ISRAEL JOURNAL OF MATHEMATICS, Vol. 20, No. 1, 1975

DIAGONAL MAPS AND DIAMETERS IN KÖTHE SPACES

BY
L. CRONE AND W. ROBINSON

ABSTRACT
First, an investigation is made of the nature of diagonal maps in Köthe spaces. The central theorem relates the existence of a non-compact map between power series spaces to the existence of a common complemented basic sequence in the two spaces.

Given a pair of locally convex spaces, a natural question to ask is: what are common properties of these spaces? Very often this can be answered by studying the types of maps between the spaces. Within the class of nuclear Köthe spaces, there have been two significant efforts recently. In [9], and [10] Ed Dubinsky studies possible embeddings of infinite type p.s.s. into arbitrary p.s.s. In [18], V. P. Zaharjuta shows that all maps from a space of type (d_2) into a space of type (d_1) are compact, and makes use of this to isomorphically distinguish products of (d_1) and (d_2) spaces.

In this paper, we first consider results on diagonal maps between nuclear Köthe spaces, and then obtain precise statements concerning pairs of infinite type p.s.s. The principal result is Theorem (3.2), which relates the existence of a non-compact map between $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$ to the existence of a non-compact, generalized, diagonal map, and this, in turn, is shown to be equivalent to the existence of a common, complemented, basic sequence in $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$. Moreover, a precise condition on α and β is given which is equivalent to these statements.

1. Preliminary definitions

By a *sequence space*, we mean a vector space λ of infinite sequences which contains ϕ, the space of finitely non-zero sequences. If a and b are sequences, $a \cdot b$ is the sequence $(a_n b_n)$, and for a sequence space λ, $a \cdot \lambda = \{a \cdot b : b \in \lambda\}$. We say that *a dominates b*, written $b \prec a$, if there exists $M > 0$ such that

Received April 30, 1974

$b_n \leqq Ma_n$ for all $n = 1,2,\cdots$ We say $b \leqq a$ if $b_n \leqq a_n$ for $n = 1,2,\cdots$. If $a < b$, a/b is the sequence whose nth coordinate is a_n/b_n, if $b_n \neq 0$, and is 0 if $b_n = 0$. λ is a *solid* sequence space (or normal) if whenever $a < b$ and $b \in \lambda$, then $a \in \lambda$. l_p is the space of sequences whose pth power is summable, for $p < \infty$, l_∞ is the space of bounded sequences, and c_0 is the space of null sequences. e^n is the sequence with 1 in the nth coordinate and zero on all other coordinates.

Let P be a collection of non-negative sequences such that (i) for each a, $b \in P$, there exists $c \in P$ with $a < c$ and $b < c$, and, (ii) for each n, there exists $a \in P$ such that $a_n \neq 0$. P is called a Köthe set. $\Lambda(P)$ is the sequence space $\Lambda(P) = \{t : \|t\|'_a = \Sigma a_n |t_n| < +\infty$ for all $a \in P\}$ topologized by the semi-norms $\| \|'_a$. $\Lambda(P)$ is a complete l.c.s., and $\Lambda(P)$ is nuclear, if and only if for each $a \in P$, there exists $b \in P$ such that $a \in b \cdot l_1$. ([14], 6.1.2)

If the Köthe set P is countable, we may assume that $P = \{a^k : a^k \leqq a^{k+1}$ for all $k\}$, and we write $\| \|'_k$ for $\| \|'_{a^k}$. For the most part, we will assume that $0 < a^k_n$ for all k and n, and in this case, we write $\Lambda(P) = \cap (1/a^k) \cdot l_1$. $\cap (1/a^k) \cdot l_1$ is called a *Köthe* space. Given a sequence space λ, the *Köthe* dual of λ is $\lambda^\times = \{t : t \cdot x \in l_1$ for all $x \in \lambda\}$ and if $\lambda^{\times\times} = \lambda$ we call λ *perfect*. If $\lambda = \cap (1/a^k) \cdot l_1$ then λ is perfect and $\lambda^\times = \cup^\infty_{a^k} \cdot l_\infty$ [13].

