that are B-measurable, and we will then prove (3) for the latter.

Pick any e_α-measurable, bounded f and any g as required. Replacing f by an $\bar{f}$ that depends only on finitely many $S^n(\eta \times \alpha)$ coordinates and that is close to f in L_2 will give, (Cauchy–Schwartz inequality)

$$
\begin{aligned}
|\mathrm{Cov}(g, S^n f) - \mathrm{Cov}(g, S^n \bar{f})|^2 &= |\mathrm{Cov}(g, S^n(f - \bar{f}))|^2 \\
&\leqq \mathrm{var}(g) \cdot \mathrm{var}(S^n(f - \bar{f})) \leqq \mathrm{var}(f - \bar{f}),
\end{aligned} \tag{5}
$$

so it is enough to prove (3) for $\bar{f}$'s. By changing conveniently the power n in (3), we may assume $\bar{f}$ is $\vee_{n=0}^{M}(S^n(\alpha \times \eta))$-measurable for some M. Hence $\bar{f}$ can be expressed as $\bar{f} = \Sigma_\sigma f_\sigma 1_{A_\sigma}$, where $A_\sigma = \{(X, T'X, T'^2X, \cdots, T'^MX) = \sigma\}$ and f_σ is a B-measurable function that agrees with $\bar{f}$ on A_σ. If $\bar{f} = \Sigma f_\sigma 1_{A_\sigma}$ is a sum of an infinite number of terms, replace it by a sum $\bar{\bar{f}}$ of a finite sub-collection that is close to $\bar{f}$ in L_2. By a computation like (5), it is enough to prove (3) for f replaced by an arbitrary component $f_\sigma 1_{A_\sigma}$. Since 1_{A_σ} is independent of (g, f_σ), $\mathrm{Cov}(g, S^n(f_\sigma 1_{A_\sigma})) = E(1_{A_\sigma}) \cdot \mathrm{Cov}(g, S^n f_\sigma)$, and we obtain finally that (3) is implied by (5) for B-measurable f's. From now on, f is such a function.

Denote $V_n = X + T'X + T'^2X + \cdots + T'^{n-1}X$.

$$\mathrm{Cov}(g, S^n f) = \mathrm{Cov}(g, T^{V_n} f) = \sum_k \mathrm{Cov}(g, T^k f) P(V_n = k). \tag{6}$$

By the lemma, for fixed $M > 0$, the expression in (6) can be made (as n becomes large) arbitrarily close to

$$
\begin{aligned}
&\sum_{m,i} \mathrm{Cov}(g, T^{mM+i} f) \cdot (1/M) \sum_{j=0}^{M-1} P(V_n = mM + j) = \\
&= \sum_{m,j} P(V_n = mM + j) \mathrm{Cov}(g, (1/M) \sum_{i=0}^{M-1} T^{mM+i} f) = \\
&= \sum_{m,j} P(V_n = mM + j) \mathrm{Cov}(g, T^{mM}((1/M) \sum_{i=0}^{M-1} T^i f)).
\end{aligned} \tag{7}
$$

The last expression is bounded in absolute value by $\mathrm{Var}((1/M)\sum_{i=0}^{M-1} T^i f)$, that can be made arbitrarily small by picking M large enough, since T is ergodic. We finished the proof that S is a k-aut. on e_α.

By theorem (13.4) in [4], to prove that S is a k-aut. on $B' \times B$, it would be enough to show the existence of a sequence $(\alpha_n)_{n=1}^{\infty}$ of finite partitions of Ω in B for which $\vee_{m=-\infty}^{\infty}(S^m(\eta \times \alpha_n))$ increase to an algebra that spans $B' \times B$.

Let $(\beta_n)_{n=1}^{\infty}$ be an increasing (i.e., successive refinements) sequence of finite partitions of Ω in B for which $\vee_{m=-\infty}^{\infty}(T^m\beta_n)$ increase to an algebra that spans B. If η is a countably *infinite* partition of Ω, let $\alpha_n = \beta_n$. If η is a *finite* partition, then the random variable X is essentially bounded and it assigns probability one to a bounded set of integers diameter (some) $d > 0$. Let $\alpha_n = \vee_{m=0}^{d} T^m \beta_n$. α_n is a "d-sweeping" of β_n.

To prove that $\vee_n \vee_k S^k(\eta \times \alpha_n) = B' \times B$, it is enough to check that it contains all sets A that belong to $\vee_{m=-M}^{M} T^m \alpha_n$ for some M, some n. And it does, since it contains for every N the set $A \cap C_N$, with $(P' \times P)(C_N) \to 1$ as $N \to \infty$, and C_N is the event: For the random walk $(V_n)_{n=-\infty}^{\infty}$ whose increments are distributed like X and $V_0 = 0$: "$\{-M, -M+1, \cdots, +M\} \subseteq \bigcup_{n=-N}^{N}\{V_n\}$" (if we are in the recurrent case) or

$$\text{“}\{-M-d, -M-d+1, \cdots, -M\} \cap \bigcup_{n=-N}^{N} \{V_n\} \neq \phi$$

and

$$\{M, M+1, \cdots, M+d\} \cap \bigcup_{n=-N}^{N} \{V_n\} \neq \phi,\text{”}$$

(if we are in the "d-sweeping" case).

COROLLARY. *Assume T to be totally ergodic. Assume X to be non--deterministic, and if it assigns probabilty one to no finite set, assume its random walk to be recurrent on the integer lattice where it lives. Then S is a k-automorphism.*

PROOF. For some integer a, $P(X = a) > 0$. Let $d = \text{g.c.d.}\{n \neq 0 \mid P(X = a + n) > 0\}$. Then the integer valued random variable $Y = (1/d)(X - a)$ is strongly aperiodic. Apply to it the Lemma as in the proof of the Theorem, and express $T^{V_n}f = T^{na}(T^d)^{V'n}f$ (where $V'n$ are the partial sums of the Y's), thus finishing the proof since T^d is ergodic.

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