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INTEGRAL AND INTEGRO-DIFFERENTIAL EQUATIONS

Control of a Boundary Value Problem for a Linear Impulsive Integro-Differential System

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1. INTRODUCTION

The theory of impulsive differential equations [1–3] includes numerous open problems related to results of the theory of integral and integro-differential equations [4–8].

The aim of the present paper is to find solvability conditions for a control boundary value problem for a linear impulsive integro-differential system. (The control problem for a linear impulsive system was earlier studied in [9].) Moreover, we obtain necessary and sufficient conditions for the solvability of a boundary value problem for a linear impulsive integro-differential system and prove results on the existence and an integral representation of solutions of integro-sum Volterra equations of the second kind and integro-differential equations with impulsive effect at given instances of time.

2. AUXILIARY ASSERTIONS

We take real numbers α and β, $\alpha < \beta$, and positive integers r and p. Let $L_2^r[\alpha,\beta]$ be the space of all square integrable functions $\varphi : [\alpha,\beta] \to R^r$, and let $D^r[1,p]$ be the set of all finite sequences $\{\xi_i\}$, $\xi_i \in R^r$, $i = 1,\dots,p$. We introduce the space $\Pi_p^r[\alpha,\beta] = L_2^r[\alpha,\beta] \times D^r[1,p]$ and denote its elements by $\{\varphi,\xi\}$. We equip this space with the inner product $\langle\{\varphi,\xi\},\{\omega,\nu\}\rangle = \int_\alpha^\beta(\varphi,\omega)dt + \sum_{i=1}^p(\xi_i,\nu_i)$, where $(\cdot,\cdot)$ is the inner product in R^r. Throughout the following, $\{\theta_i\}$, $i = 1,\dots,p$, is a given strictly increasing sequence of real numbers in the interval (α,β). By $PAC[\alpha,\beta]$ we denote the set of all piecewise absolutely continuous functions $x(t) : [\alpha,\beta] \to R^n$ that are left continuous everywhere on $[\alpha,\beta]$ and have jump discontinuities at the points $\{\theta_i\}$, $i = 1,\dots,p$.

The following lemmas are analogs of the Fubini theorem [4, p. 317].

Lemma 1. *Let D_{ij}, $i,j = 1,\dots,p$, be constant $n \times n$ matrices, and let $\{\xi_i\} \in D^n\{1,p\}$. Then*

$$\sum_{\alpha<\theta_i<t}\ \sum_{\alpha<\theta_j\le\theta_i} D_{ij}\xi_j = \sum_{\alpha<\theta_i<t}\ \sum_{\theta_i\le\theta_j<t} D_{ji}\xi_i$$

for each $t \in (\alpha,\beta)$.

The proof is by rearranging the terms.

Lemma 2. *Let $K(t,s)$ be an $n \times n$ matrix square integrable on the interval $\alpha \le s \le \beta$, and let $\varphi_i(t) \in L_2^n[\alpha,\beta]$, $i = 1,\dots,p$. Then*

$$\int_\alpha^t K(t,s)\sum_{\alpha<\theta_i<s}\varphi_i(s)ds = \sum_{\alpha<\theta_i<t}\int_{\theta_i}^t K(t,s)\varphi_i(s)ds \tag{1}$$

for each $t \in (\alpha,\beta)$.

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Proof. Consider the functions $\Phi_i(t)$, $i = 1, \ldots, p$, such that $\Phi_i(t) = 0$ for $t \le \theta_i$ and $\Phi_i(t) = \varphi_i(t)$ for $t > \theta_i$. Then the left-hand side of (1) becomes

$$\int_\alpha^t K(t,s) \sum_{\alpha<\theta_i<t} \Phi_i(s)ds = \sum_{\alpha<\theta_i<t} \int_\alpha^t K(t,s)\varphi_i(s)ds = \sum_{\alpha<\theta_i<t} \int_{\theta_i}^t K(t,s)\Phi_i(s)ds.$$

The proof of the lemma is complete.

Consider the integral equation

$$x(t) = \int_\alpha^t G(t,s)x(s)ds + \sum_{\alpha<\theta_i<t} S_i(t)x(\theta_i) + \sum_{\alpha<\theta_i<t} N_i(t)x(\theta_i+) + \sum_{\alpha<\theta_i<t} I_i + f(t), \tag{2}$$

where $x \in R^n$, $G(t,s)$ is an $n \times n$ matrix function square integrable on $[\alpha,\beta]\times[\alpha,\beta]$, $S_i(t)$ and $N_i(t)$ are $n \times n$ matrix functions whose columns, as well as the function $f(t)$, belong to $PAC[\alpha,\beta]$, and $\{I_i\} \in D^n[1,p]$. Furthermore, we assume that $\det(I - N_i(\theta_i+) + N_i(\theta_i)) \ne 0$ for all $i = 1, \ldots, p$.

