SOME EXAMPLES CONCERNING STRICTLY CONVEX NORMS ON $C(K)$ SPACES

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ABSTRACT

Some examples of $C(K)$ spaces which admit (respectively, do not admit) an equivalent strictly convex norm are given. These examples consist of ideals in $l_c^\infty(I)$ (the bounded, real-valued functions on the unit interval I having a countable support) which contain $c_0(I)$.

We are concerned in this paper with examples related to the general problem of describing those Banach spaces which admit an equivalent, strictly convex norm. (Recall that a norm is called strictly convex if the surface of its unit ball contains no line segment.) This is a problem in the study of non-separable Banach spaces since, as is well known, every separable Banach space can be strictly convexified (see Day [1]).

Clearly, the property that a Banach space be strictly convexifiable is hereditary (that is, passes from a space to its subspaces) and invariant under isomorphism. In fact, strict convexifiability has a stronger permanence property. If there is a one-to-one continuous linear operator T from a space X into a strictly convexifiable space Y then X is also strictly convexifiable. Indeed, if $\| \ \|_1$ is the given norm on X and $||| \ |||$ is a strictly convex norm on Y then $\| x \|_2 = \| x \|_1 + |||Tx|||$ is easily seen to define an equivalent strictly convex norm on X. Thus, in studying the question of strict convexifiability, it is natural to try to classify spaces according to the existence or non-existence of continuous one-to-one linear maps from one Banach space into another. Again, this classification is of interest only

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for non-separable spaces, since it is easily seen and well known that if X and Y are infinite-dimensional and separable then there always exists a one-to-one map from X into Y. (A simple way to see this is to use the fact that X is a subspace of $C(0,1)$, Y contains a normalized basic sequence $\{y_n\}$, and $T: C(0,1) \to Y$ defined by $Tf = \sum_{n=1}^{\infty} 2^{-n} f(t_n) y_n$, where $\{t_n\}_{n=1}^{\infty}$ is dense in $[0,1]$, is a one-to-one map from $C(0,1)$ into Y.)

The present paper can be considered a sequel to Day's paper [1]. In it Day gave the first example of a non-strictly convexifiable space. He showed that if $X = l_e^{\infty}(\Gamma)$, the space of all bounded functions f from an uncountable set Γ to the reals $\mathbb{R}$ such that the support of f, $\sigma(f) = \{\gamma : f(\gamma) \neq 0\}$, is countable, then X has no strictly convex norm equivalent to the sup norm. (To explain our notation, we remark that X is clearly a subspace of $l^{\infty}(\Gamma)$, the space of all bounded functions from Γ to $\mathbb{R}$; c stands for countably supported.) On the other hand, Day proved in his paper that $c_0(\Gamma)$, the subspace of $l_c^{\infty}(\Gamma)$ consisting of those f such that $\sigma_\varepsilon(f) = \{\gamma : |f(\gamma)| \geqq \varepsilon\}$ is finite for for every $\varepsilon > 0$, does admit a strictly convex norm equivalent to the sup norm. Subsequent to Day's work it was shown that a large class of Banach spaces do admit a continuous one-to-one linear map into some $c_0(\Gamma)$ and thus admit an equivalent strictly convex norm. (Such spaces include all the weakly compactly generated spaces and their duals; see [3].) However, the published results on this subject do not answer the question whether the existence of a one-to-one map from X into some $c_0(\Gamma)$ is actually a necessary condition for X to be strictly convexifiable. (This question was explicitly raised in [3, p. 259].) Likewise, it has been unknown whether the spaces $l_c^{\infty}(\Gamma)$ are the smallest examples of non-strictly convexifiable spaces, in the sense that every non-strictly convexifiable space X contains a one-to-one continuous linear image of $l_c^{\infty}(\Gamma)$ for some uncountable Γ (and thus for Γ of cardinality $\aleph_1$). In this paper we give quite natural examples of spaces which answer both questions negatively. In fact we demonstrate where, in the gap between $c_0(I)$ and $l_c^{\infty}(I)$, strict convexifiability ends and non-strict convexifiability begins for the case $\Gamma = I = [0,1]$ (which is the simplest case, modulo the continuum hypothesis).

We consider I in its natural topology, and for every subset $A \subset I$ and every countable ordinal α, we denote by $A^{(\alpha)}$ the αth derived set of A (that is, $A^{(0)} = A$, $A^{(\alpha+1)}$ is the set of all accumulation points, or cluster points, of $A^{(\alpha)}$, and for a limit ordinal α, $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$). For each countable ordinal α, we define X_α to be the subspace of $l^{\infty}(I)$ consisting of all f such that $\sigma_\varepsilon(f)^{(\alpha)} = \phi$

for every $\varepsilon > 0$ where $\sigma_\varepsilon(f) = \{t \in I : |f(t)| \geqq \varepsilon\}$. Define also $Y = \cup_{\alpha<\omega_1} X_\alpha$. Then we have

$$c_0(I) = X_1 \subset X_2 \subset \cdots \subset X_\alpha \subset X_{\alpha+1} \subset \cdots \subset Y \subset l_c^\infty(I). \tag{1}$$

Clearly, all the X_α and Y are closed linear subspaces, in fact ideals, in $l^\infty(I)$. The space Y has a simple intrinsic description. It consists of all functions in $l_c^\infty(I)$ which belong to the first Baire class, that is, are pointwise limits of sequences of continuous functions (see Proposition 1).

The main result of this note is that, from the point of view of existence of one-to-one maps, all the spaces (1) are distinct (that is, there is no continuous one-to-one linear map going in a direction opposite to the inclusion signs in (1)), that the X_α are all strictly convexifiable, and their union Y is not strictly convexifiable. The non-existence of one-to-one maps is contained in Theorem 2, a somewhat stronger result, which is a consequence of a lemma of Rosenthal [5]. The proofs of the existence, or non-existence, of strictly convex norms are based on the ideas in Day's paper (see Lemmas 6 and 11 below). The unit interval I can be replaced in these results by any uncountable compact metric space and even by more general spaces. However, since here we are simply constructing examples to (hopefully) illuminate the subject of strict convexifiability, we do not see any advantage to working in a more general setting.

