A NONREFLEXIVE BANACH SPACE THAT IS UNIFORMLY NONOCTAHEDRAL†

BY

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ABSTRACT

An example is given of a nonreflexive Banach space $\tilde{X}$ that is uniformly nonoctahedral (or uniformly non-$l_1^{(3)}$), in the sense that there is a $\lambda > 1$ such that there is no isomorphism T of $l_1^{(3)}$ into $\tilde{X}$ for which

$$\lambda^{-1}\|x\| \leqq \|T(x)\| \leqq \lambda\|x\| \quad \text{if} \quad x \in l_1^{(3)}.$$

It is known that a Banach space B is reflexive if B is uniformly convex (see [16] and [18]) or if B is uniformly nonsquare [13, Th. 1.1, p. 543], and that B^{**}/B is reflexive if B is uniformly nonoctahedral [4]. It has been conjectured that a Banach space is reflexive if it is uniformly nonoctahedral. It is known that a Banach space is super-reflexive if and only if it is isomorphic to a uniformly convex space or if and only if it is isomorphic to a uniformly nonsquare space [5]. It has been conjectured that a Banach space is super-reflexive if and only if it is isomorphic to a uniformly nonoctahedral space. The example $\tilde{X}$ shows that both of these conjectures are false.

It has also been conjectured that l_1 is finitely representable in each nonreflexive Banach space X; that is, for each n and each $\lambda > 1$ there is an isomorphism T_n of $l_1^{(n)}$ onto a subspace of X for which

$$\lambda^{-1}\|x\| \leqq \|T_n(x)\| \leqq \lambda\|x\| \quad \text{if} \quad x \in l_1^{(n)}.$$

This conjecture is known to be true if X has an unconditional basis or is a Banach lattice, in fact, X then contains a subspace isomorphic to c_0 or l_1 (see [7, Th. 6,

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p. 142], [10], [13, Th. 2.2, p. 549], [17, Th. 16, p. 647]).It is also true if X has local unconditional structure [15]; or if $R^k(X)$ is nonreflexive for all k, where $R^1(Y) = Y^{**}/Y$ and $R^{k+1}(Y) = R^1[R^k(Y)]$ for $k \geqq 1$ and Y any Banach space [4]. It was shown by A. Beck that a certain law of large numbers for random variables with values in a Banach space X is valid if and only if l_1 is not finitely representable in X; he called such a space *B-convex* [1].

The nonreflexive Banach space $\tilde{X}$ to be defined is uniformly nonoctahedral because there is a positive number Δ such that, if $\|x\| = \|y\| = \|z\| = 1$, then there is an arrangement of signs for which

$$\|x \pm y \pm z\| < 3 - \Delta. \tag{1}$$

The number Δ is rather small in comparison to the known fact that if $\delta > 3/4$, then in the unit ball of each nonreflexive Banach space there are elements x, y and z for which $\|x \pm y \pm z\| \geqq 3 - \delta$ for all choices of signs [8]. However, it is known that if $l_1^{(n)}$ is uniformly representable in a space Y (that is, l_1 is crudely finitely representable in Y in the sense that there is a $\lambda > 1$ such that for each n there is an isomorphism T_n of $l_1^{(n)}$ onto a subspace of Y for which $\lambda^{-1}\|x\| \leqq \|T_n(x)\| \leqq \lambda\|x\|$ if $x \in l_1^{(n)}$), then l_1 is finitely representable in Y [6, Th. 1, p. 62]. Thus, however small Δ may be for the space $\tilde{X}$, it follows that, for any $\lambda > 1$, there is an n for which there does not exist an isomorphism T of $l_1^{(n)}$ onto a subspace of $\tilde{X}$ with the property that

$$\lambda^{-1}\|x\| \leqq \|T(x)\| \leqq \lambda\|x\| \text{ if } x \in l_1^{(n)}.$$