Theorem 1. *System (2) has a unique piecewise continuous solution $x(t) \in PAC[\alpha,\beta]$. It can be represented in the form*

$$x(t) = \int_\alpha^t P_1(t,s)f(s)ds + \sum_{\alpha<\theta_i<t} Q_i(t)I_i + \sum_{\alpha<\theta_i<t} P_2^i(t)f(\theta_i) + f(t) + \sum_{\alpha<\theta_i<t} I_i, \tag{3}$$

where $Q_i(t)$, $P_2^i(t)$, $i = 1, \ldots, p$, and $P_1(t,s)$ are piecewise continuous $n \times n$ matrix functions.

Proof. Let $R(t,s)$ be the resolvent of the Volterra integral equation of the second kind with kernel $G(t,s)$. Then, using Lemmas 1 and 2, we find that system (2) is equivalent to the equation

$$\begin{aligned} x(t) = &\sum_{\alpha<\theta_i<t} \left[\int_{\theta_i}^t R(t,s)S_i(s)ds + S_i(t) \right] x(\theta_i) + \sum_{\alpha<\theta_i<t} \left[\int_{\theta_i}^t R(t,s)N_i(s)ds + N_i(t) \right] x(\theta_i+) \\ &+ \sum_{\alpha<\theta_i<t} \int_{\theta_i}^t R(t,s)I_i ds + \sum_{\alpha<\theta_i<t} I_i + \int_\alpha^t R(t,s)f(s)ds + f(t). \end{aligned} \tag{4}$$

Let $S_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j,s)\, S_i(s)ds$, $N_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j,s)\, N_i(s)ds$, and $p_{ij} = \int_{\theta_i}^{\theta_j} R(\theta_j,s)\, ds + E$, where E is the $n \times n$ identity matrix. Then it follows from (4) that

$$\begin{aligned} x(\theta_j) = &\sum_{\alpha<\theta_i<\theta_j} \left[(S_{ij} + S_i(\theta_j))\, x(\theta_i) + (N_{ij} + N_i(\theta_j))\, x(\theta_i+) \right] + \sum_{\alpha<\theta_i<\theta_j} p_{ij} I_i \\ &+ \int_\alpha^{\theta_j} R(\theta_j,s)\, f(s)ds + f(\theta_j). \end{aligned} \tag{5}$$

Using Eqs. (4) and (5), we obtain

$$\begin{aligned} x(\theta_j+) = &(E - N_j(\theta_j+) + N_j(\theta_j))^{-1} \Big\{ (E + S_j(\theta_j+) - S_j(\theta_j))\, x(\theta_i) \\ &+ \sum_{\alpha<\theta_i<\theta_j} [S_i(\theta_j+) - S_i(\theta_j)]\, x(\theta_i) + \sum_{\alpha<\theta_i<\theta_j} [N_i(\theta_j+) - N_i(\theta_j)]\, x(\theta_i+) \\ &+ I_j + f(\theta_j+) - f(\theta_j) \Big\}. \end{aligned} \tag{6}$$

The expressions (5) and (6) recursively define $x(\theta_j)$ and $x(\theta_j+)$. Since the nonhomogeneous part of this system is a linear combination of the vectors $\int_\alpha^{\theta_i} R(t,s)f(s)ds$, $f(\theta_i)$, and I_i, $i = 1,\dots,p$, it follows that $x(\theta_j)$ and $x(\theta_j+)$ are also linear combinations of these vectors with matrix coefficients. Substituting the expressions (5) and (6) into Eq. (4), we find that the solution of Eq. (2) has the form (3). The proof of the theorem is complete.

3. THE BOUNDARY VALUE PROBLEM

Consider the impulsive integro-differential system

$$dx/dt = A(t)x + \int_\alpha^t K(t,s)x(s)ds + f(t), \quad t \neq \theta_i,$$
$$\Delta x(\theta_i) = B_i x(\theta_i) + \sum_{\alpha<\theta_j\leq\theta_i} D_{ij}x(\theta_j) + \int_\alpha^{\theta_i} M_i(s)x(s)ds + I_i, \tag{7}$$

where $x \in R^n$, $\Delta x(\theta_i) \equiv x(\theta_i+) - x(\theta_i)$, $t \in [\alpha,\beta]$, $A(t)$, $K(t,s)$, and $M_i(t)$, $i = 1,\dots,p$, are $n \times n$ matrix functions, the columns of $A(t)$ and $M_i(t)$, $i = 1,\dots,p$, are elements of the space $L_2^n[\alpha,\beta]$, $\{f,I\} \in \Pi^n[\alpha,\beta]$, the $D_{i,j}$, $i,j = 1,\dots,p$, are constant $n \times n$ matrices, and $K(t,s)$ is a square integrable matrix function on $[\alpha,\beta] \times [\alpha,\beta]$.

