ON BANACH SPACES WITH UNCONDITIONAL BASES

BY

L. TZAFRIRI

ABSTRACT

It is shown that for every space with an unconditional basis there exists a uniformly bounded sequence of projections P_n; $n = 1, 2, \ldots$ whose ranges are uniformly isomorphic to l_p^n; $n = 1, 2, \ldots$ either for $p = 1$, or $p = 2$, or for $p = \infty$.

By now the spaces l_p; $p \geqq 1$ and c_0 have been studied in quite some detail and certainly they are the best known Banach spaces. This explains why, among the most interesting problems of the geometric theory of Banach spaces, we find questions related to the existence of subspaces isomorphic to c_0 or l_p; $p \geqq 1$, or to the existence of finite-dimensional subspaces isomorphic to l_∞^n or to l_p^n, for all $n \geqq 1$. The profoundest result in this direction is undoubtedly that of Dvoretzky [2] which shows that every infinite-dimensional Banach space contains subspaces almost isometric to l_2^n; $n = 1, 2, \cdots$.

Theorem 1 proved in this paper shows that, in a certain sense, Dvoretzky's result can be considerably improved in the case of spaces having an unconditional basis. Before stating Theorem 1 let us point out that the notation and terms used here are standard (for additional details see [4]). For the convenience of the reader we recall that two Banach spaces X and Y are called isomorphic if there exists an invertible operator from X onto Y. The Banach-Mazur distance coefficient $d(X, Y)$ of two isomorphic Banach spaces X and Y is defined by $\inf \| T \| \cdot \| T^{-1} \|$, where the infimum is taken over all invertible operators T from X onto Y.

THEOREM 1. *Let X be an infinite-dimensional Banach space with an uncon-*

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ditional basis $\{x_n\}$. *Then there exist a constant* M *and a sequence* $\{P_n\}_{n=1}^{\infty}$ *of projections in* X *such that for all* $n = 1,2,\cdots$ $\|P_n\| \leqq M$ and $d(P_nX,\ l_p^n) \leqq M$ *either for* $p = 1$, *or for* $p = 2$, *or for* $p = \infty$.

Before beginning the proof we shall make a few comments: (i) The infinite version of Theorem 1 is false in general; there are examples of spaces with unconditional basis (even Orlicz sequence spaces) which contain no complemented subspace isomorphic to either c_0 or to l_p; $p \geqq 1$ (refer to [3]). (ii) It is well known and easily seen that $l_\infty^{2^n}$ contains a subspace isometric to l_1^n and that $l_1^{2^n}$ contains a subspace close to l_2^n. This explains the claim that Theorem 1 generalizes the afore-mentioned theorem of Dvoretzky (in the case of spaces having an unconditional basis). However, the proof given here is quite elementary and does not involve any of the difficult arguments used in [2].

NOTE. It should be pointed out that our proof does not produce subspaces almost isometric to l_2^n, only subspaces whose distance from l_2^n is bounded by some constant independent of n.

We start by recalling a recent result (Proposition 2) of Brunel and Sucheston [1]. For the sake of completeness we present here its proof which is based on the following well-known combinatorial result of Ramsey [6].

PROPOSITION 2. *Let* m *and* n *be positive integers and* φ *a function defined on the unordered tuples of* n *different integers taking values in the set* $\{1,2,\cdots,m\}$. *Then there exists an infinite subset* N_0 *of positive integers such that the restriction of* φ *to the tuples constructed using only the elements of* N_0 *is constant.*

PROPOSITION 3. *Let* Y *be a Banach space with a normalized monotone Schauder basis* $\{y_n\}$. *Then, for every* $\varepsilon > 0$ *there exist a sequence of positive reals* $\{\lambda(j)\}_{j=1}^{\infty}$ *and a sub-basic sequence* $\{z_i = y_{n_i}\}_{i=1}^{\infty}$ *such that*

$$0 < \lambda(j) - \|z_{k_1} + z_{k_2} + \cdots + z_{k_j}\| < \varepsilon$$

for every set of indices $j < k_1 < k_2 < \cdots < k_j$; $j = 1,2,\cdots$. *In addition,* $\lambda(j) < \lambda(m) + \varepsilon$ *for* $j < m$.