It is known that if X is a nonreflexive Banach space and $\delta > 0$, then there are n members $\{x_i\}$ of X such that $\|x_i\| = 1$ for each i and

$$\|x_1 \pm x_2 \pm \cdots \pm x_n\| > n - \delta \tag{2}$$

for n choices of signs. These choices of signs can be the ones for which all positive signs precede all negative signs (see [13, Th. 2.1, p. 547] and [19, Th. 2.2, p. 164]). Now note that it follows by induction that if we are given $n + 1$ elements of type $x_1 \pm x_2 \pm x_3 \pm \cdots \pm x_n$, then there exist i and j for which $x_1 \pm x_i \pm x_j$ occurs as part of one of these $n + 1$ elements for each of the four possible choices of signs (if such i and j do not exist, then there must be at most $n-1$ different combinations remaining if x_n is discarded from each of the given $n+1$ combinations, but at least two of these must have occurred with both $+x_n$ and $-x_n$). Therefore if there are $n+1$ choices of signs for which (2) is satisfied and if i and j are chosen in this way, then

$$\|x_1 \pm x_i \pm x_j\| > (n-\delta)-(n-3) = 3-\delta$$

for all choices of signs. Thus it follows from the theorem of this paper that there exists a nonreflexive Banach space $\tilde{X}$ for which $l_1^{(n)}$ cannot be embedded isomorphically in X with all points on $n+1$ of the faces of the unit ball of $l_1^{(n)}$ having norms nearly equal to one.

A Banach space is said to be *stable* if for each bounded sequence $\{x_n\}$ there is a subsequence $\{x_{n_i}\}$ and an x such that, for each $\varepsilon > 0$, there is an N for which

$$\|x - n^{-1}(x_{k_1} + \cdots + x_{k_n})\| < \varepsilon$$

if $\{k_i\}$ is a strictly increasing sequence of positive integers and $n \geqq N$; the space is *ergodic for isometries* if, for each linear isometry T, the sequence of Cesaro averages $\{n^{-1}(T^1 + \cdots + T^n)\}$ converges in the strong operator topology; and the space is (r,ε)-*convex* if, whenever $\|x_i\| \leqq 1$ for $i \leqq r$, there is a choice of signs such that

$$\|x_1 \pm x_2 \pm \cdots \pm x_r\| < r(1-\varepsilon).$$

It is known [2, Th. 1] that all (r,ε)-convex spaces ($r \geqq 2$) are stable if all (r,ε)-convex spaces are ergodic for isometries. Since all stable spaces are reflexive, it follows from the theorem of this paper that if $\varepsilon < (.0432+)/3$, there is a $(3,\varepsilon)$-convex space that is not ergodic for isometries. Also, there exists a reflexive stable space which is B-convex and not super-reflexive [9, Ex. 3]. Surprisingly there is a Banach space that has an equal-signs-additive basis and is B-convex, linearly isometric to its second conjugate, and quasi-reflexive but not reflexive [3].

The reader may wish to convince himself that the example constructed in the proof of the theorem behaves much as the quasi-reflexive spaces discussed in [11] and [12]. It has a natural basis $\{e_n\}$ for which the nth component of e_n is 1 and all other components are zero. This basis is monotone and shrinking. The space is quasi-reflexive of order one; that is, its natural image in the second dual has codimension 1.

The following special conventions will be used. A *bump* is a sequence of real numbers $x = \{x_n\}$ for which there is a bounded interval I and a number a such that $x_n = a$ if $n \in I$ and $x_n = 0$ if $n \notin I$. The *altitude* of the bump is a, its *sign* is $\mathrm{sg}(a)$, and the *left* and *right ends* are the first and last integers in I. For two bumps with associated intervals I and J, the two bumps are *disjoint* if $I \cap J$ is empty, they *intersect* if $I \cap J$ is not empty, and the first bump *contains* the second if $I \supset J$. The following combinatorial lemma will be a basic tool for estimating the number Δ in (1).