Let us study the existence and uniqueness of the solution of Eq. (7) and derive solvability conditions for the boundary value problem

$$x(\alpha) = a, \qquad x(\beta) = b, \qquad a,b \in R^n, \tag{8}$$

for this system.

Theorem 2. *Let system (7) satisfy the above-mentioned conditions. Then for every $x_0 \in R^n$, there exists a unique piecewise continuous solution $x(t) \in PAC[\alpha,\beta]$, $x(\alpha) = x_0$, of this system defined on the interval $[\alpha,\beta]$.*

Proof. Differentiating and verifying the jump conditions, we can show that the integro-sum equation

$$x(t) = x_0 + \int_\alpha^t A(s)x(s)ds + \int_\alpha^t\int_\alpha^\sigma K(\sigma,s)x(s)ds\,d\sigma + \sum_{\alpha<\theta_i<t} B_i x(\theta_i) + \int_\alpha^t f(s)ds$$
$$+ \sum_{\alpha<\theta_i<t}\sum_{\alpha<\theta_j\leq\theta_i} D_{ij}x(\theta_j) + \sum_{\alpha<\theta_i<t}\int_\alpha^{\theta_i} M_i(s)x(s)ds + \sum_{\alpha<\theta_i<t} I_i \tag{9}$$

is equivalent to Eq. (7) provided that $x(\alpha) = x_0$. Using the Fubini theorem and Lemma 2, we can rewrite the latter equation in the form

$$x(t) = \int_\alpha^t \Psi(t,s)x(s)ds + \sum_{\alpha<\theta_i<t} \Phi_i x(\theta_i) + \sum_{\alpha<\theta_i<t} I_i + F(t), \tag{10}$$

where

$$\Psi(t,s) = A(s) + \int_s^t K(\sigma,s)d\sigma + \sum_{s<\theta_j<t} M_j(s),$$

$$\Phi_i = B_i + \sum_{\theta_i\leq\theta_j<t} D_{ji}, \qquad F(t) = x_0 + \int_\alpha^t f(s)ds.$$

Equation (10) is an equation of the form (2) and hence, by Theorem 1, has a unique solution. The proof of the theorem is complete.

Now let us consider the system of integro-differential equations

$$\begin{aligned} \partial h(t,s)/\partial s &= -h(t,s)A(s) - \int_s^t h(t,\sigma)K(\sigma,s)d\sigma - \sum_{s\le\theta_j<t} h(t,\theta_j)M_i(s), \qquad s\ne\theta_i, \\ \Delta h(t,\theta_i) &= -h(t,\theta_i)B_i(E+B_i)^{-1} - \sum_{\theta_i\le\theta_j<t} h(t,\theta_j+)D_{ji}(E+B_i)^{-1}, \end{aligned} \tag{11}$$

where $h\in R^n$ is a row vector, $t\in[\alpha,\beta]$, A, K, D_{ij}, M_i, and B_i are defined in the same way as in system (7), and $\Delta h(t,\theta_i)\equiv h(t,\theta_i+)-h(t,\theta_i)$. Suppose that $\det\left(E-D_{jj}(E+B_j)^{-1}\right)\ne 0$ for all $j=1,\ldots,p$.

By analogy with Theorem 2, using Theorem 1 and Lemma 2, we can show that for each $h_0\in R^n$ system (11) has a unique solution $h(t,s)$ such that $h(t,t)=h_0$.

Let $H(t,s)=\operatorname{col}(H_1,H_2,H_3,\ldots,H_n)$ be the matrix such that $H(t,t)=E$ and the rows H_i, $i=1,\ldots,n$, are solutions of system (11).