We recall that a nonempty set $A \subset I$ is said to be *dense-in-itself* if $A \subset A^{(1)}$, that is, A has no isolated points. A is *scattered* if A has no dense-in-itself subset. Equivalently, A is scattered if and only if $A^{(\alpha)} = \phi$ for some countable ordinal α. If A is scattered, then A is countable, for otherwise the condensation points in A would constitute a dense-in-itself subset (the kernel of A). The following proposition is entirely classical in nature and must certainly be known.

PROPOSITION 1. *If $f \in l^\infty(I)$ then these are equivalent:*
(2) *f is in Baire class 1 and has countable support;*
(3) $f \in Y = \cup_{\alpha<\omega_1} X_\alpha$;
(4) *for each $\varepsilon > 0$, $\sigma_\varepsilon(f) = \{t : |f(t)| \geqq \varepsilon\}$ is scattered.*

PROOF. We invoke the two classical descriptions of Baire class 1 (see [4, Ch. XV]): a function f on I is in Baire class 1 (i) if and only if $f^{-1}(F)$ is a G_δ set in I whenever F is a closed set, and (ii) if and only if for any perfect set $P \subset I$, $f|P$ has a point of continuity. Notice that a countable G_δ set A in I is always scattered, because a dense-in-itself subset $D \subset A$ would have prefect closure $\bar{D}$ in which the set $A \cap \bar{D}$ would be first category on the one hand (being countable) and a dense G_δ in $\bar{D}$ on the other hand, contradicting the Baire category theorem. This observation, combined with (i), proves (2) $\Rightarrow$ (4). Assuming (4), choose for each $n \geqq 1$ a countable ordinal α_n such that $\sigma_{1/n}(f)^{(\alpha_n)} = \phi$, and set $\alpha = \sup_n \alpha_n$. Then $f \in X_\alpha \subset Y$, thus (4) $\Rightarrow$ (3). Now assume (3), that is, $f \in X_\alpha$ for some $\alpha < \omega_1$. We prove (1). By the remarks preceding Proposition 1, $\sigma_{1/n}(f)$ is countable for each $n \geqq 1$, so $\sigma(f) = \bigcup_{n=1}^\infty \sigma_{1/n}(f)$ is countable. Let P be a perfect set in I. For every $n > 0$, $\overline{\sigma_{1/n}(f)} \cap P$ is nowhere dense in P, because if $\phi \neq V \subset \overline{\sigma_{1/n}(f)} \cap P$, with V open in P, then $\sigma_{1/n}(f) \cap V$ would be a dense-in-itself subset of $\sigma_{1/n}(f)$. Thus $\bigcup_{n=1}^\infty \overline{\sigma_{1/n}(f)} \cap P$ is first category in P, so we can choose $t_0 \in P - \bigcup_{n=1}^\infty \overline{\sigma_{1/n}(f)}$. Clearly $f(t_0) = 0$ and $f|P$ is continuous at t_0. This shows by (ii) that f is in Baire class 1, and (3) $\Rightarrow$ (2) is proved. This proves the proposition.

THEOREM 2. (i) *There is no continuous linear* $T: X_2 \to c_0(\Gamma)$ *(for any set* Γ*) such that* $T|c_0(I)$ *is one-to-one.*

(ii) *If* $\alpha < \beta$ *then there is no continuous linear* $T: X_\beta \to X_\alpha$ *such that* $T|c_0(I)$ *is one-to-one.*

(iii) *There is no continuous linear* $T: l_c^\infty(I) \to Y$ *such that* $T|c_0(I)$ *is one-to-one.*

To prove the theorem we use two lemmas. The first is a lemma of H. P. Rosenthal, which J. Kupka has recently proved (see [6]) by a very short argument.

LEMMA 3. (*Rosenthal* [5, p. 16]). *Suppose* S *is an infinite set and* $\{\mu_t : t \in S\}$ *is a family of finitely additive measures defined on all subsets of* S *such that* $\sup_{t \in S} |\mu_t|(S) < \infty$. *Then for each* $\varepsilon > 0$, *there exists a subset* $S' \subset S$ *such that* card $S' =$ card S *and* $|\mu_t|(S' - \{t\}) < \varepsilon$ *for all* $t \in S'$.

LEMMA 4. *Suppose* S *is an uncountable subset of* I *and* $\gamma: S \to I$ *is any one-to-one function. Then*

(i) *for each* $\alpha < \omega_1$ *there exists a set* $E \subset S$ *so that* $E^{(\alpha+1)} = \phi$ *and* $\gamma(E)^{(\alpha)} \neq \phi$; *and*

(ii) *there exists a countable set* $H \subset S$ *so that* $\gamma(H)$ *is dense-in-itself.*

PROOF. By the remarks preceding Proposition 1, choose a nonempty dense-in-itself subset D of the uncountable set $\gamma(S)$. For (i) it suffices to prove (5).

(5) For every $\alpha \geqq 0$ and $x \in D$ there exists a point $t \in I$ such that every open set V containing t also contains a set $E \subset V \cap S$ satisfying $E^{(\alpha)} - \{t\} = \phi$ and $x \in \gamma(E)^{(\alpha)}$.

The proof is by induction. For $\alpha = 0$, take $t = \gamma^{-1}(x)$, $E = \{t\}$. Suppose $\alpha > 0$ and (5) is true for every $\beta < \alpha$. Choose $\beta_1 \leqq \beta_2 \leqq \cdots < \alpha$ so that $\sup_n (\beta_n + 1) = \alpha$.

Choose $x \in D$, and pick a sequence $\{x_n\}$ in D such that $x_m \neq x_n$ if $m \neq n$ and $\lim x_n = x$. Let $B_\varepsilon(t)$ denote the open ball of radius ε about t. For each x_n, choose by the induction hypothesis $t_n \in I$ and a set $E_n \subset S$ such that $E_n \subset B_{1/n}(t_n)$ $E_n^{(\beta_n)} - \{t_n\} = \phi$, and $x_n \in \gamma(E_n)^{(\beta_n)}$. Then choose $n_1 < n_2 < \cdots$ so that $\{t_{n_k}\}$ is Cauchy, and let $t = \lim_{k \to \infty} t_{n_k}$. We show this is the desired $t \in I$ satisfying (5) for x.