LEMMA. *Let ξ, η, and ζ be three sequences of real numbers each of which s a finite sum of disjoint positive bumps of altitude* 1. *Let the respective numbers of bumps be p, q, and r. Then, for each arrangement of signs σ for which at most one sign is negative, there is a set of three sequences $\{\xi_\sigma, \eta_\sigma, \zeta_\sigma\}$ such that ξ_σ and ζ_σ are sums of disjoint bumps of altitude* 1, *η_σ is the sum of disjoint bumps of altitudes $+1$ or -1 according as all signs in σ are positive or exactly one sign in σ is negative, and:*

(i) $(\pm \xi \pm \eta \pm \zeta)_\sigma = \xi_\sigma + \eta_\sigma + \zeta_\sigma$;

(ii) $p_\sigma + q_\sigma + r_\sigma \leqq p + q + r$ *for each σ, where p_σ, q_σ, and r_σ are the numbers of bumps in ξ_σ, η_σ, and ζ_σ, respectively;*

(iii) $\sum_{\sigma=1}^{4} (p_\sigma + q_\sigma) \leqq 2(p + q + r)$.

PROOF. It is sufficient to prove the lemma for the case the support of $\xi + \eta + \zeta$ is an interval of integers, since otherwise ξ_σ, η_σ, and ζ_σ could be defined piece-wise on the intervals whose union is the support of $\xi + \eta + \zeta$ and which have the property that any two intervals are separated by integers at which each of ξ, η, and ζ is zero. Assuming the support of $\xi + \eta + \zeta$ is an interval of integers, we represent it as the union of intervals of maximal length which have the property that each of ξ, η, and ζ is constant on each interval. We shall let Ω denote the collection of such intervals and let α be the number of intervals in Ω on which exactly one of ξ, η, or ζ is nonzero, β the number of intervals in Ω on which exactly two of ξ, η, or ζ are nonzero, and γ the number of intervals in Ω on which all of ξ, η, and ζ are nonzero. Let us observe that

$$\alpha + \beta + \gamma \leqq 2(p + q + r) - 1. \tag{3}$$

That this inequality is true can be seen by noting that the intervals in Ω are disjoint; their union is an interval; for two consecutive intervals I and J in Ω, either the right end of I or the left end of J is an end of a bump; the left end of the first interval and the right end of the last interval are ends of bumps; and there are at most $2(p + q + r)$ ends of bumps.

If σ denotes the arrangement of signs in $\pm \xi \pm \eta \pm \zeta$ for which all signs are positive, let ξ_σ be identically 1 on the support of $\xi + \eta + \zeta$ and let η_σ be identically 1 on all intervals in Ω on which all of ξ, η, and ζ are nonzero. Then $\xi + \eta + \zeta - (\xi_\sigma + \eta_\sigma) = \zeta_\sigma$ is a sum of disjoint bumps of altitude 1. Note that ξ_σ has 1 bump and η_σ has γ bumps.

Now suppose σ denotes an arrangement of signs in $\pm\xi\pm\eta\pm\zeta$ for which exactly one sign is negative. Let ξ_σ be identically 1 on all intervals in Ω on which $(\pm\xi\pm\eta\pm\zeta)_\sigma$ is identically 2; that is, on all intervals in Ω on which the two of $\pm\xi$, $\pm\eta$, and $\pm\zeta$ with positive signs are identically 1 and the other is identically 0. Then

$$\sum_{\sigma=1}^{4} p_\sigma = 1 + \beta. \tag{4}$$

Let η_σ be identically -1 on all intervals in Ω on which $(\pm\xi\pm\eta\pm\zeta)_\sigma$ is identically -1; that is, on all intervals in Ω on which the one of $\pm\xi$, $\pm\eta$, and $\pm\zeta$ with negative sign is identically -1 and the others are identically 0. Then

$$\sum_{\sigma=1}^{4} q_\sigma = \alpha + \gamma, \tag{5}$$

and $(\pm\xi\pm\eta\pm\zeta)_\sigma - (\xi_\sigma + \eta_\sigma) = \zeta_\sigma$ is a sum of disjoint bumps of altitude 1.

To show that (ii) is satisfied, we note first that the variation of $\xi_\sigma + \eta_\sigma + \zeta_\sigma$ is $2(p_\sigma + q_\sigma + r_\sigma)$. Then it follows from (i) that the variation of $(\pm\xi\pm\eta\pm\zeta)_\sigma$ is $2(p_\sigma + q_\sigma + r_\sigma)$. Since the variations of ξ, η, and ζ are $2p$, $2q$, and $2r$, we can conclude that (ii) is satisfied. It follows from (3), (4), and (5) that

$$\sum_{\sigma=1}^{4} (p_\sigma + q_\sigma) = 1 + \alpha + \beta + \gamma \leqq 2(p + q + r),$$

so (iii) is satisfied.