Theorem 3. *Let $x(t)=x(t,\alpha,x_0)$ be a solution of the Cauchy problem for Eq. (7). Then*

$$x(t)=H(t,\alpha)x_0+\int_\alpha^t H(t,s)f(s)ds+\sum_{\alpha<\theta_i<t} H(t,\theta_j+)I_i. \tag{12}$$

Proof. Let $x(t)=x(t,\alpha,x_0)$ be a solution of Eq. (7), and let $\varphi(s)=H(t,s)x(s)$. We have [1, p. 20]

$$\varphi(t)-\varphi(\alpha)=\int_\alpha^t \varphi'(s)ds+\sum_{\alpha<\theta_i<t}\Delta\varphi(\theta_i). \tag{13}$$

We take some i. Then

$$\Delta\varphi(\theta_i)=H(t,\theta_i+)x(\theta_i+)-H(t,\theta_i)x(\theta_i)=H(t,\theta_i)\Delta x(\theta_i)+\Delta H(t,\theta_i)x(\theta_i+).$$

Summing both sides of the last relation over all i such that $\alpha<\theta_i<t$ and using Lemma 1 and the relation

$$\sum_{\alpha<\theta_i<t} H(t,\theta_i)\int_\alpha^t M_i(s)x(s)ds=\int_\alpha^t\left[\sum_{s\le\theta_i<t} H(t,\theta_i)M_i(s)\right]ds$$

(which can be proved by analogy with Lemma 2), we obtain

$$\begin{aligned} \sum_{\alpha<\theta_i<t}\Delta\varphi(\theta_i) &= \sum_{\alpha<\theta_i<t}\left[\Delta H(t,\theta_i)(E+B_i)+H(t,\theta_i)B_i\right]x(\theta_i)+\sum_{\alpha<\theta_i<t} H(t,\theta_i)\int_\alpha^{\theta_i} M_i(s)x(s)ds \\ &\quad+\sum_{\alpha<\theta_i<t}\sum_{\alpha<\theta_j\le\theta_i} H(t,\theta_i+)D_{ij}x(\theta_j)+\sum_{\alpha<\theta_i<t} H(t,\theta_i+)I_i \\ &= \sum_{\alpha<\theta_i<t}\left[\Delta H(t,\theta_i)(E+B_i)+H(t,\theta_i)B_i+\sum_{\theta_i\le\theta_j<t} H(t,\theta_j+)D_{ji}\right]x(\theta_i) \\ &\quad+\sum_{\alpha<\theta_i<t}\int_\alpha^{\theta_i} H(t,\theta_i)M_i(s)x(s)ds+\sum_{\alpha<\theta_i<t} H(t,\theta_i+)I_i. \end{aligned} \tag{14}$$

Differentiating the expression $\varphi(s) = H(t,s)x(s)$, we obtain

$$\varphi'(s) = (\partial H/\partial s)x(s) + H(t,s)\left[A(s)x(s) + \int_\alpha^t K(s,v)x(v)dv + f(s)\right].$$

This, together with the Fubini theorem, implies that

$$\int_\alpha^t \varphi'(s)ds = \int_\alpha^t \left[\partial H/\partial s + H(t,s)A(s) + \int_\alpha^t H(t,v)K(v,s)dv\right]x(s)ds + \int_\alpha^t H(t,s)f(s)ds. \tag{15}$$

Now, since H is a solution of Eq. (11), it follows from Eqs. (13)–(15) that

$$\begin{aligned}\varphi(t) - \varphi(\alpha) &= \int_\alpha^t \left[\partial H/\partial s + H(t,s)A(s) + \int_\alpha^t H(t,v)K(v,s)dv\right]x(s)ds + \int_\alpha^t H(t,s)f(s)ds \\ &+ \sum_{\theta_i\le\theta_i<t}\left[\Delta H(t,\theta_i)(E+B_i) + H(t,\theta_i)B_i + H(t,\theta_i+)D_{ij}\right]x(\theta_i) \\ &+ \int_\alpha^t \sum_{s\le\theta_i<t} H(t,\theta_i)M_i(s)x(s)ds + \sum_{\alpha<\theta_i<t} H(t,\theta_i+)I_i \\ &= \int_\alpha^t H(t,s)f(s)ds + \sum_{\alpha<\theta_i<t} H(t,\theta_i+)I_i.\end{aligned}$$

The proof of the theorem is complete.

Now we consider system (7) with the boundary conditions

$$x(\alpha) = 0, \qquad x(\beta) = 0. \tag{16}$$

The following assertion is an easy consequence of Theorem 3.

Theorem 4. *The boundary value problem* (7), (16) *is solvable if and only if*

$$\langle\{H_j(\beta,s), H_j(\beta,\theta_j)\}, \{f, I_j\}\rangle = 0$$

for all $j = 1, \ldots, p$.