If E is a l.c.s., a sequence (x^n) in E is a (*absolute*) *basis* for E if for each $x \in E$, there exists a unique sequence of scalars (t_n) such that $x = \Sigma t_n x^n$, with (absolute) convergence in the topology of E. Thus, any l.c.s. E with a basis (x^n) can be identified with the associated sequence space $\{(t_n) : \Sigma t_n x^n$ converges in $E\}$. Two bases are said to be *equivalent* if they have the same associated sequence space. In particular, any Frechet space E with a continuous norm and an absolute basis (x^n) can be identified with a Köthe space. In fact, one may choose any increasing sequence of norms $(\| \|_k)$, defining the topology of E and let $\|x^n\|_k = a^k_n$. Then the associated sequence space is $\cap (1/a^k) \cdot l_1$. Bessaga calls (a^k_n) a matrix representation of $(E, (x^n))$ [1].

Of particular interest in this paper will be the power series spaces (p.s.s.) $\Lambda_1(\alpha)$ and $\Lambda_\infty(\alpha)$. Given $0 < \alpha_1 \leqq \alpha_2 \leqq \cdots$, the finite type p.s.s. generated by α is the Köthe space $\Lambda_1(\alpha)$ whose representing matrix is $(a^k_n) = ((e^{-1/k})^{\alpha_n})$; and the infinite–type p.s.s. generated by α is the Köthe space, $\Lambda_\infty(\alpha)$, whose representing matrix is $(a^k_n) = ((e^k)^{\alpha_n})$. Given $R > 0$, we write R^α for the sequence (R^{α_n}).

The following theorems are well-known.

THEOREM (1.1). [14] *A Köthe space* $\cap (1/a^k) \cdot l_1$ *is nuclear if and only if the system of norms* $\|x\|_k = \sup_n a^k_n |x_n|$ *is equivalent to the system of norms* $\|x\|'_k = \Sigma_n a^k_n |x_n|$.

THEOREM (1.2). [13] *A Köthe space* $\cap(1/a^k)\cdot l_1$ *is a Montel space if and only if for each k there exists m such that* $a_k/a_m \in c_0$.

If (y^n) is a sequence in a l.c.s.which is a basis for its closed linear span $[y^n]$, then (y^n) is said to be a basic sequence. If (y^n) is a basic sequence and if $[y^n]$ is complemented, (y^n) is said to be a *complemented basic sequence* (CBS). The following important theorem is due to Bessaga in [1] (cf. also [7]).

THEOREM (1.3). *Let* λ *be a nuclear Köthe space with representing matrix* (a_n^k). *Let* (y^n) *be a CBS in* λ. *There exists a sequence* (d_n) *of positive numbers and integers* $k_n \to \infty$ *such that* $(d_n a_{k_n}^k)$ *is a representing matrix for* $([y^n],(y^n))$.

This result, and similar considerations led Dragilev to define the following invariant of a nuclear Köthe space λ. $K(\lambda)=\{[d_n e^{k_n}]: d_n>0$ for all n, and such that $\lim k_n=\infty\}$. ([8]). If $k_n<k_{n+1}$ for all n, and if $I=(k_n)$ we will write $\lambda_I=[e^{k_n}]$. λ_I is called a *step space* of λ.

We will say that a Köthe space λ is (D_1) if $\lambda = \cap(1/a^k)\cdot l_1$, where the matrix (a_n^k) satisfies these conditions: (i) for all $k,n,(a_n^k)/(a_n^{k+1})\geqq(a_{n+1}^k)/(a_{n+1}^{k+1})$; (ii) $a_n^1=1$ for all n, and (iii) for all k there exists p such that $a^k\cdot a^k\leqq a^p$. A representing matrix (a_n^k) satisfying condition (i) is said to be *regular* [7].