THEOREM. *Let* Δ *satisfy* $\Delta < 3 - (3^{\frac{1}{2}} + \frac{1}{2}\cdot 6^{\frac{1}{2}}) = .0432+$. *Then there is a nonreflexive Banach space* $\tilde{X}$ *such that, if* x, y, *and* z *are members of* $\tilde{X}$ *for which* $\|x\| = \|y\| = \|z\| = 1$, *then there is an arrangement of signs for which*

$$\|x \pm y \pm z\| < 3 - \Delta.$$

PROOF. Choose θ so that $0 < \theta < 1$ and

$$3 - (3^{\frac{1}{2}} + \tfrac{1}{2}\cdot 6^{\frac{1}{2}})\theta^{-7} > \Delta. \tag{6}$$

Since $u = v = w = (1 + 2\theta^3)^{-1}$ is a solution of the system

$$u + \theta^3 v + \theta^3 w = 1, \quad \theta^3 u + v + \theta^3 w = 1, \quad \theta^3 u + \theta^3 v + w = 1,$$

we can choose λ so that $\theta^3 < \lambda < 1$ and, if $\theta^3 < a_{ij} \leqq \theta^3/\lambda$ for all i and j, then there are positive values of u, v, and w for which

$$u + a_{12}v + a_{13}w = 1,\ a_{21}u + v + a_{23}w = 1,\ a_{31}u + a_{32}v + w = 1. \tag{7}$$

Now define a functional $[\![\ \ ]\!]$ whose domain is the set of all sequences x of real numbers which have support in a bounded interval I and also have representations as $\Sigma_1^\infty x^i$, where each x^i is the sum of m_i disjoint bumps with support in I and equal altitudes a_i and the quotient of the altitude of a bump of x^j and the altitude of a bump of x^i for $i < j$ is λ^p for some positive integer p (and therefore the bumps of x^i and of x^j have the same sign for all i and j). For each such x, let

$$[\![x]\!] = \inf\left\{\left(\sum_1^\infty m_i a_i^2\right)^{\frac{1}{2}} : x = \sum_1^\infty x^i \text{ as described above}\right\}. \tag{8}$$

For an arbitrary sequence x with finite support, let

$$\|x\| = \inf\left\{\sum_1^m [\![x^k]\!] : x = \sum_1^m x^k\right\},$$

and let X be the resulting normed linear space. Let us verify simultaneously that $\|x\| > 0$ if $x \neq 0$ and that the completion $\tilde{X}$ of X is not reflexive (see [14, Th. 3(23), p. 109]), by noting that:

(a) $\|\{x_i\}\| \leqq 1$ if $x_i = 1$ when $i \leqq n$ and $x_i = 0$ when $i > n$;

(b) $\|x\| \geqq (1-\lambda)\sup\{|x_i|\}$.

To establish (b), observe that if the altitudes of the bumps making up an $x^k = \{x_i^k\}$ are $a\lambda^{j-1}$ for $j \geqq 1$, then we have

$$\sup\{|x_i^k|\} \leqq \left|\sum_{j=1}^\infty a\lambda^{j-1}\right| = \frac{|a|}{1-\lambda} \leqq \frac{[\![x^k]\!]}{1-\lambda},$$

so that $[\![x^k]\!] \geqq (1-\lambda)\sup\{|x_i^k|\}$ and it follows from the definition of $\|\cdot\|$ that $\|x\| \geqq (1-\lambda)\sup\{|x_i|\}$.

