A NON-STANDARD REPRESENTATION FOR BROWNIAN MOTION AND ITÔ INTEGRATION

BY

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ABSTRACT

In a recent paper [10], Peter A. Loeb showed how to convert non-standard measure spaces into standard ones and gave applications to probability theory. We apply these results to Brownian Motion and Itô integration. We first develop a number of new tools about Loeb spaces. We then show that Brownian Motion can be obtained as the Loeb process corresponding to a non-standard random walk obtained from a *-finite number of coin tosses. This permits a very constructive proof of a special case of Donsker's Theorem. The Itô integral with respect to this Brownian Motion is a non-standard Stieltjes integral with respect to the random walk. As a consequence, an easy proof of Itô's Lemma is possible. The results in this paper were announced in [1].

1. Introduction

Non-standard analysis, introduced by Abraham Robinson in 1960 [14], provides a rigorous means of developing analysis using infinitesimals. It is particularly attractive as a means of reducing continuous processes to discrete ones. For this reason, a number of authors have applied non-standard analysis to problems in measure and probability theory. The following are of special relevance to the subject matter of this paper.

Bernstein and Wattenberg [2] showed that Lebesgue measure on $[0,1]$ can be realized as counting measure on a *-finite collection of points in ${}^*[0,1]$. More specifically, they proved that there exists an internal subset F of ${}^*[0,1]$ such that, for all Lebesgue measurable sets B, the Lebesgue measure of B is given by ${}^\circ(|F \cap {}^*B|/|F|)$, where $|\circ|$ denotes (internal) cardinality.

D. W. Müller [13] used non-standard analysis to give a new proof of the Donsker–Prokhorov invariance principle, and thus establish the existence of

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Brownian Motion. His proof relied on a non-standard characterization of tight measures; it thus did not provide a non-standard representation of Brownian Motion *per se.*

Reuben Hersh [5] produced a discrete non-standard analogue of Wiener measure. In an approach reminiscent of that of Bernstein and Wattenberg, he constructed a *-finite set F of polygonal functions and set $\mu(B) = (|F \cap {}^*B| / |F|)$ for any subset B of $C([0,1])$. However, Hersh's measure is not countably additive on the Borel subsets of $C([0,1])$ and, indeed, is supported on a countable set. Our approach will make use of a related set of polygonal functions; however, we shall induce a countably additive extension of Wiener measure on $C([0,1])$.

Allan F. Abrahamse [0] considered Brownian Motion and White Noise in the context of generalized random processes. He constructed an internal stochastic process defined on $^*R$ by summing a *-finite number of independent random variables, and demonstrated that this process generates essentially the same generalized random process as Brownian Motion. This is a reasonable characterization of Brownian Motion among standard stochastic processes. However, it leaves room for very bad local behaviour in non-standard stochastic processes.[†]

Hersh and P. Greenwood [6] used non-standard increments to obtain some interesting results about standard increments (including quadratic variation) in Brownian Motion and other stochastic processes. Their methods, however, did not produce a non-standard formulation of the Itô integral or a proof of Itô's Lemma.

Peter A. Loeb [10] recently introduced a new technique for formulating probabilistic processes in non-standard terms. He showed how to convert a non-standard measure space $(X, \mathcal{A}, \nu)$ into a standard space $(X, \sigma(\mathcal{A}), L(\nu))$ which inherits most of the structural properties of the original space. This is particularly useful if $\mathcal{A}$ is *-finite. Loeb gave applications to coin tossing and the Poisson process. H. Jerome Keisler has used Loeb spaces to facilitate a synthesis of continuous and discrete processes in Economics.