Let $V \subset I$ be open with $t \in V$. Since $E_n \subset B_{1/n}(t_n)$ and $t_{n_k} \to t$, there exists $k_0 \geqq 1$ so that $E_{n_k} \subset V$ for $k > k_0$. Define $E = \bigcup_{k \geqq k_0} E_{n_k} \subset V \cap S$. Since each $E_n^{(\beta_n)}$ is at most a singleton, we have $E_n^{(\alpha)} = \phi$, $n = 1, 2, \cdots$, thus for $k_1 \geqq k_0$ we have

$$E^{(\alpha)} \subset E_{n_{k_0}}^{(\alpha)} \cup \cdots \cup E_{n_{k_1}}^{(\alpha)} \bigcup \left(\bigcup_{k > k_1} E_{n_k} \right)^{(\alpha)} \subset \overline{\bigcup_{k > k_1} E_{n_k}}.$$

For any open set $W \subset I$ containing t, $E_{n_k} \subset W$ for large k, so $E^{(\alpha)} \subset \overline{W}$. Thus $E^{(\alpha)} - \{t\} = \phi$. Finally, for each $\beta < \alpha$, eventually $\beta_{n_k} \geqq \beta$ and

$$x_{n_k} \in \gamma(E_{n_k})^{(\beta_{n_k})} \subset \gamma(E)^{(\beta_{n_k})} \subset \gamma(E)^{(\beta)}.$$

Since $x_{n_k} \to x$, we have $x \in \bigcap_{\beta < \alpha} \gamma(E)^{(\beta + 1)} = \gamma(E)^{(\alpha)}$. This proves (5), and (i) is proved.

To prove (ii), just let $H = \gamma^{-1}(C)$, where $C \subset D$ is some countable set dense in D. Then H is countable and $\gamma(H) = C$ is dense-in-itself. This proves Lemma 4.

Proof of Theorem 2. Suppose Z is a Banach space with $c_0(I) \subset Z \subset l^\infty(I)$ and $T: Z \to l^\infty(\Gamma)$, for some set Γ, is a continuous operator such that $T \mid c_0(I)$ is one-to-one. We denote by k_E the characteristic function of the set E. By Zorn's lemma, choose a maximal subset $I_0 \subset I$ which is the domain of some one-to-one function $\gamma: I_0 \to \Gamma$ satisfying $Tk_{(t)}(\gamma_t) \neq 0$ for all $t \in I_0$. By the maximality of I_0 we must have $Tk_{(t)}$ supported in $\gamma(I_0)$ for all $t \in I - I_0$, thus for all $f \in c_0(I - I_0)$ we have $\sigma(Tf) \subset \gamma(I_0)$. Since $T \mid c_0(I)$ is one-to-one, the maps $f \to Tf(\gamma_t)$, $t \in I_0$, form a total set of linear functionals over $c_0(I - I_0)$. Thus I_0 must be uncountable, otherwise $I - I_0$ would be uncountable and $c_0(I - I_0)^*$ would contain a countable total set, a contradiction. Thus there exists $\varepsilon > 0$ so that, for all t in some uncountable subset $I_1 \subset I_0$, $|Tk_{(t)}(\gamma_t)| > \varepsilon$. For $t \in I_1$, let $\delta_{\gamma_t} \in l^\infty(\Gamma)^*$ be an evaluation at γ_t, and by the Hahn-Banach theorem choose $\mu_t \in l^\infty(I)^*$ such that $\mu_t \mid Z = T^*\delta_{\gamma_t}$ and $\| \mu_t \| = \| T^*\delta_{\gamma_t} \| \leqq \| T \|$. Regarding the μ_t as finitely additive measures on the subsets of I, we find by Rosenthal's lemma an uncountable subset $I_2 \subset I_1$

such that $|\mu_t|\,(I_2 - \{t\}) < \frac{1}{2}\varepsilon$ for all $t \in I_2$. If $E \subset I_2$ is any subset with $k_E \in Z$ then, for all $t \in E$,

$$\begin{aligned} |Tk_E(\gamma_t)| &= |T^*\delta_{\gamma_t}(k_E)| = |\mu_t(E)| \\ &\geqq |\mu_t(\{t\})| - |\mu_t|\,(E - \{t\}) \\ &\geqq |Tk_{\{t\}}(\gamma_t)| - |\mu_t|(I_2 - \{t\}) \\ &> \varepsilon - \tfrac{1}{2}\varepsilon = \tfrac{1}{2}\varepsilon. \end{aligned}$$

Thus $\gamma(E) = \{\gamma_t : t \in E\} \subset \sigma_{\frac{1}{2}\varepsilon}(Tk_E)$.

To prove (*i*), suppose $X_2 \subset Z$ and choose E to be any one-to-one Cauchy sequence in I_2. Then $k_E \in X_2 \subset Z$ and $\gamma(E)$ is infinite, γ being one-to-one. Thus $\sigma_{\frac{1}{2}\varepsilon}(Tk_E)$ is infinite, so $Tk_E \notin c_0(\Gamma)$ and (i) is proved.

Now suppose $\Gamma = I$. To prove (ii), suppose $\alpha < \beta$ and $X_\beta \subset Z$. By Lemma 4, choose $E \subset I_2$ so that $E^{(\alpha+1)} = \phi$ and $\gamma(E)^{(\alpha)} \neq \phi$. Then $k_E \in X_{\alpha+1} \subset Z$ and $\sigma_{\frac{1}{2}\varepsilon}(Tk_E)^{(\alpha)} \neq \phi$. Thus $Tk_E \notin X_\alpha$, and (ii) is proved. To prove (iii), suppose $l_c^\infty(I) \subset Z$ and choose, again by Lemma 4, a countable set $H \subset I_2$ such that $\gamma(H)$ is dense-in-itself. Then $k_H \in l_c^\infty(I) \subset Z$ but $\sigma_{\frac{1}{2}\varepsilon}(Tk_H)$ is not scattered since it contains $\gamma(H)$. Thus, by Proposition 1, $Tk_H \notin Y$, and (iii) is proved. This proves Theorem 2.