Theorem 5. *Problem* (7), (8) *is solvable if and only if*

$$\langle\{H_j(\beta,s), H_j(\beta,\theta_j+)\}, \{f(s), I_j\}\rangle = H_j(\beta,\beta-)b - H_j(\beta,\alpha+)a, \qquad j = 1,\ldots,p.$$

Proof. We claim that there exists a continuous function $\varphi(t)$ such that $\varphi(\theta_j) = 0$, $j = 1,\ldots,p$, and the substitution $x(t) = y(t) + \varphi(t)$ reduces problem (7), (8) to the system

$$\begin{aligned}dy/dt &= A(t)y + \int_\alpha^t K(t,s)y(s)ds + f(t) - \left[\varphi'(t) - A(t)\varphi(t) - \int_\alpha^t K(t,s)\varphi(s)ds\right], \qquad t \ne \theta_i, \\ \Delta y(\theta_i) &= B_i y(\theta_i) + \sum_{\alpha<\theta_j\le\theta_i} D_{ij}y(\theta_j) + \int_\alpha^{\theta_i} M_i(s)x(s)ds + I_i\end{aligned} \tag{17}$$

with the boundary conditions

$$y(\alpha) = 0, \qquad y(\beta) = 0. \tag{18}$$

Indeed, let $\varphi_1(t)$ be the Lagrange polynomial such that $\varphi_1(\alpha) = a$, $\varphi_1(\beta) = b$, and $\varphi_1(\theta_j) = 0$, $j = 1, \ldots, p$. Substituting $x(t) = z(t) + \varphi_1(t)$ into Eqs. (7) and (8), we obtain

$$\begin{gathered} dz/dt = A(t)z + \int_\alpha^t K(t,s)z(s)ds + f(t) - \left[\varphi_1'(t) - A(t)\varphi_1(t) - \int_\alpha^t K(t,s)\varphi_1(s)ds\right], \qquad t \neq \theta_i, \\ \Delta z(\theta_i) = B_i z(\theta_i) + \sum_{\alpha<\theta_j\le\theta_i} D_{ij} z(\theta_j) + \int_\alpha^{\theta_i} M_i(s)z(s)ds + \int_\alpha^{\theta_i} M_i(s)\varphi_1(s)ds + I_i \end{gathered} \tag{19}$$

and

$$z(\alpha) = 0, \qquad z(\beta) = 0. \tag{20}$$

Now we take a Lagrange polynomial $\varphi_2^0(t)$ such that $\varphi_2^0(\alpha) = 0$, $\varphi_2^0(\beta) = 0$, and $\varphi_2^0(\theta_j) = 0$, $j = 1, \ldots, p$. Using this polynomial, we construct a function $\varphi_2(t)$ as follows. Let

$$k_1 = \int_\alpha^{\theta_1} M_1(s)\varphi_1(s)ds, \qquad p_1 = \int_\alpha^{\theta_i} M_1(s)\varphi_2^0(s)ds.$$

Then we set $\varphi_2(t) = -k_1 p_1^{-1}\varphi_2^0(t)$ for $t \in [\alpha, \theta_1]$. Now let

$$k_2 = \int_\alpha^{\theta_2} M_2(s)\varphi_1(s)ds + \int_\alpha^{\theta_1} M_2(s)\varphi_2(s)ds, \qquad p_2 = \int_{\theta_1}^{\theta_2} M_2(s)\varphi_2^0(s)ds.$$

We set $\varphi_2(t) = -k_2 p_2^{-1}\varphi_2^0(t)$ for $t \in (\theta_1, \theta_2]$.

Proceeding this way, we define $\varphi_2(t)$ on the entire interval $[\alpha, \beta]$ so that the substitution $z(t) = y(t) + \varphi_2(t)$ reduces problem (19), (20) to system (17) with the boundary conditions (18). Hence $\varphi(t) = \varphi_1(t) + \varphi_2(t)$.

By Theorem 4, problem (17), (18) is solvable if and only if

$$\langle \{H_j(\beta,s), H_j(\beta,\theta_j+)\}, \{F, I_j\}\rangle = 0, \qquad j = 1, \ldots, p, \tag{21}$$

where $F(t) = -\varphi_1'(t) + A(t)\varphi(t) + \int_\alpha^t K(t,s)\varphi(t)ds$. Integrating by parts in (21) and using the Fubini theorem, we complete the proof.