For the definition of $\|\ \ \|$ to be valid, it is necessary that the domain of $[\![\ \ ]\!]$ be such that each x in X is the sum of members of the domain of $[\![\ \ ]\!]$. This is rather clearly satisfied, since each x in X has only finitely many nonzero components and each x with only one nonzero component is in the domain of $[\![\ \ ]\!]$. Actually, the domain of $[\![\ \ ]\!]$ is the set of all $x \in X$ whose nonzero components all have the same sign. To see this, suppose $x = \{x_i\}$ with $x_i \geqq 0$ for all i and choose any

a so that $a(\Sigma_0^\infty \lambda^i) > \sup\{x_i\}$. Now define x^1 by letting $x^1 = \{\varepsilon_i^1 a\}$, where $\varepsilon_i^1 = 1$ if $x_i \geqq a$ and $\varepsilon_i^1 = 0$ otherwise; then let $x^2 = \{\varepsilon_i^2 a\lambda\}$, where $\varepsilon_i^2 = 1$ if $x_i - x_i^1 \geqq a\lambda$ and $\varepsilon_i^2 = 0$ otherwise; and, in general, let $x^k = \{\varepsilon_i^k a\lambda^{k-1}\}$, where $\varepsilon_i^k = 1$ if $x_i - \Sigma_{j=1}^{k-1} x_i^j \geqq a\lambda^{k-1}$ and $\varepsilon_i^k = 0$ otherwise.

If $[\![x]\!]$ exists, $[\![x]\!]$ is actually attained as $(\Sigma_1^\infty m_i a_i^2)^{\frac{1}{2}}$. To see this, note first that if $x_i = 0$ when $i > N$, then in (8) we can always have $m_i \leqq N$. Therefore, if

$$\lim_{k\to\infty}\left[\sum_{i=1}^{\infty} m_i^k(\lambda^{i-1}a_k)^2\right]^{\frac{1}{2}} = [\![x]\!]$$

with $m_1^k \neq 0$ for all k, and if a is an accumulation point of $\{a_k : k \geqq 1\}$, then there are numbers $m_i \geqq 0$ for which

$$[\![x]\!] = \left[\sum_1^\infty m_i(\lambda^{i-1}a)^2\right]^{\frac{1}{2}}.$$

A brief sketch will be given of a proof that $[\![x+y]\!] \leqq [\![x]\!] + [\![y]\!]$ need not be true even if x, y, and $x+y$ are in the domain of $[\![\quad]\!]$. The purpose of this is to show that the introduction of $\|\quad\|$ was badly needed, both because the domain of $[\![\quad]\!]$ is not X and because $[\![\quad]\!]$ does not satisfy the triangle inequality. It can be shown that, subject to $m_1 \geqq 1$, m_1 and each m_i being a nonnegative integer, and

$$a(1 + \sum_1^\infty \varepsilon_i\lambda^i) = 1 \text{ with } \varepsilon_i = 0 \text{ or } 1$$

according as $m_i > 0$ or $m_i = 0$, the minimum of $a^2(m_1 + \Sigma_1^\infty m_i\lambda^{2i})$ is $(1-\lambda)(1+\lambda)^{-1}$ and this minimum is attained only when $m_i = 1$ for all i and $a = 1-\lambda$. Now choose $\delta < 1$ and an integer n so that $n[\delta(1-\lambda)] = 1$. For each $k \leqq n$, let x^k have constant value $\delta(1-\lambda)$ on the interval $[1,k]$ and zero terms otherwise. Then $[\![x^k]\!] = \delta(1-\lambda)^{3/2}(1+\lambda)^{-1/2}$. Also, if $[\![x+y]\!] \leqq [\![x]\!] + [\![y]\!]$ whenever x, y, and $x+y$ are in the domain of $[\![\quad]\!]$, then

$$[\![\sum_1^n x^k]\!] \leqq (1-\lambda)^{\frac{1}{2}}(1+\lambda)^{-\frac{1}{2}}.$$

However, there is a number a for which there are bumps with altitudes $a\lambda^{p(i)}$ such that $p(0) = 0$ and

$$\left[\!\!\left[\sum_1^n x^k\right]\!\!\right] = a\left[1 + \sum_1^\infty \lambda^{2p(i)}\right]^{\frac{1}{2}}, \; a\left[1 + \sum_1^\infty \lambda^{p(i)}\right] = 1.$$

But then $p(i) = i$ for all i, $a = 1-\lambda$, and $\Sigma_1^n x^k$ is the sum of bumps of altitudes

$(1-\lambda)\lambda^i$ for $i \geqq 0$, as well as the sum of n unequal bumps of altitudes $\delta(1-\lambda) < 1-\lambda$. This is not possible.

