CONSTANTS OF SIMULTANEOUS EXTENSION OF CONTINUOUS FUNCTIONS†

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ABSTRACT

It is shown that if B is the unit ball of a non-separable Hilbert space with its weak topology, then for every number $\lambda \geqq 1$, there exists a space K_λ containing B, such that the constant of simultaneous extension from $C(B)$ to $C(K_\lambda)$ is exactly λ. This gives a negative answer to the question whether the constants of simultaneous extension ought to be odd integers, as was suggested by examples of Corson-Lindenstrauss and Corson-Pelczynski.

Let K be a compact Hausdorff space, and S a closed subset of K. A bounded linear operator T from $C(S)$ into $C(K)$ is called a simultaneous extension operator (*seo*) if for every f in $C(S)$, Tf is an extension of f to a continuous function on K.

Set $\eta(S,K) = \inf\{\| T \|: T \text{ is } seo \text{ from } C(S) \text{ to } C(K)\}$.

The Borsuk-Kakutani theorem (see [3]), ensures the existence of a norm one *seo* provided S is metrizable. On the other hand there are known examples where $\eta(S,K) = \infty$, that is, there exists no bounded extension operator from $C(S)$ to $C(K)$.

Corson and Lindenstrauss [1] were the first to compute the constants of simultaneous extension $\eta(S,K)$ for a pair S, K where none of these extreme cases happen. They showed that if S is the one point compactification of an uncountable discrete set, then for every K containing S, $\eta(S,K)$ is an odd integer (or infinity). Moreover, for every integer n there exists a K_n containing S, with $\eta(S,K_n) = 2n+1$.

Another example of a space S with extension constants $\eta(S,K)$ different from

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1 or ∞, was constructed by Corson and Pelczynski [2]. In their example as well, the only possible values of $\eta(S, K)$ for different K are odd integers.

These two examples led to the question ([2] Prob. 11), whether the numbers $\eta(S,K)$ are always odd integers.

It is the purpose of this note to show that the answer to this question is negative, and that every $\lambda \geqq 1$ can be attained as an extension constant $\eta(S,K)$ for some pair S and K.

For a compact Hausdorff K, we shall identify $C(K)^*$ with the space of all finite regular Borel measures on K. The measure of unit mass concentrated at a point k will be denoted by $\delta(k)$.

THEOREM 1. *Let B be the unit ball of a non-separable Hilbert space with its weak topology. For every $\lambda \geqq 1$, there exists a compact Hausdorff K_λ containing B, such that $\eta(B,K_\lambda) = \lambda$.*

Denote by Σ the unit ball of $C(B)^*$ with its ω^*-topology. If K contains B, then $\eta(B,K) \leqq \lambda$ iff there exists a ω^*-continuous function f mapping K into $\lambda\Sigma$ such that $f(b) = \delta(b)$ for every b in B. In particular, if we take K_λ to be $\lambda\Sigma$ and embed B into $\lambda\Sigma$ canonically by $b \to \delta(b)$, we have that $\eta(B,\lambda\Sigma) \leqq \lambda$. In order to prove that $\eta(B,\lambda\Sigma) = \lambda$, we only have to show that for every $\lambda_1 < \lambda$ there exists no ω^*-continuous map of $\lambda\Sigma$ into $\lambda_1\Sigma$ with $\psi(\delta(b)) = \delta(b)$ for $b \in B$.

Let $\{e_\alpha\}_{\alpha \in A}$ be an orthonormal basis for the Hilbert space whose unit ball is B. Theorem 1 follows easily from the following two propositions.

PROPOSITION 2. *Let n be a natural number and $\lambda > 1$. There exist open sets V_α in $\lambda\Sigma$ with $\delta(n^{-1}e_\alpha) \in V_\alpha$ such that the intersection of any $\frac{1}{2}(\lambda + 1)n^2 + 2$ of them is empty.*

PROPOSITION 3. *Let n be a natural number and $\lambda > 1$. If $\{G_\alpha\}$ is any system of open sets in $\lambda\Sigma$ with $\delta(n^{-1}e_\alpha) \in G_\alpha$, there exist $\frac{1}{2}(\lambda + 1)n^2 - 1$ different G_α's with non-empty intersection.*