4. CONTROLLABILITY OF THE BOUNDARY VALUE PROBLEM

Consider the boundary value problem (8) for the system

$$\begin{gathered} dx/dt = A(t)x + \int_\alpha^t K(t,s)x(s)ds + C(t)u(t) + f(t), \qquad t \neq \theta_i, \\ \Delta x(\theta_i) = B_i x(\theta_i) + \sum_{\alpha<\theta_j\le\theta_i} D_{ij} x(\theta_j) + \int_\alpha^{\theta_i} M_i(s)x(s)ds + Q_i v_i + I_i. \end{gathered} \tag{22}$$

Here $x \in R^n$, A, K, M_i, and B_i, $i = 1, \ldots, p$, are the same matrices as in Eq. (7), $C(t)$ and Q_i, $i = 1, \ldots, p$, are $n \times m$ matrices, m is a given positive integer, the columns of $C(t)$ belong to $L_2^n[\alpha, \beta]$, the Q_i, $i = 1, \ldots, p$, are constant matrices, and the solutions of system (22) belong to $PAC[\alpha, \beta]$.

If for each element $\{f, I\} \in \Pi^n[\alpha, \beta]$ and for all $a, b \in R^n$, there exists a control $\{u, v\} \in \Pi^m[\alpha, \beta]$ such that problem (22), (8) is solvable, then we say that the *control problem* γ_1 is solvable. The problem γ_1 with $a = 0$ and $b = 0$ will be referred to as the *control problem* γ_2.

Lemma 3. *The control problem γ_1 is solvable if and only if so is the control problem γ_2.*

Proof. Let the problem γ_1 be solvable. Since the problem γ_2 is a special case of the problem γ_1, we find it is also solvable. Conversely, suppose that the problem γ_2 is solvable. Let $\varphi(t)$ be the same function as in the proof of Theorem 5. Replacing $x(t)$ by $y(t)+\varphi(t)$, we find that $y(t)$ satisfies the system

$$dy/dt = A(t)y + \int_\alpha^t K(t,s)y(s)ds + C(t)u(t) + [f(t) - \varphi'(t) + A(t)\varphi(t)], \qquad t \neq \theta_i,$$

$$\Delta y(\theta_i) = B_i y(\theta_i) + \sum_{\alpha<\theta_j\le\theta_i} D_{ij} y(\theta_j) + \int_\alpha^{\theta_i} M_i(s)y(s)ds + Q_i v_i + I_i$$

and the boundary conditions $y(\alpha) = 0$ and $y(\beta) = 0$. This problem is solvable by assumption. The proof is complete.

Theorem 6. *The problem γ_1 is solvable if and only if the trivial solution of system* (11) *satisfies the relation*

$$\langle \{Cu, Q_i v_i\}, \{h^{\mathrm T}, h^{\mathrm T}\}\rangle = 0 \qquad \forall \{u, v\} \in \Pi_p^m[\alpha,\beta]. \tag{23}$$

Proof. Sufficiency. Let $h(t,s)$, $h(t,t) = h$, $h \in R^n$, be a solution of system (11); then $h(t,s) = hH(t,s)$. Therefore, by the assumptions of the theorem, the infinite system

$$\langle \{Cu, Qv\}, \{H^{\mathrm T}(\beta,s)h^{\mathrm T}, H^{\mathrm T}(\beta,\theta_i)h^{\mathrm T}\}\rangle = 0, \qquad \forall \{u,v\} \in \Pi_p^m[\alpha,\beta],$$

has only the trivial solution $h = 0$. Let us show that there exist n elements $\{u^k, v^k\} \in \Pi_p^m[\alpha,\beta]$, $k = 1,\ldots,n$, such that $N = \left\langle \{Cu^k, Qv^k\}, \left\{H_j^{\mathrm T}, H_j^{\mathrm T}\right\}\right\rangle_{jk}$, $j,k = 1,\ldots,n$, is a nondegenerate matrix.

Suppose the contrary. We take some $\{u^k, v^k\} \in \Pi^m[\alpha,\beta]$, $k = 1,\ldots,n$. Without loss of generality, we can assume that the last row of N can be represented as a linear combination of the remaining rows. By h^0 we denote a nontrivial solution of the system

$$\langle \{Cu^k, Qv^k\}, \{H^{\mathrm T}h^{\mathrm T}, H^{\mathrm T}h^{\mathrm T}\}\rangle = 0, \qquad k = 1,\ldots,n-1. \tag{24}$$

For every $\{u,v\}$, there exist numbers μ_k, $k = 1,\ldots,n-1$, such that

$$\left\langle \{Cu, Qv\}, \left\{H_j^{\mathrm T}, H_j^{\mathrm T}\right\}\right\rangle = \sum_{k=1}^{n-1} \mu_k \left\langle \{Cu^k, Qv^k\}, \left\{H_j^{\mathrm T}, H_j^{\mathrm T}\right\}\right\rangle, \qquad j = 1,\ldots,n-1.$$

Therefore, relation (24) implies relation (23), where $h = h^0$.