PROOF. For every integer $k > 1$ we consider a fixed partition

$$\lambda_0^{(k)} = 1 < \lambda_1^{(k)} < \lambda_2^{(k)} < \cdots < \lambda_{s(k)}^{(k)} = k + 1$$

with the property that $\lambda_j^{(k)} - \lambda_{j-1}^{(k)} < \varepsilon$; $j = 1,2,\cdots s(k)$; $k = 1,2,\cdots$. Once the partition has been chosen we define a function φ_k from all the unordered tuples of k different integers into the numbers $\{1,2,\cdots,s(k)\}$, by setting $\varphi_k(\{n_1,n_2,\cdots,n_k\}) = j$ if $\lambda_{j-1}^{(k)} \leqq \| y_{n_1} + y_{n_2} + \cdots + y_{n_k} \| < \lambda_j^{(k)}$.

Let $N = N^{(1)}$ denote the set of all positive integers and $\lambda(1) = 1$. Applying Proposition 2 to the function φ_2 we obtain an infinite subset of integers $N^{(2)} \subset N^{(1)}$ (on which φ_2 is constant) and a number $\lambda(2)$ which is defined as follows: if φ_2, restricted to pairs of integers from $N^{(2)}$, is equal to j then $\lambda(2) = \lambda_j^{(2)}$. Again we apply Proposition 2, this time for the function φ_3 restricted to tuples constructed with the elements of $N^{(2)}$. In this way we obtain another infinite subset of integers $N^{(3)} \subset N^{(2)}$ (on which φ_3 is constant) and a number $\lambda(3)$ such that φ_3 restricted to $N^{(3)}$ is equal to j where $\lambda_j^{(3)} = \lambda(3)$.

Continuing so and using a standard diagonal argument for

$$N = N^{(1)} \supset N^{(2)} \supset N^{(3)} \supset \cdots \supset N^{(k)} \supset \cdots$$

we construct an infinite subsequence of positive integers $\{n_1 < n_2 < \cdots < n_l < \cdots\}$ and a sequence of positive reals $\{\lambda(j)\}_{j=1}^{\infty}$ having all the properties required in the statement of Proposition 3. The last assertion follows from the monotony of the basis.

PROPOSITION 4. *Fix $0 < \varepsilon < 1$ in Proposition 3 and let $\{\lambda(j)\}_{j=1}^{\infty}$ and $\{z_i\}_{i=1}^{\infty}$ be the two sequences defined there. Assume the existence of an integer $h > 1$ such that*

$$\frac{\lambda(hn)}{\lambda(n)} \geqq 1 + \varepsilon; \; n = 1,2,\cdots.$$

Then there exist a constant A and a number $q > 2$ such that

$$\Big\| \sum_{j=1}^{n} a_j z_{m+k_j} \Big\| / \lambda(n) \leqq A \Big(\sum_{j=1}^{n} |a_j|^q \Big)^{1/q} / n^{1/q}$$

for every $n = 1,2,\cdots$; $m > n$; $0 < k_1 < k_2 < \cdots < k_n$ and every sequence of scalars $\{a_1, a_2, \cdots, a_n\}$.

PROOF. First, we choose $r > 2$ such that $1 < h^{1/r} < 1 + \varepsilon$. Then we have $\lambda(hn)/\lambda(n) > h^{1/r}$; $n = 1,2,\cdots$, which implies

$$\frac{\lambda(h^s n)}{\lambda(n)} > (h^s)^{1/r}; \; n, s = 1,2,\cdots.$$

Now, if $\alpha > \beta$ are two integers and $h^{i-1} < \alpha \leqq h^i$; $h^{j-1} < \beta \leqq h^j$, for some i and j, then either $i > j$ and

$$\frac{\lambda(\alpha)}{\lambda(\beta)} \geqq \frac{1}{(1+\varepsilon)^2}\,\frac{\lambda(h^{i-1})}{\lambda(h^j)} \geqq \frac{1}{4}\,\frac{\lambda(h^{i-j-1}h^j)}{\lambda(h^j)} \geqq \frac{1}{4}(h^{i-j-1})^{1/r} \geqq \frac{1}{4h^{2/r}}\left(\frac{\alpha}{\beta}\right)^{1/r}$$