Our next result gives a stronger version of part (iii) of the preceding theorem. It is similar to the classical stationarity principle for monotone transfinite sequences of Baire class 1 functions (see Kuratowski [2, p. 420]). We denote by $\mathscr{B}_1$ the space of all bounded Baire class 1 functions on I. For $x \in l_c^\infty(\Gamma)$, $\|x\| = 1$, we let F_x denote the facet of the unit ball determined by x, that is,

$$F_x = \{y \in l_c^\infty(\Gamma) : \|y\| = 1,\ y(\gamma) = x(\gamma) \text{ if } x(\gamma) \neq 0\}.$$

THEOREM 5. *If Γ is uncountable and $T : l_c^\infty(\Gamma) \to \mathscr{B}_1$ is any continuous linear operator, then there exists an $x \in l_c^\infty(\Gamma)$ of norm 1 such that $Ty = Tx$ for all $y \in F_x$.*

Before proving the theorem, we need a lemma which will also be used to prove Theorem 9. If F is a convex set in a linear space and $x \in F$, we say that F is *symmetric about* x if whenever $y \in F$ then $y' = 2x - y$ is also in F (hence $x = \frac{1}{2}[y + y']$). A function $\rho : F \to R$ is *convex* if, for all $x, y \in F$ and $t \in [0,1]$, $\rho(tx + (1-t)y) \leqq t\rho(x) + (1-t)\rho(y)$.

LEMMA 6. *Suppose $F_1 \supset F_2 \supset F_3 \supset \cdots$ is a decreasing sequence of convex sets in a linear space, $x_n \in F_n$, and each F_n is symmetric about x_n. Let $\rho : F_1 \to \mathrm{R}$ be a bounded convex function, and let*

$$M_n = \sup\{\rho(z): z \in F_n\} \text{ and } m_n = \inf\{\rho(z): z \in F_n\}.$$

If

$$\rho(x_n) \geqq \tfrac{3}{4} M_{n-1} + \tfrac{1}{4} m_{n-1}, \qquad n = 2, 3, \cdots,$$

then ρ *is constant on* $F = \bigcap_{n=1}^{\infty} F_n$.

PROOF. The idea comes from Day [1, p. 521–522]. First choose $n \geqq 1$ and $\varepsilon > 0$. Pick $y \in F_n$ so that $\rho(y) < m_n + 2\varepsilon$. Since $2x_n - y \in F_n$ and $x_n = \frac{1}{2}(2x_n - y) + \frac{1}{2}y$, we have $\rho(x_n) < \frac{1}{2}M_n + \frac{1}{2}m_n + \varepsilon$, and ε being arbitrary, we rewrite this as $m_n \geqq 2\rho(x_n) - M_n$. For $n > 1$, we then have

$$\begin{aligned} M_n - m_n \leqq M_n - 2\rho(x_n) + M_n &\leqq 2M_{n-1} - \tfrac{3}{2}M_{n-1} - \tfrac{1}{2}m_{n-1} \\ &= \tfrac{1}{2}[M_{n-1} - m_{n-1}]. \end{aligned}$$

By induction we obtain $M_n - m_n \leqq (2^{-n+1})[M_1 - m_1]$, $n = 2, 3, \cdots$, so $M_n - m_n \to 0$ and ρ must be constant on $F = \bigcap_{n=1}^{\infty} F_n$. This proves Lemma 6.

COROLLARY 7. *If* $x \in l_c^{\infty}(\Gamma)$, $\|x\| = 1$, *and* $\rho_1, \rho_2, \cdots$ *are bounded convex functions on* F_x, *then there exists a fixed* $y \in F_x$ *such that all* ρ_m *are constant on* F_y.

PROOF. First prove it for a single ρ. Choose inductively a sequence $x_1, x_2, \cdots$ so that $x_1 = x$, $\|x_n\| = 1$, $F_{x_1} \supset F_{x_2} \supset \cdots$, and for $n = 2, 3, \cdots$,

$$\rho(x_n) \geqq \tfrac{3}{4}\sup\{\rho(z): z \in F_{x_{n-1}}\} + \tfrac{1}{4}\inf\{\rho(z): z \in F_{x_{n-1}}\}.$$

Clearly, the sequence $\{x_n\}$ is pointwise convergent on Γ, and if $y = \lim x_n$ then $F_y = \bigcap_{n=1}^{\infty} F_{x_n}$. By Lemma 6, ρ is constant on F_y. For a sequence $\rho_1, \rho_2, \cdots$ choose inductively $y_0 = x$ and $y_m \in F_{y_{m-1}}$ so that ρ_m is constant on F_{y_m}. Then all ρ_m are constant on $F_y = \bigcap_{m=1}^{\infty} F_{y_m}$, where $y = \lim y_m$ (pointwise).

PROOF OF THEOREM 5. For $n \geqq 1$ and $y \in l_c^{\infty}(\Gamma)$, $\|y\| = 1$, let

$$S_{n,y} = \{t \in I: \sup_{z \in F_y} Tz(t) - \inf_{z \in F_y} Tz(t) < 1/n\}.$$

We first prove:

(i) For every closed set $K \subset I$, every $n \geqq 1$, and every $x \in l_c^{\infty}(\Gamma)$ with $\|x\| = 1$, there exists a $y \in F_x$ such that $S_{n,y}$ contains a nonempty relatively open subset of K.