Given Köthe spaces λ and μ we will denote the collection of continuous linear (compact) maps from λ to μ by $L(\lambda,\mu)$ (respectively $LC(\lambda,\mu)$). If $L(\lambda,\mu)=LC(\lambda,\mu)$, we say the pair (λ,μ) is in relation R and write $(\lambda,\mu)\in R$ (cf [16] for a discussion of Köthe spaces in relation R). For each $T\in L(\lambda,\mu)$, we say T is represented by the infinite matrix (t_{ij}) if for all i,j, $\langle Te^i,e^j\rangle=t_{ij}$. We write $T\sim(t_{ij})$. T is a diagonal map if $t_{ij}=0$ whenever $i\neq j$. If $t_i=t_{ii}$, then $Tx=t\cdot x$ for all $x\in\lambda$. The space of diagonal maps from λ to μ is then $D(\lambda,\mu)=\{t: t\cdot x\in\mu$ for all $x\in\lambda\}$. Since Köthe spaces are perfect, $D(\lambda,\mu)=(\lambda\cdot\mu^\times)^\times$ [3]. We will use the notation $DC(\lambda,\mu)$ for the space of diagonal compact maps from λ to μ.

2. Diagonal maps

We begin with an investigation of diagonal compact maps between Köthe spaces. The first lemma is implicitly in [18].

LEMMA (2.1). *Let* λ *and* μ *be Köthe spaces.*

a) $T\in L(\lambda,\mu)$ *if and only if for each m there exists k such that*

$$\sup_n \frac{\| Te^n \|_m}{\| e^n \|_k} < +\infty.$$

b) *If μ is Montel, then $T \in LC(\lambda, \mu)$ if and only if there exists k such that for all m,*

$$\sup \frac{\| Te^n \|_m}{\| e^n \|_k} < +\infty.$$

PROOF. We prove (b) and note that a proof of (a) is similar. $T \in LC(\lambda, \mu)$ if and only if there exists k such that $\{Tx : \| x \|_k \leqq 1\}$ is bounded, since in Montel spaces bounded sets and precompact sets coincide [13]. Thus $T \in LC(\lambda, \mu)$ if and only if there exists k such that for all m,

$$\sup \left\{ \frac{\| Tx \|_m}{\| x \|_k} : x \in \lambda, x \neq 0 \right\} < +\infty,$$

so in particular

$$\sup_n \left\{ \frac{\| Te^n \|_m}{\| e^n \|_k} \right\} < +\infty.$$

Now suppose there exists k such that for all m,

$$\sup_n \frac{\| Te^n \|_m}{\| e^n \|_k} = C(m) < +\infty.$$

Let $x \in \lambda$. Then $\| Tx \|_m = \| \Sigma_n x_n Te^n \|_m \leqq C(m) \sum_n |x_n| \, \| e^n \|_k = C(m) \| x \|_k$, so $T \in LC(\lambda, \mu)$.

LEMMA (2.2). *Let λ and μ be Köthe spaces with μ a nuclear space. Then $DC(\lambda, \mu) = \mu \cdot \lambda^\times$.*

PROOF. By (2.1) $t \in DC(\lambda, \mu)$ if and only if there exists k such that for all m,

$$\sup \frac{\| t \cdot e^n \|_m}{\| e^n \|_k} < +\infty$$

if and only if there exists k such that for all m, $t / a^k \cdot b^m \in l_\infty$, so that

$$t \in \bigcup_k \bigcap_m \frac{a^k}{b^m} \cdot l_\infty = \lambda^\times \cdot \mu.$$

LEMMA (2.3). *If λ and μ are Köthe spaces such that $\lambda \cdot \mu^{\times}$ is perfect, then $DC(\mu,\lambda) = D(\mu,\lambda)$ whenever $DC(\lambda,\mu) = D(\lambda,\mu)$.*

PROOF. $DC(\mu,\lambda) = \mu^{\times} \cdot \lambda = (\mu^{\times} \cdot \lambda)^{\times\times} = D(\lambda,\mu)^{\times} = DC(\lambda,\mu)^{\times} = (\lambda^{\times} \cdot \mu)^{\times}$
$= D(\mu,\lambda)$.