or $i = j$ and

$$\frac{\lambda(\alpha)}{\lambda(\beta)} \geqq \frac{1}{2} \geqq \frac{1}{2h^{1/r}}\left(\frac{h^i}{h^{i-1}}\right)^{1/r} \geqq \frac{1}{2h^{1/r}}\left(\frac{\alpha}{\beta}\right)^{1/r}.$$

Thus, in both cases, we have

$$\frac{\lambda(\alpha)}{\lambda(\beta)} \geqq \frac{1}{4h^{2/r}}\left(\frac{\alpha}{\beta}\right)^{1/r}.$$

Set $q > r$ and $A = 16h^{2/r}/(1 - q'/r')^{1/q'}$, where $1/q + 1/q' = 1$ and $1/r + 1/r' = 1$. Fix integers $n < m$ and $k_1 < k_2 < \cdots < k_n$. Then, for every set of coefficients a_j; $j = 1, 2, \cdots, n$, we have

$$\left\| \sum_{j=1}^{n} a_j z_{m+k_j} \right\| \leqq \sum_{s=1}^{4} \left\| \sum_{j=1}^{n} b_j^{(s)} z_{m+k_j} \right\|$$

where $a_j = (b_j^{(1)} - b_j^{(2)}) + i(b_j^{(3)} - b_j^{(4)})$; $j = 1, 2, \cdots, n$ and $0 \leqq b_j^{(s)} \leqq |a_j|$; $s = 1, 2, 3, 4$; $j = 1, 2, \cdots, n$.

Let $\sum_{j=1}^{n} b_j z_{m+k_j}$ be any of these four sums where $0 \leqq b_j \leqq |a_j|$; $j = 1, 2, \cdots, n$. Let π be a permutation of the integers $\{1, 2, \cdots, n\}$ so that $b_{\pi(1)} \geqq b_{\pi(2)} \geqq \cdots \geqq b_{\pi(n)} \geqq 0$. Then,

$$\left\| \sum_{j=1}^{n} b_j z_{m+k_j} \right\| = \left\| \sum_{j=1}^{n} b_{\pi(j)} z_{m+k_{\pi(j)}} \right\|$$

$$\leqq (b_{\pi(1)} - b_{\pi(2)})\lambda(1) + (b_{\pi(2)} - b_{\pi(3)})\lambda(2) + \cdots + (b_{\pi(n-1)} - b_{\pi(n)})\lambda(n-1) + b_{\pi(n)}\lambda(n).$$

Thus,

$$\left\| \sum_{j=1}^{n} b_j z_{m+k_j} \right\| / \lambda(n) \leqq 4h^{2/r}\left\{(b_{\pi(1)} - b_{\pi(2)})\left(\frac{1}{n}\right)^{1/r} + (b_{\pi(2)} - b_{\pi(3)})\left(\frac{2}{n}\right)^{1/r} + \cdots\right.$$

$$\left. + (b_{\pi(n-1)} - b_{\pi(n)})\left(\frac{n-1}{n}\right)^{1/r} + b_{\pi(n)}^{1/r}\right\} \leqq 4h^{2/r} \cdot \sum_{j=1}^{n} b_{\pi(j)}\left[\left(\frac{j}{n}\right)^{1/r} - \left(\frac{j-1}{n}\right)^{1/r}\right].$$