To deduce the theorem, assume that $\lambda > \lambda_1$ and that ψ is a continuous map from $\lambda\Sigma$ into $\lambda_1\Sigma$ with $\psi(\delta(b)) = \delta(b)$ for $b \in B$. Choose n large enough so that $\frac{1}{2}(\lambda + 1)n^2 > \frac{1}{2}(\lambda_1 + 1)n^2 + 4$ and let V_α be the neighbourhoods of $\delta(n^{-1}e_\alpha)$ in $\lambda_1\Sigma$ given by Proposition 2, that is, the intersection of every $\frac{1}{2}(\lambda_1 + 1)n^2 + 2$ different V_α is empty. Since $\psi(\delta(n^{-1}e_\alpha)) = \delta(n^{-1}e_\alpha)$ we obtain that $G_\alpha = \psi^{-1}(V_\alpha)$ are open neighbourhoods of $\delta(n^{-1}e_\alpha)$ in $\lambda\Sigma$, and clearly every $\frac{1}{2}(\lambda_1 + 1)n^2 + 2$ of them has empty intersection. By the choice of n this is a contradiction to Proposition 3.

For the proof of the first proposition we shall need the following lemma.

LEMMA 4. *For every* $\lambda < \infty$, $\delta > 0$, $0 < a < 1$ *and every open sub-interval* I *of* $[-1,1]$ *containing* a, *there exist* $\varepsilon > 0$ *and an integer* k *such that for every measure* μ *on* $[-1,1]$ *satisfying*

(i) $\| \mu \| \leqq \lambda$ *and*

(ii) $\left| \int x^j d\mu - a^j \right| < \varepsilon$ *for* $j = 0, 1, \cdots, k$

we have $\mu^+(I) > 1 - \delta$ (where μ^+ denotes the positive part of μ).

PROOF. If the lemma were false, we would have a $\delta > 0$, sequences $\varepsilon_n \to 0$ and $k_n \to \infty$, and measures μ_n satisfying (i) and (ii) for ε_n, k_n such that $\mu_n^+(I) \leqq 1-\delta$. By passsing to a subsequence, if necessary, we can assume that the sequence μ_n converges in the ω^*-topology to a measure ν. Since ν satisfies (ii) with $\varepsilon = 0$ and for every j, we obtain, by the density of the polynomials in $C[-1,1]$, that $\nu = \delta(a)$. Let f be a non-negative continuous function on $[-1,1]$, supported in I, such that $f(a) = \| f \| = 1$. Then we have

$$1 = f(a) = \lim \int f d\mu_n \leqq \liminf \int f d\mu_n^+ \leqq 1 - \delta, \text{ a contradiction.} \quad \blacksquare$$

PROOF OF PROPOSITION 2. Suppose $\lambda > 1$, $\delta > 0$, and n are given, and let ε, k be those attained by Lemma 4 for $a = 1/n$ and $I = ((n+\delta)^{-1}, (n-\delta)^{-1})$.

Define $V_\alpha = \{\mu \in \lambda\Sigma : | \int x_\alpha^j d\mu - n^{-j} | < \varepsilon \quad j = 0, \cdots, k\}$ where x_α is the α-coordinate function on B, that is, if $b = \{b_\alpha\} \in B$ then $x_\alpha(b) = b_\alpha$.

Let $\pi_\alpha : B \to [-1,1]$ be the natural projection of B on the α-coordinate. If $\mu \in V_\alpha$ and μ_α is its image under π_α, then μ_α satisfies the conditions of Lemma 4, and thus we have that $\mu_\alpha^+(I) > 1 - \delta$.

Set $B_\alpha = \{b \in B : (n+\delta)^{-1} < b_\alpha < (n-\delta)^{-1}\} = \pi_\alpha^{-1}(I)$; then clearly

$$\mu^+(B_\alpha) \geqq \mu_\alpha^+(I) > 1 - \delta.$$

Suppose now that $\mu \in V_{\alpha_1} \cap \cdots \cap V_{\alpha_m}$, and let $\{C_k\}$ be the atoms of the partition generated by $B_{\alpha_1}, \cdots, B_{\alpha_m}$. Since the intersection of every $(n+\delta)^2 + 1$ different $B_\alpha' s$ is empty, we obtain that each C_k is contained in at most $(n+\delta)^2$ different $B_\alpha' s$. Hence

$$m(1-\delta) \leqq \sum_{j=1}^{m} \mu^+(B_{\alpha_j}) = \sum_j \sum_{C_k \subseteq B_j} \mu^+(C_k) \leqq (n+\delta)^2 \sum_k \mu^+(C_k) \leqq (n+\delta)^2 \| \mu^+ \|.$$

But $\| \mu \| \leqq \lambda$ and $| \int 1 d\mu - 1 | < \varepsilon$ imply that $\| \mu^+ \| \leqq \frac{1}{2}(\lambda + 1 + \varepsilon)$, and thus we get that $m \leqq \frac{1}{2}(n+\delta)^2(\lambda+1+\varepsilon)/(1-\delta) \leqq \frac{1}{2}(\lambda+1)n^2+1$ provided δ is small enough. ■

For the proof of Proposition 3 we shall need the following simple combinatorial lemma [1].