Indeed, let $h^0 = (h_1^0, h_2^0, \ldots, h_n^0)$, $h_i^0 \in R$, $i = 1,\ldots,n$. Then we can write $h^0 H = \sum_{j=1}^n h_j^0 H_j(t,s)$. Hence

$$\begin{aligned}
\langle \{Cu, Qv\}, \{H^{\mathrm T}h^{0\mathrm T}, H^{\mathrm T}h^{0\mathrm T}\}\rangle &= \left\langle \{Cu, Qv\}, \left\{\sum_{j=1}^n H_j^{\mathrm T}(t,s)h_j^{0\mathrm T}, \sum_{j=1}^n H_j^{\mathrm T}(t,s)h_j^{0\mathrm T}\right\}\right\rangle \\
&= \sum_{k=1}^n \left\langle \{Cu, Qv\}, \left\{H_j^{\mathrm T}, H_j^{\mathrm T}\right\}\right\rangle h_j^{0\mathrm T} = \sum_{j=1}^n \left[\sum_{k=1}^{n-1} \mu_k \left\langle \{Cu^k, Qv^k\}, \left\{H_j^{\mathrm T}, H_j^{\mathrm T}\right\}\right\rangle\right] h_j^{0\mathrm T} \\
&= \sum_{k=1}^{n-1} \mu_k \sum_{j=1}^n \left\langle \{Cu^k, Qv^k\}, \left\{H_j^{\mathrm T}, H_j^{\mathrm T}\right\}\right\rangle h_j^{0\mathrm T} = \sum_{k=1}^{n-1} \mu_k \langle \{Cu^k, Qv^k\}, \{H^{\mathrm T}h^{0\mathrm T}, H^{\mathrm T}h^{0\mathrm T}\}\rangle \\
&= \sum_{k=1}^{n-1} \mu_k \times 0 = 0.
\end{aligned}$$

Thus, $h^0 H$ is a nontrivial solution of Eq. (11) satisfying (23). We see that N is necessarily a nondegenerate matrix. Now we consider the boundary value problem (16) for the system

$$\begin{aligned} dx/dt &= A(t)x + \int_\alpha^{\mathrm{T}} K(t,s)x(s)ds - C(t)\sum_{j=1}^{n} m_k u^k + f(t), \qquad t \neq \theta_i, \\ \Delta x(\theta_i) &= B_i x(\theta_i) + \sum_{\alpha<\theta_j\le\theta_i} D_{ij} x(\theta_j) + \int_\alpha^{\theta_i} M_i(s)x(s)ds - Q_i \sum_{j=1}^{n} m_k v_i^k + I_i, \end{aligned} \tag{25}$$

where the $\{u^k, v^k\} \in \Pi^m[\alpha,\beta]$ have been defined above and $m_k \in R^1$, $k = 1,\dots,n$. Since N is a nondegenerate matrix, it follows that the system

$$\sum_{k=1}^{n} \left\langle \{Cu^k, Qv^k\}, \left\{H_j^{\mathrm{T}}, H_j^{\mathrm{T}}\right\}\right\rangle m_k = \left\langle \{f, I\}, \left\{H_j^{\mathrm{T}}, H_j^{\mathrm{T}}\right\}\right\rangle, \qquad j = 1,\dots,n,$$

is solvable for the m_k; therefore, by Theorem 4, problem (25), (16) is solvable.

Necessity. Suppose the contrary: the problem γ_1 is solvable and system (11) has a nontrivial solution h satisfying Eq. (23). We can readily show that there exists an element $\{f, I\} \in \Pi^n[\alpha,\beta]$ such that $\langle \{f, I\}, \{h^{\mathrm{T}}, h^{\mathrm{T}}\}\rangle \neq 0$. We take some element with this property. Then, adding the last inequality to Eq. (23), we see that the pair $\{f, I\} \in \Pi^n[\alpha,\beta]$ satisfies $\langle \{Cu + f, Qv + I\}, \{h^{\mathrm{T}}, h^{\mathrm{T}}\}\rangle \neq 0$ for all $\{u, v\}$. Since this contradicts Theorem 4, we arrive at the desired assertion.

It follows from the last theorem that the problems γ_1 and γ_2 are solvable if and only if the system $\langle \{Cu, Qv\}, \{H^{\mathrm{T}}(\beta,s)h^{\mathrm{T}}, H^{\mathrm{T}}(\beta,\theta_i)\, h^{\mathrm{T}}\}\rangle = 0$ has only the trivial solution for $h \in R^n$ for every $\{u, v\} \in \Pi_p^m[\alpha,\beta]$. Moreover, the problems γ_1 and γ_2 are solvable if and only if $\det(H(\beta,t)C(t)) \neq 0$ and $\det(H(\beta,\theta_i)\, Q_i) \neq 0$, $i = 1,\dots,p$, for all $t \in [\alpha,\beta]$.