Using the fact that $j^{1/r} - (j-1)^{1/r} \leqq 1/(j^{1/r'})$ and applying Holder's inequality for q and q' we obtain

$$\left\| \sum_{j=1}^{n} b_j z_{m+k_j} \right\| / \lambda(n) \leqq \frac{4h^{2/r}}{n^{1/r}} \sum_{j=1}^{n} \frac{b_{\pi(j)}}{j^{1/r'}}$$

$$\leqq \frac{4h^{2/r}}{n^{1/r}} \left(\sum_{j=1}^{n} | b_{\pi(j)} |^q \right)^{1/q} \left(\sum_{j=1}^{n} \frac{1}{j^{q'/r'}} \right)^{1/q'}$$

$$\leqq \frac{4h^{2/r}}{n^{1/r}} \left(\sum_{j=1}^{n} | a_j |^q \right)^{1/q} \left(\frac{1}{1 - q'/r'} \right)^{1/q'} \cdot n^{(1-q'/r')(1/q')}$$

the last inequality being obtained by integrating the function $1/x^{q'/r'}$ between 0 and n. Therefore

$$\left\| \sum_{j=1}^{n} b_j z_{m+k_j} \right\| / \lambda(n) \leqq \frac{4h^{2/r}}{(1 - q'/r')^{1/q'}} \left(\sum_{j=1}^{n} | a_j |^q \right)^{1/q} / n^{1/q}$$

and this proves completely the proposition.

PROPOSITION 5. *Let V be a 2^n-dimensional Banach space generated by a system of vectors $\{v_1, v_2, \cdots, v_{2^n}\}$. Suppose there exist constants $K > 1$ and $p > 2$ such that*

$$K^{-1} \left(\sum_{j=1}^{2^n} | a_j |^{p'} \right)^{1/p'} \Big/ (2^n)^{1/p'} \leqq \left\| \sum_{j=1}^{2^n} a_j v_j \right\| \Big/ \left\| \sum_{j=1}^{2^n} v_j \right\|$$

$$\leqq K \left(\sum_{j=1}^{2^n} | a_j |^p \right)^{1/p} \Big/ (2^n)^{1/p}$$

for every set of scalars a_j; $j = 1, 2, \cdots, 2^n$ (where $1/p + 1/p' = 1$). *Then there is a constant $M = M(K, p)$, depending only on K and p (that is, independent of V and n), and a projection P in V such that $\| P \| \leqq M$ and $d(PV, l_2^n) \leqq M$.*

PROOF. Let χ_δ denote the characteristic function of a set $\delta \subset [0, 1]$.

Put
$$\varepsilon_{k_j} = \begin{cases} 1 & (2h-2)2^{n-k} + 1 \leqq j \leqq (2h-1)2^{n-k} \\ -1 & (2h-1)2^{n-k} + 1 \leqq j \leqq 2h \cdot 2^{n-k} \end{cases}$$

for $j = 1, 2, \cdots, 2^n$, $k = 1, 2, \cdots, n$, and $h = 1, 2, \cdots, 2^{k-1}$. Consider the functions

$$r_k = \sum_{j=1}^{2^n} \varepsilon_{kj} \chi_{[(j-1)/2^n, j/2^n]}; \; k = 1, 2, \cdots, n$$

and the vectors

$$w_k = \sum_{j=1}^{2^n} \varepsilon_{kj} v_j; \; k = 1, 2, \cdots, n.$$

One can easily recognize that $\{r_1, r_2, \cdots, r_n\}$ are exactly the first n Rademacher functions on $[0,1]$ while $\{w_1, w_2, \cdots, w_n\}$ forms the so-called Rademacher system in V. (This terminology as well as some ideas used in the proof are taken from [5].)

By the well-known Khintchin inequality there is, for every $r \geqq 1$, a constant K_r depending only on r and such that for every choice of a_k

$$K_r^{-1}\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \leqq \left\| \sum_{k=1}^{n} a_k r_k \right\|_r \leqq K_r \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}},$$

where $\|f\|_r$ denotes the norm of a function f in $L_r(0,1)$. It follows immediately that

$$K^{-1}K_{p'}^{-1}\left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}} \leqq \left\| \sum_{k=1}^{n} a_k w_k \right\| \Big/ \left\| \sum_{j=1}^{2^n} v_j \right\| \leqq K K_p \left(\sum_{k=1}^{n} |a_k|^2\right)^{\frac{1}{2}}$$

which shows that $d(W, l_2^n) \leqq K^2 K_p K_{p'}$, where $W = \operatorname{span}_{1 \leqq k \leqq n}\{w_k\}$.