Now let x, y, and z be arbitrary members of X for which $\|x\| = \|y\| = \|z\| = 1$. Choose representations for x, y, and z so that $x = \Sigma_1^p x_1^k$, $y = \Sigma_1^q y_1^k$, $z = \Sigma_1^r z_1^k$ and

$$\|x\| > \theta \sum_1^p [\![x_1^k]\!],\ \|y\| > \theta \sum_1^q [\![y_1^k]\!],\ \ \|z\| > \theta \sum_1^r [\![z_1^k]\!].$$

Next, replace these representations by $\{(x_2^k, y_2^k, z_2^k): k \leqq s\}$, so that

$$\|x\| > \theta \sum_1^s [\![x_2^k]\!],\ \ \|y\| > \theta \sum_1^s [\![y_2^k]\!],\ \ \|z\| > \theta \sum_1^s [\![z_2^k]\!],$$

where $x = \Sigma_1^s x_2^k$, $y = \Sigma_1^s y_2^k$, $z = \Sigma_1^s z_2^k$, and

$$\inf\{[\![x_2^k]\!], [\![y_2^k]\!], [\![z_2^k]\!]\} \geqq \theta^3 \sup\{[\![x_2^k]\!], [\![y_2^k]\!], [\![z_2^k]\!]\}. \tag{9}$$

This can be done by choosing s large enough so that, for each x_1^k, there is an integer n_k such that $\Sigma_1^p n_k = s$ and

$$\frac{\theta}{s} \leqq \frac{\theta \sum_1^p [\![x_1^k]\!]}{s} < \frac{[\![x_1^k]\!]}{n_k} < \frac{\theta^{-1} \sum_1^p [\![x_1^k]\!]}{s} < \frac{\theta^{-2}}{s},$$

with similar statements for y_1^k and z_1^k. Then divide each x_1^k into n_k vectors, each equal to x_1^k/n_k, with similar divisions of each y_1^k and z_1^k. These give the $\{x_2^k\}$, $\{y_2^k\}$, and $\{z_2^k\}$. They satisfy (9) however they are ordered.

Finally, choose $\{(x_k, y_k, z_k): 1 \leqq i \leqq n\}$ so that

$$\|x\| > \theta \sum_1^n [\![x^k]\!],\ \|y\| > \theta \sum_1^n [\![y^k]\!],\ \|z\| > \theta \sum_1^n [\![z^k]\!], \tag{10}$$

where $x = \Sigma_1^n x^k$, $y = \Sigma_1^n y^k$, $z = \Sigma_1^n z^k$,

$$\inf\{[\![x^k]\!], [\![y^k]\!], [\![z^k]\!]\} \geqq \theta^6 \cdot \sup\{[\![x^k]\!], [\![y^k]\!], [\![z^k]\!]\}, \tag{11}$$

and there are representations, $x^k = \Sigma_{i=1}^\infty z^{k,i}$, $y^k = \Sigma_{i=1}^\infty y^{k,i}$, $z^k = \Sigma_{i=1}^\infty z^{k,i}$, for which $[\![x^k]\!]$, $[\![y^k]\!]$, and $[\![z^k]\!]$ can be evaluated by use of (8) and x^k, y^k, and z^k have λ-compatible altitudes (that is, the quotient of the altitudes of two bumps is $\pm\lambda^p$ for some integer p if each of these bumps is used in one of x^k, y^k, or z^k). This is done as follows. For each k, choose $\{a_{ij}\}$ so that $\theta^3 < a_{ij} \leqq \theta^3/\lambda$; $(x_2^k, a_{21}y_2^k, a_{31}z_2^k)$ have λ-compatible altitudes; $(a_{12}x_2^k, y_2^k, a_{32}z_2^k)$ have λ-compatible altitudes; and $(a_{13}x_2^k, a_{23}y_2^k, z_2^k)$ have λ-compatible altitudes. Choose a solution (u, v, w) of (7) for which u, v, and w are positive and let each of $(ux_2^k, ua_{21}y_2^k, ua_{31}z_2^k)$, $(va_{12}x_2^k, vy_2^k, va_{32}z_2^k)$, $(wa_{13}x_2^k, wa_{23}y_2^k, wz_2^k)$ form a triple of type $(x^\kappa, y^\kappa, z^\kappa)$. The vector sum of these three new triples is (x_2^k, y_2^k, z_2^k) Inequality (11) follows from (9) and the fact that $\theta^3 < a_{ij} < 1$ for each a_{ij}.