Fix K, n, and x. Let $G_1, G_2, \cdots$ denote a base of open sets in I. If we define the functions $\bar{\varphi}_m$, $\underline{\varphi}_m$ on F_x by $\bar{\varphi}_m(z) = \sup_{t \in K \cap G_m} Tz(t)$ and $\underline{\varphi}_m(z) = \inf_{t \in K \cap G_m} Tz(t)$, $z \in F_x$, where m runs over the integers satisfying $K \cap G_m \neq \phi$, then $\bar{\varphi}_m$ and $-\underline{\varphi}_m$ are bounded convex functions on F_x. By Corollary 7 we choose $y \in F_x$ so that, on F_y, all these functions are constant. Since $Ty \in \mathscr{B}_1$, there exists in K a point of relative continuity of Ty, so there exists some G_{m_0} such that $K \cap G_{m_0} \neq \phi$ and $\bar{\varphi}_{m_0}(y) - \underline{\varphi}_{m_0}(y) < 1/n$. Since $\bar{\varphi}_{m_0}$ and $\underline{\varphi}_{m_0}$ are constant on F_y, we have for each $b \in K \cap G_{m_0}$

$$\sup_{z \in F_y} Tz(b) - \inf_{z \in F_y} Tz(b) \leqq \sup_{z \in F_y} \bar{\varphi}_{m_0}(z) - \inf_{z \in F_y} \underline{\varphi}_{m_0}(z)$$

$$= \bar{\varphi}_{m_0}(y) - \underline{\varphi}_{m_0}(y) < 1/n,$$

so $b \in S_{n,y}$. Therefore $K \cap G_{m_0} \subset S_{n,y}$, which proves (i).

We next prove:

(ii) For each $n \geqq 1$ and $x \in l_c^\infty(\Gamma)$ with $\| x \| = 1$, there exists a $y \in F_x$ such that $S_{n,y} = I$.

To prove (ii), fix n and x. Choose G_{m_1} to be the first G_m contained in any set $S_{n,z}$ for $z \in F_x$ (G_{m_1} exists by part (i) applied to the case $K = I$), and choose $y_1 \in F_x$ such that $G_{m_1} \subset S_{n,y_1}$. Now let m_2 be the next index past m_1 such that G_{m_2} is contained in some $S_{n,z}$, $z \in F_{y_1}$. Choose then $y_2 \in F_{y_1}$ such that $G_{m_2} \subset S_{n,y_2}$. Continuing inductively, we obtain a sequence $\{y_k\}$ such that $y_1 \in F_x$ and $y_{k+1} \in F_{y_k}$ and a subsequence $G_{m_1}, G_{m_2}, \cdots$ of basic open sets with $G_{m_k} \subset S_{n,y_k}$. Now let $y = \lim y_k$; we show $S_{n,y} = I$. Indeed, if $G = \bigcup_{k=1}^\infty G_{m_k}$ then $G \subset S_{n,y}$ because $G_{m_k} \subset S_{n,y_k} \subset S_{n,y}$. It suffices to show $G = I$. If not, then $K = I - G \neq \phi$, and by (i), some $S_{n,z}$, $z \in F_y$, must contain a nonempty set of the form $K \cap G_m$. But $G_m = (K \cup G_m) \cup (G_m - K)$, so $G_m \subset S_{n,z} \cup G \subset S_{n,z} \cup S_{n,y} = S_{n,z}$. Since $z \in F_{y_k}$ for each k, G_m must have been included in the subsequence $G_{m_1}, G_{m_2}, \cdots$. Thus $G_m \subset G$, contradicting $K \cap G = \phi$. This proves (ii).

The theorem now follows quickly. By (ii) choose inductively a sequence $\{y_n\}$ with $y_{n+1} \in F_{y_n}$ and $S_{n,y_n} = I$, $n = 1, 2, \cdots$. Taking $x = \lim y_n$, we have $Ty = Tx$ for all $y \in F_x$. This proves Theorem 5.

COROLLARY 8. *There is no continuous linear one-to-one map from Baire class 2 into Baire class* 1.

PROOF. $l_c^\infty(I)$ is contained in Baire class 2.

REMARK. By Theorem 5, there exists for any $T: l_c^\infty(\Gamma) \to \mathscr{B}_1$ a countable set $\sigma(x) = \Gamma_0 \subset \Gamma$ such that $Tz = 0$ if $z \in l_c^\infty(\Gamma)$ is supported off Γ_0. Thus $T | c_0(\Gamma)$ cannot be one-to-one, and part (iii) of Theorem 2 is strengthened.

THEOREM 9. *The space* $Y = \bigcup_{\alpha < \omega_1} X_\alpha$ *has no strictly convex norm equivalent to the sup norm.*

PROOF. Suppose N is a continuous norm on Y. Choose $f_1 \in Y$ with $\|f_1\| = 1$, and let $P_1 = I$. Define

$$F_1 = \{f \in Y: \|f\| = 1, \, f(t) = f_1(t) \text{ if } f_1(t) \neq 0\}.$$

Suppose $n > 1$ and $f_{n-1} \in Y$, $P_{n-1} \subset I$, and $F_{n-1} \subset F_1$ are defined such that P_{n-1} is perfect. Then choose $f_n \in F_{n-1}$ so that $N(f_n) \geqq \frac{3}{4} \sup\{N(f): f \in F_{n-1}\} + \frac{1}{4} \inf\{N(f): f \in F_{n-1}\}$. Since $\sigma(f_n)$ is countable, we can choose P_n to be a perfect subset of the uncountable Borel set $P_{n-1} - \sigma(f_n)$ ([2, p. 447]). Then define $F_n = \{f \in Y: \|f\| = 1, \, f(t) = f_n(t) \text{ if } f_n(t) \neq 0, \, \sigma(f - f_n) \subset P_n\}$. Clearly, $F_1 \supset F_2 \supset F_3 \supset \cdots$, $P_1 \supset P_2 \supset P_3 \supset \cdots$, and $\sigma(f_{n+1} - f_n) \subset P_n - P_{n+1}$, $n = 1, 2, \cdots$. Obviously each F_n is convex and symmetric about f_n, so Lemma 6 implies that N is constant on $F = \bigcap_{n=1}^{\infty} F_n$. It remains to show that F contains at least two points of Y.