Now, let Q be the orthogonal projection in $L_2(0,1)$ whose range is $\operatorname{span}_{1 \leqq k \leqq n}\{r_k\}$; it is well known that Q acts as a bounded linear projection in every $L_r(0,1)$; $r > 1$ (see, for example, [4 II.3.c]) and the norm $\|Q\|_r$ of Q in $L_r(0,1)$ is independent of n. If

$$Q\left(\sum_{j=1}^{2^n} a_j \chi_{[(j-1)/2^n, j/2^n]}\right) = \sum_{k=1}^{n} b_k r_k$$

then we set

$$P\left(\sum_{j=1}^{2^n} a_j v_j\right) = \sum_{k=1}^{n} b_k w_k.$$

One can easily check that the mapping P, defined in this manner, is a linear projection in V whose range is exactly W. Moreover,

$$\begin{aligned}\left\|P\left(\sum_{j=1}^{2^n} a_j v_j\right)\right\| &= \left\|\sum_{k=1}^{n} b_k w_k\right\| \leqq K K_p \left\|\sum_{j=1}^{2^n} v_j\right\| \cdot \left(\sum_{k=1}^{n} |b_k|^2\right) \\ &\leqq K K_p K_{p'} \left\|\sum_{j=1}^{2^n} v_j\right\| \cdot \left\|\sum_{k=1}^{n} b_k r_k\right\|_{p'} \leqq \|Q\|_{p'} K K_p K_{p'} \left\|\sum_{j=1}^{2^n} v_j\right\| \cdot \\ &\cdot \left(\sum_{j=1}^{2^n} |a_j|^{p'}\right)^{1/p'} \big/ (2^n)^{1/p'} \leqq \|Q\|_{p'} K^2 K_p K_{p'} \left\|\sum_{j=1}^{2^n} a_j v_j\right\|.\end{aligned}$$

In conclusion we have just shown that $\|P\| \leqq M$ and $d(PV, l_2^n) \leqq M$ where $M = \|Q\|_{p'} \cdot K^2 \cdot K_p \cdot K_{p'}$.

PROOF OF THEOREM 1. We start by assuming, with no loss of generality, that the unconditional constant of $\{x_n\}$ is equal to 1.

Fix $0 < \varepsilon < 1$; then, using Proposition 3 for $\{x_n\}$ as well as for $\{x_n^*\}$ (the sequence of the biorthogonal functionals associated to $\{x_n\}$) we can construct two sequences of reals $\{\lambda(j)\}_{j=1}^{\infty}$ and $\{\mu(j)\}_{j=1}^{\infty}$, and two subsequences $\{z_i = x_{n_i}\}_{i=1}^{\infty}$ and

$$\{z_i^* = x_{n_i}^*\}_{i=1}^{\infty}$$

so that

$$0 < \lambda(j) - \| z_{k_1} + z_{k_2} + \cdots + z_{k_j} \| < \varepsilon$$

$$0 < \mu(j) - \| z_{k_1}^* + z_{k_2}^* + \cdots + z_{k_j}^* \| < \varepsilon$$

for every set of indices satisfying $j < k_1 < k_2 < \cdots < k_j$; $j = 1, 2, \cdots$.

We shall distinguish three cases.

Case I. For every integer $h > 1$ there exists an integer $n = n(h)$ such that $\lambda(hn)/\lambda(n) < 1 + \varepsilon$. In this case, after fixing h and $n = n(h)$, we shall consider the following vectors:

$$\begin{aligned} u_1 &= (z_{hn+1} + \cdots + z_{hn+n})/\lambda(n) \\ u_2 &= (z_{hn+n+1} + \cdots + z_{hn+2n})/\lambda(n) \\ &\vdots \\ u_h &= (z_{hn+(h-1)n+1} + \cdots + z_{hn+hn})/\lambda(n). \end{aligned}$$