In what follows, we let (ξ, η, ζ) denote $(\pm x^k, \pm y^k, \pm z^k)$ for a particular value of k and a particular arrangement of signs. Then

$$\inf\{[\![\xi]\!], [\![\eta]\!], [\![\zeta]\!]\} > \theta^6 \cdot \sup\{[\![\xi]\!], [\![\eta]\!], [\![\zeta]\!]\}. \tag{12}$$

Also, ξ, η, and ζ have representations as $\xi = \Sigma_1^\infty \xi^i$, $\eta = \Sigma_1^\infty \eta^i$, $\zeta = \Sigma_1^\infty \zeta^i$ for which

$$[\![\xi]\!] = \left[\sum_1^\infty p_i(a_i)^2\right]^{\frac{1}{2}}, [\![\eta]\!] = \left[\sum_1^\infty q_i(a_i)^2\right]^{\frac{1}{2}}, [\![\zeta]\!] = \left[\sum_1^\infty r_i(a_i)^2\right]^{\frac{1}{2}},$$

where ξ^i, η^i, and ζ^i have bumps of altitude a_i and the respective numbers of bumps are p_i, q_i and r_i for each i. The quotient a_j/a_i for $i < j$ is λ^p for some positive integer p. We assume that the signs in $(\pm x^k, \pm y^k, \pm z^k)$ have been chosen so that $a_i > 0$ and all bumps used in ξ^i, η^i, or ζ^i have positive altitudes for all i.

It follows from the lemma that, for each i and each arrangement of signs σ for which at most one sign is negative, we can replace $\{\xi^i, \eta^i, \zeta^i\}$ by $\{\xi^i_\sigma, \eta^i_\sigma, \zeta^i_\sigma\}$ so that ξ^i_σ and ζ^i_σ are sums of disjoint bumps of altitude a_i, η^i_σ is the sum of disjoint bumps of altitude a_i or $-a_i$ according as all signs in σ are positive or exactly one sign is negative, and:

(a) $(\pm \xi^i \pm \eta^i \pm \zeta^i)_\sigma = \xi^i_\sigma + \eta^i_\sigma + \zeta^i_\sigma$;

(b) $p_{\sigma,i} + q_{\sigma,i} + r_{\sigma,i} \leqq p_i + q_i + r_i$ for each σ, where $p_{\sigma,i}$, $q_{\sigma,i}$, and $r_{\sigma,i}$ are the numbers of bumps in ξ^i_σ, η^i_σ, and ζ^i_σ, respectively;

(c) $\Sigma_{\sigma=1}^4 (p_{\sigma,i} + q_{\sigma,i}) \leqq 2(p_i + q_i + r_i)$.

Let $(A_\sigma)^2 = \Sigma_{i=1}^\infty p_{\sigma,i}(a_i)^2$, $(B_\sigma)^2 = \Sigma_{i=1}^\infty q_{\sigma,i}(a_i)^2$, and $(C_\sigma)^2 = \Sigma_{i=1}^\infty r_{\sigma,i}(a_i)^2$. Then it follows from (a) that

$$\|(\pm \xi \pm \eta \pm \zeta)_\sigma\| \leqq A_\sigma + B_\sigma + C_\sigma.$$

It follows from (b) that $A_\sigma^2 + B_\sigma^2 + C_\sigma^2 \leqq \Sigma_{i=1}^\infty (p_i + q_i + r_i)a_i^2 = M^2$, and therefore

$$\|(\pm \xi \pm \eta \pm \zeta)_\sigma\| \leqq A_\sigma + B_\sigma + [M^2 - (A_\sigma^2 + B_\sigma^2)]^{\frac{1}{2}}.$$