Clearly the sequence $\{f_n\}$ is pointwise convergent, so we let $f = \lim f_n$. We show $f \in Y$ by showing that all $\sigma_\varepsilon(f)$, $\varepsilon > 0$, are scattered and apply Proposition 1. Since $\sigma_\varepsilon(f) = \sigma_\varepsilon(f_1) \cup \bigcup_{n=1}^{\infty} \sigma_\varepsilon(f_{n+1} - f_n)$, it suffices to show $\bigcup_{n=1}^{\infty} \sigma_\varepsilon(f_{n+1} - f_n)$ is scattered. For any subset $D \subset \bigcup_{n=1}^{\infty} \sigma_\varepsilon(f_{n+1} - f_n)$, we let $D_n = D \cap \sigma_\varepsilon(f_{n+1} - f_n)$, $n = 1, 2, \cdots$, and observe that $D_n \subset \sigma(f_{n+1} - f_n) \subset P_n - P_{n+1}$. If D_{n_0} is the first nonempty set D_n for $n \geqq 1$, then $D - D_{n_0} = \bigcup_{n > n_0} D_n \subset \bigcup_{n > n_0} P_n = P_{n_0 + 1}$, so $\overline{D - D_{n_0}} \subset P_{n_0 + 1}$. But $D_{n_0} \subset P_{n_0} - P_{n_0 + 1}$, so $D_{n_0} \cap \overline{D - D_{n_0}} = \phi$, which implies that D_{n_0} is dense-in-itself if D is. But $D_{n_0} \subset \sigma_\varepsilon(f_{n_0 + 1} - f_{n_0})$, which is scattered, thus D_{n_0} cannot be dense-in-itself, hence neither can D. This shows that $\sigma_\varepsilon(f)$ is scattered and $f \in Y$. Obviously $f \in F$. Furthermore, since the P_n are compact, we can find $g \in F$, $g \neq f$, by choosing $t_0 \in \bigcap_{n=1}^{\infty} P_n$ and defining $g(t_0) = 1$ and $g(t) = f(t)$ for $t \neq t_0$. This proves Theorem 9.

THEOREM 10. *For each countable ordinal $\lambda > 0$, the space X_λ is strictly convexifiable.*

PROOF. The fundamental building block is Day's norm, which can be defined for any bounded scalar-valued function f by

$$D(f) = \sup \left(\sum_{n=1}^{\infty} \frac{1}{4^n} |f(s_n)|^2 \right)^{\frac{1}{2}},$$

where the supremum ranges over all one-to-one sequences $\{s_n\}$ in the domain of f. We choose a basis $G_1, G_2, \cdots$ of open sets in I, and for each $f \in l^\infty(I)$ let $D_n(f) = D(f \mid G_n)$, $n = 1, 2, \cdots$. For $f \in l^\infty(I)$ define the function $\hat{f} \in l^\infty(I)$ by $\hat{f}(t) = \limsup \{|f(s)|: s \to t, \; s \neq t\}$, $t \in I$. Let $f^{(0)} = |f|$, $f^{(\mu+1)} = \hat{f}^{(\mu)}$ and $f^{(\mu)}$

$= \inf_{\alpha<\mu} f^{(\alpha)}$ if μ is a limit ordinal. Consider the family Φ_λ of functions φ on $l^\infty(I)$, defined for $f \in l^\infty(I)$ by

$$\varphi(f) = D_n\left(\frac{1}{k}\sum_{i=1}^{k} f^{(\alpha_i)}\right)$$

with n ranging over the integers and $\{\alpha_i\}^k_{i=1}$ ranging over all finite sequences of ordinals such that $0 \leqq \alpha_1 < \alpha_2 < \cdots < \alpha_k < \lambda$. Φ_λ is a countable family, and we arrange it in a simple sequence $\varphi_1, \varphi_2, \cdots$. Then define, for each $f \in X_\lambda$

$$N_\lambda(f) = \left(\sum_{j=0}^{\infty} \frac{1}{4^j}\varphi_j(f)^2\right)^{\frac{1}{2}},$$

where $\varphi_0(f) = \|f\|_\infty$. We will show that N_λ is the desired norm on X_λ. It is trivial to verify that N_λ is a norm equivalent to the sup norm.

We remark here for later use that $f \in X_\alpha$ if and only if $f^{(\alpha)} = 0$. This follows immediately from the relation $\sigma_\varepsilon(f)^{(\alpha)} = \sigma_\varepsilon(f^{(\alpha)})$, which is proved by a straightforward induction.

Before proving that N_λ is strictly convex, we need two lemmas.

LEMMA 11. *Suppose Γ is a set, $f, g \in l^\infty(\Gamma)$, $D(f) = D(g)$, and $D(f+g) = D(f) + D(g)$. Then $f(t) = g(t)$ at any point $t \in \Gamma$ such that either*

$$|f(t)| > \sup_{s \in \Gamma - \{t\}} |f(s)| \quad \text{or} \quad |g(t)| > \sup_{s \in \Gamma - \{t\}} |g(s)|.$$

PROOF. By normalizing, we can assume $D(f) = D(g) = 1$ and $D(f+g) = 2$. For each integer $i \geqq 1$, choose a one-to-one sequence $\{s_1^i, s_2^i, \cdots\} \subset \Gamma$ so that

$$\left(\sum_{n=1}^{\infty} \frac{1}{4^n}|f(s_n^i) + g(s_n^i)|^2\right)^{\frac{1}{2}} > D(f+g) - 1/i.$$

For each $i \geqq 1$, define elements x^i and y^i in l_2 by $x_n^i = (1/2^n)f(s_n^i)$ and $y_n^i = (1/2^n)g(s_n^i)$, $n = 1, 2, \cdots$. Then $\|x^i\|_2 \leqq D(f) = 1$, $\|y^i\|_2 \leqq D(g) = 1$, and

$$2 - 1/i = D(f+g) - 1/i < \|x^i + y^i\|_2 \leqq \|x^i\|_2 + \|y^i\|_2 \leqq D(f) + D(g) = 2.$$