Then, using the fact that the unconditional constant of $\{x_n\}$ is equal to 1 and $\| u_i \| \geqq \frac{1}{2}$; $i = 1, 2, \cdots, h$, we have

$$\frac{1}{2} \max_{1 \leqq i \leqq h} |a_i| \leqq \left\| \sum_{i=1}^{h} a_i u_i \right\| \leqq \left(\max_{1 \leqq i \leqq h} |a_i| \right) \cdot \left\| \sum_{i=1}^{h} u_i \right\|$$

$$= \left(\max_{1 \leqq i \leqq h} |a_i| \right) \cdot \left\| \sum_{j=1}^{hn} z_{hn+j} \right\| / \lambda(n) \leqq \left(\max_{1 \leqq i \leqq h} |a_i| \right) \frac{\lambda(hn)}{\lambda(n)} \leqq 2 \max_{1 \leqq i \leqq h} |a_i| .$$

This shows that $d(X_h, l_\infty^h) \leqq 4$, where $X_h = \mathrm{span}_{1 \leqq i \leqq h}\{u_i\}$. Since l_∞ is an injective space it follows immediately that there are projections P_h in X such that $P_h X = X_h$ and $\| P_h \| \leqq 4$; $h = 1, 2, \cdots$.

Case II. For every $h > 1$ there exists an integer $n = n(h)$ such that $\mu(hn)/\mu(n) < 1 + \varepsilon$. Using the results obtained in Case I we first establish that for every h there are functionals $\{y_1^*, y_2^*, \cdots, y_h^*\}$ in X^* which have disjoint supports (relative to the basis $\{x_n\}$) and such that

$$\frac{1}{2} \max_{1 \leqq i \leqq h} |a_i| \leqq \left\| \sum_{i=1}^{h} a_i y_i^* \right\| \leqq 2 \max_{1 \leqq i \leqq h} |a_i| ,$$

for every choice of $\{a_i\}$.

Choosing now vectors $y_i \in X$; $i = 1,2,\cdots,h$ such that the support of y_i is contained in that of y_i^*; $\| y_i \| = 1$ and $2 \geqq y_i^* \cdot y_i \geqq \frac{1}{2}$; $i = 1,2,\cdots,h$, we have

$$\left\| \sum_{i=1}^{h} b_i y_i \right\| \leqq \sum_{i=1}^{h} |b_i| \leqq 2 \left(\sum_{i=1}^{h} (\operatorname{sgn} b_i) y_i^* \right) \left(\sum_{i=1}^{h} b_i y_i \right) \leqq 4 \left\| \sum_{i=1}^{h} b_i y_i \right\|$$

which shows that $d(Y_h, l_1^h) \leqq 4$, where $Y_h = \operatorname{span}_{1 \leqq i \leqq h} \{y_i\}$. Furthermore, for $x \in X$, let us define

$$P_h x = \sum_{i=1}^{h} \frac{y_i^*(x)}{y_i^*(y_i)} y_i.$$

Then P_h is a projection from X onto Y_h for which

$$\| P_h x \| \leqq \sum_{i=1}^{h} \frac{|y_i^*(x)|}{y_i^*(y_i)} = \left(\sum_{i=1}^{h} \frac{\operatorname{sgn} y_i^*(x)}{y_i^*(y_i)} y_i^* \right) x \leqq 4 \| x \|,$$

i.e., again $\| P_h \| \leqq 4$; $h = 1,2,\cdots$

Case III. If both the conditions characterizing Cases I and II do not hold we can apply Proposition 4 to both X and X^* thus obtaining constants $B > 1$ and $p > 2$ such that, simultaneously,

$$\left\| \sum_{j=1}^{n} a_j z_{n+j} \right\| / \lambda(n) \leqq B \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} / n^{1/p}$$

and

$$\left\| \sum_{j=1}^{n} a_j z_{n+j}^* \right\| / \mu(n) \leqq B \left(\sum_{j=1}^{n} |a_j|^p \right)^{1/p} / n^{1/p}$$

for every $n = 1,2,\cdots$ and for every choice of $\{a_j\}$.