We now consider the special case in which λ and μ are power series spaces.

THEOREM (2.4). *If $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$ are nuclear and if $\alpha/\beta \in c_0$ then $D(\Lambda_\infty(\alpha),\Lambda_\infty(\beta)) = \Lambda_\infty(\beta) = DC(\Lambda_\infty(\alpha),\Lambda_\infty(\beta))$ and $D(\Lambda_\infty(\beta),\Lambda_\infty(\alpha)) = DC(\Lambda_\infty(\beta),\Lambda_\infty(\alpha)) = \Lambda_\infty(\beta)^{\times}$.*

PROOF. First we compute $\Lambda_\infty(\alpha) \cdot \Lambda_\infty(\beta)^{\times} = \{t$: there exists k such that for all m, $t \cdot m^{\alpha}/k^{\beta} \in l_\infty\} = \{t$: there exists k such that for all m there exists $M > 0$ such that for all n, $\|t_n| \leqq M\, k^{\beta_n}/m^{\alpha_n}\} = \{t$: there exists k such that for all m, $\overline{\lim}_n |t_n|^{1/\beta_n} \leqq k \quad \overline{\lim}_n (1/m)^{\alpha_n/\beta_n} = k\} = \Lambda_\infty(\beta)^{\times}$. Thus $D(\Lambda_\infty(\alpha), \Lambda_\infty(\beta)) = \Lambda_\infty(\beta)^{\times\times} = \Lambda_\infty(\beta)$. Next observe that $\Lambda_\infty(\beta) \cdot \Lambda_\infty(\alpha)^{\times} = \Lambda_\infty(\beta)$ by a similar computation so the first claim is true. For the second, we apply Lemma (2.3) with $\lambda = \Lambda_\infty(\alpha)$, $\mu = \Lambda_\infty(\beta)$.

We are now in a position to characterize those pairs of α and β for which $D(\Lambda_\infty(\alpha), \Lambda_\infty(\beta)) = DC(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$.

THEOREM (2.5). *Let $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$ be nuclear p.s.s. The following are equivalent:*

1) $D(\Lambda_\infty(\alpha), \Lambda_\infty(\beta)) = DC(\Lambda_\infty(\beta), \Lambda_\infty(\alpha))$.

2) $D(\Lambda_\infty(\beta), \Lambda_\infty(\alpha)) = DC(\Lambda_\infty(\beta), \Lambda_\infty(\alpha))$.

3) *$\alpha/\beta \in c_0$, or $\beta/\alpha \in c_0$, or $N = I_1 \cup I_2, I_1, I_2$ infinite disjoint sequences such that $\lim_{n\in I_1} \alpha_n/\beta_n = 0$ and $\lim_{n\in I_2} \alpha_n/\beta_n = \infty$.*

PROOF. 3) $\to$ 1) and 2) by means of Theorem 2.4. Suppose there exists an infinite subsequence $I \subseteq N$ such that $0 < \underline{\lim}_{n\in I} \alpha_n/\beta_n \leqq \overline{\lim}_{n\in I} \alpha_n/\beta_n < +\infty$. Then $\Lambda_\infty(\alpha)_I = \Lambda_\infty(\beta)_I$ so $D(\Lambda_\infty(\alpha), \Lambda_\infty(\beta)) \neq DC(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$. Hence, if 1) is true we obtain the fact that for every infinite subsequence $I \subseteq N$ either $\underline{\lim}\, (\alpha_n/\beta_n) = 0$ or $\overline{\lim}\, (\alpha_n)/(\beta_n) = \infty$. Let $I_1 = \{n : (\alpha_n/\beta_n) \leqq 1\}$ and $I_2 = \{n : (\alpha_n/\beta_n) > 1\}$. If either set is finite then we know that $\alpha/\beta \in c_0$ or $\beta/\alpha \in c_0$. Suppose both I_1 and I_2 are infinite. Then for all $I \subseteq I_1$, $\underline{\lim}_{n\in I} (\alpha_n/\beta_n) = 0$, so $\lim_{n\in I_1} (\alpha_n/\beta_n) = 0$, and similarly $\lim_{n\in I_2} (\alpha_n/\beta_n) = \infty$. Thus, 1) $\to$ 3). By a symmetric argument 2) $\to$ 3).