Thus $\|x^i + y^i\|_2 \to 2$ as $i \to \infty$, and by the uniform convexity of the l_2 norm, $\|x^i - y^i\|_2 \to 0$. In particular, $|x_1^i - y_1^i| \to 0$, hence $|f(s_1^i) - g(s_1^i)| \to 0$ as $i \to \infty$. Now suppose $t \in \Gamma$ satisfies $\delta = \sup_{s\in\Gamma-\{t\}}|f(s)| < |f(t)|$. If $n > 2$ and $|a| < |b|$, then $(\frac{1}{4})^2a^2 + (\frac{1}{4})^n b^2 < (\frac{1}{4})^2 b^2 + (\frac{1}{4})^n a^2$, so that if $i \geqq 1$ and $s_1^i \neq t$, then

$$
\begin{aligned}
D(f)^2 - \| x^i \|_2^2 &= D(f)^2 - \sum_{n=1}^{\infty} \frac{1}{4^n} f(s_n^i)^2 \\
&\geqq \left[\frac{1}{4} f(t)^2 + \frac{1}{4^2} f(s_1^i)^2 + \sum_{n=3}^{\infty} \frac{1}{4^n} f(\hat{s}_n^i)^2\right] \\
&\quad - \left[\frac{1}{4} f(s_1^i)^2 + \frac{1}{4^2} f(t)^2 + \sum_{n=3}^{\infty} \frac{1}{4^n} f(\hat{s}_n^i)^2\right] \\
&= (f(t)^2 - f(s_1^i)^2)\left(\frac{1}{4} - \frac{1}{4^2}\right) \geqq (f(t)^2 - \delta^2)\left(\frac{3}{16}\right) > 0
\end{aligned}
$$

(where $\hat{s}_n^i = s_n^i$ unless $s_n^i = t$, and $\hat{s}_n^i = s_2^i$ if $s_n^i = t$). Thus $\| x^i \|_2$ is bounded away from $D(f)$ unless $s_1^i = t$, and since $\| x^i \|_2 \to D(f)$ we must have eventually $s_1^i = t$. Since $|f(s_1^i) - g(s_1^i)| \to 0$, we therefore have $f(t) = g(t)$. This proves Lemma 11.

LEMMA 12. *Suppose $\lambda > 0$ is some ordinal number and $\{a_\alpha : 0 \leqq \alpha \leqq \lambda\}$ is a nonincreasing family of real numbers (that is, if $0 \leqq \alpha \leqq \beta \leqq \lambda$ then $a_\alpha \geqq a_\beta$) such that for $0 < \beta \leqq \lambda$, $a_\beta = \inf\{a_{\alpha+1} : \alpha < \beta\}$. Then for each $\varepsilon > 0$, there exists an increasing finite sequence of ordinals $\alpha_0 < \alpha_1 < \cdots < \alpha_k < \alpha_{k+1}$ such that $\alpha_0 = 0$, $\alpha_{k+1} = \lambda$, and*

$$
\sum_{i=0}^{k} (a_{\alpha_i+1} - a_{\alpha_{i+1}}) < \varepsilon.
$$

PROOF. A simple induction on λ.

To prove that N_λ is strictly convex, assume that $N_\lambda(f) = N_\lambda(g)$ and $N_\lambda(f+g) = N_\lambda(f) + N_\lambda(g)$. We show $f = g$.

Observe that

$$
\begin{aligned}
N_\lambda(f+g) &= \left\| \left\{\frac{1}{2^j} \varphi_j(f+g)\right\} \right\|_2 \leqq \left\| \left\{\frac{1}{2^j} \varphi_j(f) + \frac{1}{2^j} \varphi_j(g)\right\} \right\|_2 \\
&\leqq \left\| \left\{\frac{1}{2^j} \varphi_j(f)\right\} \right\|_2 + \left\| \left\{\frac{1}{2^j} \varphi_j(g)\right\} \right\|_2 = N_\lambda(f) + N_\lambda(g).
\end{aligned}
$$

Since $N_\lambda(f+g) = N_\lambda(f) + N_\lambda(g)$, the $\leqq$ sign can be stated as an $=$ sign in both places. The first equality implies that $\varphi_j(f+g) = \varphi_j(f) + \varphi_j(g)$ and the second that $\varphi_j(f) = \varphi_j(g)$, $j = 1, 2, \cdots$, by the strict convexity of the l_2 norm.

Pick $t \in I$. We must show $f(t) = g(t)$. Notice that $\hat{f}(t) \geqq \hat{\hat{f}}(t) \geqq \cdots \geqq f^{(\lambda)}(t) = 0$. We first show that for each $\alpha \geqq 1$, $f^{(\alpha)}(t) = g^{(\alpha)}(t)$. Indeed, either $f^{(\alpha)}(t) = 0 = g^{(\alpha)}(t)$, or else there exists a least ordinal $\beta \geqq \alpha$, $\beta < \lambda$, such that either $f^{(\beta)}(t) > f^{(\beta+1)}(t)$ or $g^{(\beta)}(t) > g^{(\beta+1)}(t)$. Then $f^{(\alpha)}(t) = f^{(\beta)}(t)$ and $g^{(\alpha)}(t) = g^{(\beta)}(t)$. Suppose $f^{(\beta)}(t) > f^{(\beta+1)}(t)$. Choose a basic open set G_n containing t such that $f^{(\beta)}(t) > \sup_{s \in G_n - \{t\}} f^{(\beta)}(s)$. Since

$$D_n((f+g)^{(\beta)}) \leqq D_n(f^{(\beta)} + g^{(\beta)}) \leqq D_n(f^{(\beta)}) + D_n(g^{(\beta)}),$$

and by assumption, equality holds and $D_n(f^{(\beta)}) = D_n(g^{(\beta)})$, we can apply Lemma 11 to obtain $f^{(\beta)}(t) = g^{(\beta)}(t)$. The same argument applies if $g^{(\beta)}(t) > g^{(\beta+1)}(t)$, so $f^{(\alpha)}(t) = g^{(\alpha)}(t)$ in either case.