Fix n and let $x = \sum_{j=1}^{n} b_j z_{n+j}$ be a vector in X such that $\| x \| = 1$ and

$$\left\| \sum_{j=1}^{n} z_{n+j}^* \right\| = \sum_{j=1}^{n} b_j.$$

Choose an integer C such that $C > (8B)^p$. If

$$\eta = \left\{ j;\, 1 \leqq j \leqq n;\, |b_j| \geqq \frac{8C}{\lambda(n)} \right\}$$

and s is the number of the elements of η then $n > Cs$. Indeed, if $n \leqq Cs$ then

$$\lambda(n) \leqq 2 \left\| \sum_{j=1}^{n} z_{Cs+j} \right\| \leqq 2 \left\| \sum_{j=1}^{Cs} z_{Cs+j} \right\| \leqq 2C \lambda(s)$$

which contradicts the fact that

$$1 = \| x \| \geqq \left\| \sum_{j \in \eta} |b_j| z_{n+j} \right\| \geqq \frac{8C}{\lambda(n)} \left\| \sum_{j \in \eta} z_{n+j} \right\| \geqq 4C \frac{\lambda(s)}{\lambda(n)}.$$

Using the fact that $s/n < 1/C$ and the second inequality characterizing Case III we have

$$\frac{\mu(s)}{\mu(n)} \leqq 2\left\|\sum_{j=1}^{s} z_{n+j}^*\right\|/\mu(n) \leqq 2B\left(\frac{s}{n}\right)^{1/p} \leqq \frac{2B}{C^{1/p}} < \frac{1}{4}.$$

Hence

$$\begin{aligned} \mu(n) &\leqq 2\sum_{j=1}^{n} b_j \leqq 16C\frac{n}{\lambda(n)} + 2\sum_{j\in\eta} b_j \leqq 16C\frac{n}{\lambda(n)} + 2\left(\sum_{j\in\eta} z_{n+j}^*\right)x \\ &\leqq 16C\frac{n}{\lambda(n)} + 2\mu(s) \leqq 16C\frac{n}{\lambda(n)} + \frac{\mu(n)}{2} \end{aligned}$$

that is,

$$\mu(n) \leqq 32C\frac{n}{\lambda(n)}\,;\ n = 1,2,\cdots.$$

Consequently, for $1/p + 1/p' = 1$, we have

$$\begin{aligned} \left\|\sum_{j=1}^{n} a_j z_{n+j}\right\| &\geqq \sum_{j=1}^{n} |a_j|^{p'} \Big/ \left\|\sum_{j=1}^{n} |a_j|^{p'-1}(\operatorname{sgn} a_j) z_{n+j}^*\right\| \\ &\geqq \frac{1}{B}\, n^{1/p}\left(\sum_{j=1}^{n} |a_j|^{p'}\right)\Big/\mu(n)\left(\sum_{j=1}^{n} |a_j|^{(p'-1)p}\right)^{1/p} \\ &\geqq \frac{1}{32BC}\,\frac{\lambda(n)}{n^{1/p'}}\left(\sum_{j=1}^{n} |a_j|^{p'}\right)^{1/p'}. \end{aligned}$$

We complete the proof of Theorem 1 by applying Proposition 5 since B, C and p are independent of n.

Added in Proof

1. Brunel and Sucheston, *On J-convexity and some ergodic super-properties of Banach spaces* (to appear), have proved that for every infinite dimensional Banach space X there exists an infinite dimensional Banach space Y which has an unconditional basis and which is finitely representable in X (i.e., for every $\varepsilon > 0$ and every finite dimensional subspace Y_0 of Y there exists a subspace X_0 of X such that $d(X_0, Y_0) < 1 + \varepsilon$). This result combined with Theorem 1 and the remarks thereafter give a new and elementary proof of a slightly weaker version of Dvoretzky's thorrem [2]: Every infinite dimensional Banach space contains subspaces uniformly isomorphic to l_2^n; $n = 1, 2, \cdots$.

2. W. B. Johnson, *On finite dimensional subspaces of Banach spaces with local unconditional structure* (to appear), has shown independently that when

X is not uniformly convexifiable then Theorem 1 holds with $p = 1$ or $p = \infty$ (possibly also with $p = 2$).

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The Hebrew University of Jerusalem
Jerusalem, Israel

and

California Institute of Technology
Pasadena, California, U. S. A.