REMARK (2.6). 1) The symmetric nature of (2.5) is somewhat surprising and is in marked contrast to the situation involving the sequence spaces l_p, $p > 1$, where $L(l_p, l_q) = LC(l_p, l_q)$ if and only if $p > q$ [11].

2) Since $\Lambda_\infty(\alpha)$, $\Lambda_\infty(\beta)$ are nuclear, we know that the compact maps are in fact nuclear maps [4].

3) The condition that all diagonal maps from $\Lambda_\infty(\alpha)$ to $\Lambda_\infty(\beta)$ are compact is not sufficient to conclude that all linear maps from $\Lambda_\infty(\alpha)$ to $\Lambda_\infty(\beta)$ are compact. Indeed, for any α choose β to be any subsequence of α for which $\alpha/\beta \in c_0$. Then by Theorem (2.5), $D(\Lambda_\infty(\alpha), \Lambda_\infty(\beta)) = DC(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$. However, $\Lambda_\infty(\beta)$ is a complemented subspace of $\Lambda_\infty(\alpha)$, so there exists a non-compact projection of $\Lambda_\infty(\alpha)$ onto $\Lambda_\infty(\beta)$. More generally, it is easy to see that if $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$ have a common complemented subspace, then there exists non-compact maps from $\Lambda_\infty(\alpha)$ to $\Lambda_\infty(\beta)$ and from $\Lambda_\infty(\beta)$ to $\Lambda_\infty(\alpha)$. We investigate the converse to this fact below.

4) It is straightforward to prove that Theorem (2.5) is true with $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$ replaced by $\Lambda_1(\alpha)$ and $\Lambda_1(\beta)$. As a consequence of a theorem of Zaharjuta [18], it is known that $L(\Lambda_1(\alpha), \Lambda_\infty(\beta)) = LC(\Lambda_1(\alpha), \Lambda_\infty(\beta))$ for all pairs of α and β. However, there can be non-compact maps, even isomorphisms, from certain spaces $\Lambda_\infty(\alpha)$ to subspaces of certain of the spaces $\Lambda_1(\beta)$ (cf [16], [9]).

3. Generalized diagonal maps and complemented subspaces

DEFINITION (3.1). Let λ and μ be sequence spaces. A linear map $T: \lambda \to \mu$ is said to be a *generalized diagonal map* if there exists a sequence $t \in \omega$ and an injection $\sigma: N \to N$ such that $Te^n = t_n e^{\sigma(n)}$ for every $n \in N$.

THEOREM (3.2). *Let $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$ be nuclear p.s.s. Let (m_i) and (n_i) be increasing sequences of positive integers such that $\beta_{m_i-1} \leqq \alpha_{n_{i-1}} \leqq \alpha_{n_{i-1}+1} \leqq \cdots \leqq \alpha_{n_i-1} < \beta_{m_i} \leqq \cdots \leqq \beta_{m_{i+1}-1} \leqq \alpha_{n_i} \leqq \cdots$ for all i. The following statements are equivalent:*

1) $(\Lambda_\infty(\alpha), \Lambda_\infty(\beta)) \in R$.
2) $(\Lambda_\infty(\beta), \Lambda_\infty(\alpha)) \in R$.
3) *No step space of $\Lambda_\infty(\alpha)$ is isomorphic to a step space of $\Lambda_\infty(\beta)$.*
4) *Every generalized diagonal from $\Lambda_\infty(\alpha)$ to $\Lambda_\infty(\beta)$ is compact.*
5) *Every generalized diagonal from $\Lambda_\infty(\beta)$ to $\Lambda_\infty(\alpha)$ is compact.*
6) (a) $\lim_i (\beta_{m_i}/\alpha_{n_i-1}) = \infty$ *and* (b) $\lim_i (\alpha_{n_i}/\beta_{m_{i+1}-1}) = \infty$.
7) $K(\Lambda_\infty(\alpha)) \cap K(\Lambda_\infty(\beta)) = \varnothing$.
8) *No CBS in $\Lambda_\infty(\alpha)$ is equivalent to a CBS in $\Lambda_\infty(\beta)$.*