Now let $A^2 = \Sigma_{\sigma=1}^4 A_\sigma^2$, $B^2 = \Sigma_{\sigma=1}^4 B_\sigma^2$, and use the elementary inequality $\Sigma_{\sigma=1}^4 x_\sigma \leqq 2(\Sigma_{\sigma=1}^4 x_\sigma^2)^{\frac{1}{2}}$ to obtain

$$\begin{aligned}\sum_{\sigma=1}^{4} \left\| (\pm \xi \pm \eta \pm \zeta)_\sigma \right\| &\leqq \sum_{\sigma=1}^{4} (A_\sigma + B_\sigma) + \sum_{\sigma=1}^{4} [M^2 - (A_\sigma^2 + B_\sigma^2)]^{\frac{1}{2}} \\ &\leqq 2(A+B) + 2[4M^2 - (A^2+B^2)]^{\frac{1}{2}} \\ &\leqq 2[2(A^2+B^2)]^{\frac{1}{2}} + 2[4M^2 - (A^2+B^2)]^{\frac{1}{2}}.\end{aligned}$$

The rate of change of the last member of this inequality with respect to $A^2 + B^2$ is positive if $A^2 + B^2 < 8M^2/3$. Since it follows from (c) that $A^2 + B^2 \leqq 2M^2$, we have

$$\sum_{\sigma=1}^{4} \left\| (\pm \xi \pm \eta \pm \zeta)_\sigma \right\| \leqq (4 + 2 \cdot 2^{\frac{1}{2}})M. \tag{13}$$

Now, for each k, let $M_k^2 = [\![x^k]\!]^2 + [\![y^k]\!]^2 + [\![z^k]\!]^2$. Then it follows from (11) that

$$M_k \leqq 3^{\frac{1}{2}}\theta^{-6}[\![x^k]\!] \text{ if } 1 \leqq k \leqq n,$$

and it follows from (13) that

$$\sum_{\sigma=1}^{4} \left\| \pm x^k \pm y^k \pm z^k \right\| \leqq (4 \cdot 3^{\frac{1}{2}} + 2 \cdot 6^{\frac{1}{2}})\theta^{-6}[\![x^k]\!],$$

where $\Sigma_{\sigma=1}^4$ indicates the sum over the four arrangements of signs in $\pm x^k \pm y^k \pm z^k$ for which at most one of $\pm x^k, \pm y^k$, and $\pm z^k$ has negative bumps. However, this could as well be the sum over the four arrangements of signs for which at most one sign is negative, so we have

$$\begin{aligned}\sum_{\sigma=1}^{4} \left\| \pm x \pm y \pm z \right\| &= \sum_{\sigma=1}^{4} \left\| \sum_{k=1}^{n} (\pm x^k \pm y^k \pm z^k) \right\| \\ &\leqq \sum_{k=1}^{n} \sum_{\sigma=1}^{4} \left\| \pm x^k \pm y^k \pm z^k \right\| \\ &\leqq (4 \cdot 3^{\frac{1}{2}} + 2 \cdot 6^{\frac{1}{2}})\theta^{-6} \sum_{k=1}^{n} [\![x^k]\!],\end{aligned}$$

and it follows from (10) and $\| x \| = 1$ that

$$\begin{aligned}\sum_{\sigma=1}^{4} \left\| \pm x \pm y \pm z \right\| &\leqq (4 \cdot 3^{\frac{1}{2}} + 2 \cdot 6^{\frac{1}{2}})\theta^{-7} \| x \| = (4 \cdot 3^{\frac{1}{2}} + 2 \cdot 6^{\frac{1}{2}})\theta^{-7} \\ &= 12 - [12 - (4 \cdot 3^{\frac{1}{2}} + 2 \cdot 6^{\frac{1}{2}})\theta^{-7}].\end{aligned}$$

Therefore there is an arrangement of signs for which

$$\|x \pm y \pm z\| \leqq 3 - [3 - (3^{\frac{1}{2}} + \tfrac{1}{2} \cdot 6^{\frac{1}{2}})\theta^{-7}],$$

and it then follows from (6) that, for this arrangement of signs,

$$\|x \pm y \pm z\| < 3 - \Delta.$$

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