Now let $\mathbf{\Gamma}$ be the Gram matrix of the elements $\{H_j C, H_j(\beta,\theta_j+)\, Q_j\}$, $j = 1,\dots,n$, i.e.,

$$\mathbf{\Gamma} = \int_\alpha^\beta H(\beta,t)C(t)C^{\mathrm{T}}(t)H^{\mathrm{T}}(\beta,t)dt + \sum_{i=1}^{p} H(\beta,\theta_i+)\, Q_i Q_i^{\mathrm{T}} H^{\mathrm{T}}(\beta,\theta_i).$$

Theorem 7. *The problem γ_1 is solvable if and only if the Gram matrix $\mathbf{\Gamma}$ is nondegenerate.*

Proof. Let the problem γ_1 be solvable. By Theorem 6, system (23) has only the trivial solution $h = 0$. Setting $\{u, v\} = \{C^{\mathrm{T}}(t)H^{\mathrm{T}}(\beta,t), Q_i^{\mathrm{T}} H^{\mathrm{T}}(\beta,\theta_i)\}$ in this equation, we find that the system $h\mathbf{\Gamma} = 0$ has only the trivial solution. Conversely, if the equation $h\mathbf{\Gamma} = 0$ has only the trivial solution, then system (23) has only the solution $h = 0$. The proof is complete. Let

$$\mathbf{K} = \mathbf{\Gamma}^{-1}\left\{ H(\beta,\beta)b - H(\beta,\alpha)a - \int_\alpha^\beta H(\beta,t)f(t)dt - \sum_{i=1}^{p} H(\beta,\theta_i+)\, I_i \right\},$$

$$S(t) = H(\beta,t)C(t), \qquad P_i = H(\beta,\theta_i+)\, Q_i.$$

From Theorem 6, we obtain the following assertion.

Theorem 8. *Suppose that the problem γ_1 is solvable. Then the control $\{U, V\}$, where*

$$U = S^{\mathrm{T}}(t)\mathbf{K}, \qquad V_i = P_i^{\mathrm{T}}\mathbf{K},$$

is a solution of the problem γ_1.

Proof. By Theorem 5, the problem γ_1 is solvable if and only if

$$\int_\alpha^\beta H(\beta,t)[f(t)+C(t)u(t)]dt+\sum_{i=1}^{p} H(\beta,\theta_i)\left[I_i+Q_i v_i\right]=H(\beta,\beta-)b-H(\beta,\alpha+)a. \tag{26}$$

Substituting the expressions

$$U=C^{\mathrm{T}}(t)H^{\mathrm{T}}(\beta,t)h^{\mathrm{T}}, \qquad V_i=Q_i H^{\mathrm{T}}(\beta,\theta_i)\,h^{\mathrm{T}} \tag{27}$$

for $\{u,v\}$ into Eq. (26), we obtain a system of linear equations for h. Using the solution of this system in Eq. (27), we obtain the desired expression for $\{U,V\}$. The proof of the theorem is complete.

The control $\{U,V\}$ allows one to describe the set of all controls solving the problem γ_1.

Theorem 9. *A control $\{u,v\}$ solves the problem γ_1 if and only if it has the form $u=U+\xi$, $v_i=V_i+\nu_i$, where $\{\xi,\nu\}\in\Pi^m[\alpha,\beta]$ is orthogonal to all columns of the matrix $\{S^{\mathrm{T}}(t),P_i^{\mathrm{T}}\}$ in $\Pi^m[\alpha,\beta]$.*

Proof. Indeed, let $\{u,v\}$ be a control solving the problem γ_1. Then

$$\int_\alpha^\beta S(t)(u(t)-U(t))dt+\sum_{i=1}^{p} P_i\left(v_i-V_i\right)=0.$$

If we assume that $\xi=u-U$ and $\nu_i=v_i-V_i$, then we obtain the desired assertion.

Conversely, suppose that $u=U+\xi$ and $v_i=V_i+\nu_i$. Then condition (26) is satisfied and the control $\{u,v\}$ solves the problem γ_1.

We equip $\Pi^m[\alpha,\beta]$ with the norm $\|\{u,v\}\|_m=\langle\{u,v\},\{u,v\}\rangle^{1/2}$. Following [10, p. 157], we can show that $\{U,V\}$ has the least norm in $\Pi^m[\alpha,\beta]$ of all controls solving the problem γ_1.

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