PROOF. Clearly 1) → 3) and 2) → 3) as in Remark (2.6), (4). To see that 3) → 4), we suppose that there exists a generalized diagonal map $T\colon \Lambda_\infty(\alpha) \to \Lambda_\infty(\beta)$ which is not compact, say $Te^n = t_n \cdot e^{\sigma(n)}$; for all n. Let $I = \{n : t_n \neq 0\}$. I must be infinite, so write $I = (i_n)$, $i_n < i_{n+1} < \cdots$. Let $\alpha'_n = \alpha_{i_n}$ and $\beta'_n = \beta_{\sigma(i_n)}$. Since T is non-compact, $D(\Lambda_\infty(\alpha'), \Lambda_\infty(\beta')) \neq DC(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$. By Theorem 2.5, 3) → 1), we obtain an infinite set $I' \subseteq I$ such that $0 < \underline{\lim}_{n\in I'} (\alpha'_n/\beta'_n) \leqq \overline{\lim}_{n\in I'} (\alpha'_n/\beta'_n) < \infty$, so $\Lambda_\infty(\alpha')_I = \Lambda_\infty(\beta')_I$. Hence $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$ have a common stepspace. A parallel argument will establish that 3) → 5).

Now assume that 6) fails. If $\lim_i (\beta_{m_i}/\alpha_{n_i-1}) \neq \infty$, then there exists an infinite set $I \subseteq N$ and $1 < M < +\infty$ such that $1 \leqq (\alpha_{n_i}/\beta_{m_{i+1}-1}) \leqq M$ for all $\sigma \in I$. Thus $\Lambda_\infty((\alpha_{n_i-1})_{i\in I}) = \Lambda_\infty((\beta_{m_i})_{i\in I})$.

Moreover, if $\lim_i (\alpha_{n_i}/\beta_{m_{i+1}-1}) \neq \infty$, then there exists an infinite set $I \subseteq N$ and $1 < M < \infty$ such that $1 \leqq (\alpha_{n_i}/\beta_{m_{i+1}-1}) \leqq M$, so that $\Lambda_\infty((\alpha_{n_i})_{i\in I}) = \Lambda_\infty((\beta_{m_{i+1}-1})_{i\in I})$. Thus if 6) fails we see that there exist infinite sets $I = (i_n)$ and $J = (j_n)$ in N such that $\Lambda_\infty(\alpha)_I = \Lambda_\infty(\beta)_J$.

Define

$$Te^j = \begin{cases} e^{i_n} & \text{if } j = j_n \\ 0 & \text{if } j \notin J. \end{cases}$$

Then it is easy to check that T is a non-compact generalized diagonal map from $\Lambda_\infty(\alpha)$ to $\Lambda_\infty(\beta)$. Thus 4) → 6). By adjusting T in an obvious way we also obtain (5) → (6).

Next, we establish (6) → (1).