LEMMA 5. *Let A be an uncountable set, and m an integer. For every α in A, let $\phi(\alpha)$ be a subset of A whose complement is finite. Then there exist $\{\alpha_i\}_{i=1}^m$ in A such that $\alpha_i \in \phi(\alpha_j)$ for every $i \neq j$.*

PROOF OF PROPOSITION 3. Let $\lambda > 1$, n and a rational number $1 < s/t \leqq \frac{1}{2}(\lambda+1)$ be given, and suppose $\{G_\alpha\}$ are open neighbourhoods of $\delta(n^{-1}e_\alpha)$ in $\lambda\Sigma$.

The algebra generated by the coordinate functions, $\{x_\alpha\}$ is, by the Stone-Weierstrass theorem, dense in $C(B)$. Thus every G_α contains a subset of the form

$$G'_\alpha = \left\{\mu \in \lambda\Sigma : \left|\int x_\alpha^j d\mu - n^{-j}\right| < \varepsilon_\alpha;\ \left|\int P_j\, d\mu\right| < \varepsilon_\alpha,\ j = 0,\cdots,k_\alpha\right\}$$

where each P_j is a monomial in the coordinate functions, such that at least one of its variables is different from x_α.

Let A_α be the finite set of all the indices of the variables appearing in the definition of G'_α, and apply Lemma 5 with $\phi(\alpha_j) = A \setminus A_\alpha$ and $m = [(s/t)n^2]$. We thus get a subset $\{\alpha_i\}_{i=1}^m$ of A such that for every $i \neq j$, $\alpha_i \notin A_{\alpha_j}$. Define now

$$z_k = \frac{1}{n}\sum_{r=1}^{n^2} e_{(r,k)} \text{ and } \mu = \left(1 - \frac{s}{t}\right)\delta(0) + \frac{1}{t}\sum_{k=1}^{s}\delta(z_k)$$

where (r,k) is an enumeration of $\alpha_1,\cdots,\alpha_m$, such that every α_i appears exactly t times and for t different k. (It should be understood that if $t[(s/t)n^2] < sn^2$, we shall have only n^2-1 summands for some z_k). One such ordering is given by $(r,k) = \alpha_j$, where $j = [(r-1)s + (k-1]/)t + 1$.

We shall show that $\mu \in G'_{\alpha_1} \cap \cdots \cap G'_{\alpha_m}$, and thus, provided only that s/t is a good enough approximation of $\frac{1}{2}(\lambda+1)$ we shall get that $m = [(s/t)n^2] \geqq \frac{1}{2}(\lambda+1)n^2 - 1$.

We proceed to show that $\mu \in G'_{\alpha_i}$ for every $i \leqq m$.

(i) $$\|\mu\| = \left(\frac{s}{t} - 1\right) + \frac{s}{t} = 2\frac{s}{t} - 1 \leqq 2\,\frac{\lambda+1}{2} - 1 = \lambda.$$

(ii) $$\int 1 d\mu = \left(1 - \frac{s}{t}\right) + \frac{s}{t} = 1.$$

(iii) Since α_i appears for exactly t different k and exactly once for every such k we obtain that

$$\int x^j_{\alpha_i} d\mu = \frac{1}{t} \sum_{k=1}^{s} x^j_{\alpha_i}\left(\frac{1}{n} \sum_{r=1}^{n^2} e_{(r,k)}\right) = \frac{1}{t} \cdot t \left(\frac{1}{n}\right)^j = n^{-j}.$$

(iv) Every monomial P_l appearing in the definition of G'_{α_i} has at least one variable x_β with $\beta \neq \alpha_i$. Since by definition $\beta \in A_{\alpha_i}$, we obtain, by the choice of $\{\alpha_j\}_1^m$, that $\beta \neq \alpha_j$ for every $j \leqq m$. Thus the function x_β and clearly also P_l, is identically zero on the support of μ which implies that $\int P_l d\mu = 0$. ■

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