We now show that $|f(t)| = |g(t)|$. We can assume that either $f(t) \neq 0$ or $g(t) \neq 0$; say $f(t) \neq 0$. By Lemma 12, select a finite sequence $\alpha_0 = 0 < \alpha_1 < \cdots < \alpha_{k+1} = \lambda$ such that

$$\sum_{i=0}^{k} [f^{(\alpha_i+1)}(t) - f^{(\alpha_{i+1})}(t)] < \tfrac{1}{2}|f(t)|.$$

Let h be the function

$$h = \sum_{i=0}^{k} f^{(\alpha_i)}.$$

Recall that $f^{(\lambda)} = 0$. Then

$$\begin{aligned} h(t) - \hat{h}(t) &\geqq \sum_{i=0}^{k} f^{(\alpha_i)}(t) - \sum_{i=0}^{k} f^{(\alpha_i+1)}(t) \\ &= f^{(\alpha_0)}(t) - f^{(\alpha_{k+1})}(t) - \sum_{i=0}^{k} [f^{(\alpha_i+1)}(t) - f^{(\alpha_{i+1})}(t)] \\ &> |f(t)| - f^{(\lambda)}(t) - \tfrac{1}{2}|f(t)| = \tfrac{1}{2}|f(t)| > 0. \end{aligned}$$

Thus $h(t) > \hat{h}(t)$, so we choose an open set G_n such that $h(t) > \sup_{s \in G_n - \{t\}} |h(s)|$. Let $\varphi \in \Phi_\lambda$ be the semi-norm defined for $u \in X_\lambda$ by

$$\varphi(u) = D_n\left(\frac{1}{k+1} \sum_{i=0}^{k} u^{(\alpha_i)}\right).$$

By assumption $\varphi(f+g) = \varphi(f) + \varphi(g)$, and

$$\varphi(f) = \varphi(g) = D_n\left(\frac{1}{k+1} h\right) = D_n\left(\frac{1}{k+1} \sum_{i=0}^{k} g^{(\alpha_i)}\right).$$

Applying Lemma 11, we obtain $h(t) = \sum_{i=0}^{k} g^{(\alpha_i)}(t)$. But for $\alpha > 0, f^{(\alpha_i)}(t) = g^{(\alpha_i)}(t)$ therefore we must have $f^{(\alpha_0)}(t) = g^{(\alpha_0)}(t) = |f(t)| = |g(t)|$.

This shows that the assumptions $N_\lambda(f+g) = N_\lambda(f) + N_\lambda(g)$ and $N_\lambda(f) = N_\lambda(g)$ imply $|f| = |g|$.

Put now $w = \frac{1}{2}(f+g)$; then $N_\lambda(w) = N_\lambda(f)$ and $N_\lambda(w+f) = N_\lambda(w) + N_\lambda(f)$ so $|w| = |f|$. This implies $f(t) = g(t)$ for all t, and Theorem 10 is proved.

Concluding remarks

The spaces X_α and Y, being closed ideals in $l^\infty(I)$, are isometric to spaces of the form $C_0(K)$, the space of all continuous functions on a compact Hausdorff space K which vanish at some fixed $k_0 \in K$. They are also clearly isomorphic to $C(K)$ spaces. Thus our results show that, even in the class of $C(K)$ spaces, the question whether or not a given space is strictly convexifiable is quite delicate. We do not know of a way to characterize intrinsically those compact Hausdorff spaces K for which $C(K)$ is strictly convexifiable. (And of course we do not know of a characterization of strict convexifiability of general Banach spaces.)

Let us just mention the following observation. Assume that the spaces X_α and Y are constructed over the Cantor set C instead of I; the above theorems will still hold. For any family $\{Z_t : t \in C\}$ of Banach spaces, let

$$Z = \left(\sum_{t \in C} \oplus Z_t\right)_{X^\alpha}$$

denote the Banach space of all functions f on C such that $f_t \in Z_t$ for each $t \in C$, and if f' is defined by $f'(t) = \|f_t\|$, $t \in C$, then $f' \in X_\alpha$; the norm is $\|f\| = \sup_{t \in C} f'(t) = \|f'\|$. If each Z_t is isomorphic to a strictly convex space Z_t', then the isomorphisms can be taken to have uniform upper and lower bounds. Then Z is isomorphic to $Z' = (\sum_{t \in C} \oplus Z_t')_{N_\alpha}$, which is defined as above except that the norm on Z' is $\|f\| = N_\alpha(f')$, where N_α is the strictly convex norm on X_α given by Theorem 10. It is easy to verify that Z' is strictly convex; see Day [1, Th. 6]. Thus $Z = (\sum_{t \in C} \oplus Z_t)_{X_\alpha}$ is strictly convexifiable if each Z_t is. Furthermore, if each Z_t is a subspace of Y, then there is an isometric embedding of Z into Y. Indeed, since C is homeomorphic to $C \times C$, we regard Y as a space of functions on $C \times C$ and each Z_t, $t \in C$, as a space of functions on C. Then define $T: Z \to Y$ by $(Tf)(s,t) = f_t(s)$ for $(s,t) \in C \times C$ and $f \in Z$. T is clearly an isometry since $\sup_{s,t} |(Tf)(s,t)| = \sup_t \|f_t\| = \|f\|$. (It must be verified that Tf is in fact an element of Y. This is done by showing that $\sigma_\varepsilon(Tf)$ is scattered for all $\varepsilon > 0$, which comes from the fact that $\sigma_\varepsilon(f')$ and $\sigma_\varepsilon(f_t)$, $t \in C$, are scattered. We omit the straightforward details.)

In particular, there exists a strictly convexifiable subspace of Y, isomorphic to a $C(K)$ space, which contains subspaces isometric to X_α for every countable α. More generally, given $Z_t \subset Y$, $t \in C$, and Z_t strictly convexifiable, we have constructed strictly convexifiable subspaces $W_\alpha \subset Y$ (for $\alpha < \omega_1$) such that $W_1 \subset W_2 \subset \cdots \subset W_\alpha \subset \cdots$ and each W_α is isometric to $(\sum_{t \in C} \oplus Z_t)_{X_-}$ and therefore isometrically contains each Z_t. In this way, we can repeatedly construct larger, strictly convexifiable subspaces of Y, including each time in the space constructed any given family of $2^{\aleph_0}$ strictly convexifiable subspaces of Y, and thereby strictly convexify all of the spaces in the given family simultaneously.

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