First, observe that $L(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$ is solid; i.e., if $T \sim (t_{ij})$ and if $|s_{ij}| \leqq |t_{ij}|$ for all i, j, then the map $S \sim (s_{ij}) \in L(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$ whenever $T \in L(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$. Thus, it suffices to deal with $T \sim (t_{ij})$ where $t_{ij} > 0$ for all i, j. Next write $t_{ij} = \exp(-r_{ij}\beta_i)$. Since $Te^j = \Sigma_i t_{ij} e^i \in \Lambda_\infty(\beta)$, we must have $\lim_i r_{ij} = \infty$ for each j. $T \in L(\Lambda_\infty(\alpha), \Lambda_\infty(\beta))$, if and only if for all p there exists q such that $\sup_j \|Te^j\|_p / \|e^j\|_q < +\infty$, by Lemma 2.1. But $\|Te^j\|_p = \exp\{(p - r_{i(p,j),j})\beta_{i(p,j)}\}$ for a suitable choice of $i(p,j)$, for each p and j. Now we write $p - r_{i(p,j),j} = c_{p,j}(\alpha_j/\beta_{i(p,j)})$, and we see that for all p, there exists q such that $\sup_j (c_{p,j} - q)\alpha_j < +\infty$, so that for all p, $\sup_j (c_{p,j}) < +\infty$. But for each j, the sequence $(\|Te^j\|_p)_p$ is non-decreasing in p, so for each j, $(c_{pj})_p$ is non-decreasing in p. By definition of $i(p,j)$, we have for each p and each j, $\sup_i (p - r_{ij})\beta_i = c_{pj}\alpha_j$. Then for all s, p, and j, $(p - r_{i(s,j),j})\beta_{i(s,j)} \leqq c_{pj}\alpha_j$, so that $(p - s)\beta_{i(s,j)} \leqq (c_{pj} - c_{sj})\alpha_j$. Then $(\beta_{i(s,j)})_j < (\alpha_j)_j$ for each s. But if $\beta_{i(s,j)} > \alpha_j$ infinitely often, condition (a) of (6) implies that $\lim_j \beta_{i(s,j)}/\alpha_j = \infty$. Hence, for each s, $\beta_{i(s,j)} \leqq \alpha_j$ for sufficiently large j. Now if $s > p$, we know that for all j $(\beta_{i(s,j)}/\alpha_j) \geqq (c_{sj} - c_{pj})/(s-p) \geqq 0$, and condition (b) of (6) implies that $\lim_j (c_{sj} - c_{pj}) = 0$.

Finally we suppose that T is not compact. Then for all q, there exists p such that $\sup_j (c_{pj} - q)\ \alpha_j = \infty$. Hence for all q, there exists p and an infinite set I such that $c_{pj} > q$ for $j \in I$. Choose some p_0 and let $q \geqq \sup c_{p_0,j}$. Then, there exists $p > p_0$ and an infinite set I such that $c_{pj} > 2q$ for $j \in I$. But then $c_{pj} - c_{p_0j} > q$ infinitely often, contrary to the fact that $\lim_j (c_{pj} - c_{p_0j}) = 0$. Hence $6) \to 1)$, and a symmetric argument shows that $6) \to 2)$. Thus we have established the equivalence of the first six statements.

$(6) \to (7)$: Let (k_n) and (k'_n) be non-decreasing sequences of positive integers converging to ∞ and $[d_n e^{k_n}] \in K(\Lambda_\infty(\alpha))$, $[d'_n e^{k'_n}] \in K(\Lambda_\infty(\beta))$. $[d_n e^{k_n}] \cong [d'_n e^{k'_n}]$ if and only if $[e^{k_n}] = [e^{k'_n}]$ [2], and this is true if and only if $0 < \inf_n (\alpha_{k_n}/\beta_{k'_n}) \leqq \sup_n (\alpha_{k_n}/\beta_{k'_n}) < +\infty$.

Now if $\alpha_{k_n} \leqq \beta_{k'_n}$ infinitely often, then by (a) we have $\inf_n (\alpha_{k_n}/\beta_{k'_n}) = 0$, and if $\alpha_{k_n} \geqq \beta_{k'_n}$ infinitely often, then by (b) we have $\sup_n (\alpha_{k_n}/\beta_{k'_n}) = \infty$. Hence $K(\Lambda_\infty(\alpha)) \cap K(\Lambda_\infty(\beta)) = \emptyset$.

$(7) \to (8)$ is a consequence of Bessaga's Theorem (1.3). $(8) \to (3)$ trivially, so the proof is complete.

REMARK (3.3). It can be shown that Theorem (3.2) is valid with the infinite type p.s.s. $\Lambda_\infty(\alpha)$ and $\Lambda_\infty(\beta)$ replaced by nuclear finite type p.s.s. $\Lambda_1(\alpha)$ and $\Lambda_1(\beta)$, respectively.

4. Diametral dimension and $\lambda \cdot \lambda^\times$

In (2.3), we made use of the hypothesis that $\lambda \cdot \mu^\times$ was perfect. It is easy to see that it is always normal, and by making a connection between $\lambda \cdot \lambda^\times$ and diametral dimension, we show that in general $\lambda \cdot \lambda^\times$ is not perfect.

DEFINITION (4.1). We say that the Köthe space $\lambda = \cap (1/a_p) \cdot l_1$ is *regular* if for all p and q, $(a_{pn}/a_{qn})_n$ is monotone. Let $U_p = \{t \in \lambda : \|t\|_p \leqq 1\}$. Then whenever λ is regular, $d_{n-1}(U_q, U_p) = (a_{pn}/a_{qn})$ for all n and for all $q \geqq p$ [7]. The diametral dimension $\delta(\lambda)$ (of [2]) is then equal to $\{t$: there exists p such that for all q, $t \cdot (a_q/a_p) \in l_1\} = \cup_p \cap_q (a_p/a_q) \cdot l_1 = \lambda \cdot \lambda^\times.\}$. Also the diametral dimension $\Gamma(\lambda) = \cap_p \cup_q (a_q/a_p) \cdot l_1 = (\lambda \cdot \lambda^\times)^\times = D(\lambda, \lambda)$. Observe that $\delta(\lambda)^\times = \Gamma(\lambda)$. In [6], Dragilev shows that the diametral dimension Γ does not distinguish the class of spaces with a regular basis, by constructing two regular spaces λ and μ with regular bases such that λ is not isomorphic to μ but such that

$\Gamma(\lambda) = \Gamma(\mu)$. (Theorem 2, [6]). Moreover, μ is a (D_1) space so (by Prop. 2.5 of [15]) $\mu \cdot \mu^{\times} = \mu$. We claim that the space λ is such that $\lambda \cdot \lambda^{\times}$ is not perfect. For if $\lambda \cdot \lambda^{\times}$ is perfect, then the fact that $\Gamma(\lambda) = \Gamma(\mu)$ implies that $\lambda \cdot \lambda^{\times} = (\lambda \cdot \lambda^{\times})^{\times\times} = (\mu \cdot \mu^{\times})^{\times\times} = \mu$. But, by applying Lemma 2 of [5], we see that there exist $d_n > 0$ such that $\lambda = d \cdot \mu$, which is a contradiction.

REMARK (4.2). We now give an example of a nuclear sequence space $\Lambda(P)$ which has the property that $\Lambda(P)'_b = \bigcup_{a \in P} a \cdot l_\infty \not\subseteq \Lambda(P)^{\times}$. Such spaces are mentioned in [17] and [12], but to our knowledge, no examples have been presented. For this, we let $P = \{a \colon a \in \lambda \cdot \lambda^{\times}, a \geqq 0\}$, where $\lambda = \bigcap (1/a_K) \cdot l_1$, the space in (4.1). Then $\Lambda(P) = (\lambda \cdot \lambda^{\times})^{\times}$ and $\Lambda(P)'_b = \lambda \cdot \lambda^{\times} = \bigcup_{a \in P} a \cdot l_\infty \not\subseteq \Lambda(P)^{\times}$. To see that $\Lambda(P)$ is nuclear, we apply the Grothendieck-Pietsch criterion [14]. In fact, observe that if $a \in P$, then there exists k such that $a \in a^k \cdot \lambda$. Now if $t \in \lambda$, $(n^2 t_n) \in \lambda$ by nuclearity of λ [14], so $(n^2 a_n) \in a^k \cdot \lambda$. Thus, for all $a \in P$, there exists $b \in P$ such that $a/b \in l_1$.

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DEPARTMENT OF MATHEMATICS
CLARKSON COLLEGE OF TECHNOLOGY
POTSDAM, N.Y. 13673, U.S.A.