In the next section, we develop a number of tools which will be needed for the application of Loeb spaces to probabilistic problems. We first extend Loeb's integration theory to unbounded functions and to spaces X with $L(\nu)(X) = +\infty$. In particular, we define $SL^p(X, \mathcal{A}, \nu)$, a well-behaved factor space of a subspace of $^*L^p(X, \mathcal{A}, \nu)$ and show (Theorem 11) that it is isometrically isomorphic to $L^p(X, L(\mathcal{A}), L(\nu))$ via the standard part map; here $(X, L(\mathcal{A}), L(\nu))$ denotes the

[†] It can be shown that Abrahamse's process satisfies a continuity condition similar to our Theorem 27. Thus, his process could have been used to induce a Brownian Motion on R.

completion of $(X, \sigma(\mathscr{A}), L(\nu))$. This permits us to transform functions from one space to the other while preserving integration properties. We show (Corollary 17) that the Lebesgue measure space is the image under a measure-preserving transformation of a *-finite Loeb space. This may be thought of as an easier alternative to the Bernstein–Wattenberg [2] construction of Lebesgue measure. The construction may be generalized to any σ-finite Radon measure on an arbitrary Hausdorff space; the details will be presented in another article. We show (Theorem 22) that the Loeb space of a product $X \times Y$ is closely related to the product of the Loeb spaces of X and Y. Finally, we prove an analogue of the Central Limit Theorem (Theorem 21).

In the third section, we show that Loeb spaces permit a natural representation of Brownian Motion. Our method will be to consider a non-standard *-finite random walk χ. An easy computation shows that β, the standard part of this random walk, is Brownian Motion (Theorem 26). Path continuity follows easily (Theorem 27); hence we get Wiener measure on $C([0,1])$. In addition, this provides a natural setting for proving a special case of Donsker's Theorem (Theorem 29). The weak convergence of the measures induced by the finite random walks is demonstrated by exhibiting their weak limit as the measure induced by χ. In another article, we shall show how the argument can be generalized to give a natural characterization of weak convergence; similar results have been obtained independently by Salim Rashid.

In the fourth section we use the *-finite random walk formulation to develop the theory of Itô integration. The main difficulty in the standard theory is that almost all paths of Brownian Motion are of unbounded variation. However, every path of χ is of *-bounded variation. Hence, given a *-integrable function g, we can define a *-Lebesgue-Stieltjes integral $\int g d\chi$. We shall show (Theorem 33) that, if f is Itô-integrable with respect to β, there is a lifting g of f such that ${}^\circ g = f$ and $\int f d\beta = {}^\circ \int g d\chi$. It then follows that almost every path of $\int f d\beta$ is continuous (Theorem 35) and that $\int f d\beta$ is itself Itô-integrable (Theorem 36). Moreover, χ has the property that, over appropriate infinitesimal intervals $[a, b]$, $(d\chi)^2 = (\chi(b) - \chi(a))^2 = b - a = dt$. This simple observation leads to a direct and easy proof of Itô's Lemma (Theorem 37).

Further applications to the theory of stochastic integration and stochastic differential equations will be given in another article.

2. Properties of Loeb measure spaces

In this section, we study the properties of Loeb spaces. We assume throughout that we have a structure containing the real numbers R, and a denumerably comprehensive enlargement of this structure. Assume that X is an internal set in this enlargement, $\mathscr{A}$ an internal algebra of subsets of X, and ν an internal finitely additive set function $\nu: \mathscr{A} \to {}^*R^+$.

Loeb showed that ${}^\circ\nu$ must necessarily be countably additive and used the Caratheodory Extension Theorem to form a standard measure space $(X, \sigma(\mathscr{A}), L(\nu))$ where $\sigma(\mathscr{A})$ is the smallest σ-algebra containing $\mathscr{A}$, and $L(\nu)$ is the unique extension of ${}^\circ\nu$ to $\sigma(\mathscr{A})$.[†] We shall find it convenient to consider instead the completion of $(X, \sigma(\mathscr{A}), L(\nu))$; we denote this complete measure space by $(X, L(\mathscr{A}), L(\nu))$, and refer to it as the Loeb space of $(X, \mathscr{A}, \nu)$.

Definition 1. $f: X \to {}^*R$ is *finite* if
i) f is $\mathscr{A}$-measurable,
ii) there exists $n \in N$ such that $|f(x)| < n$ for all $x \in X$,
iii) ${}^\circ\nu(\{x : f(x) \neq 0\}) < +\infty$.

Remark 2. Loeb showed that, for each $B \in \sigma(A)$ with $L(\nu)(B) < +\infty$, there exists $A \in \mathscr{A}$ such that $L(\nu)(B \Delta A) = 0$, where $B \Delta A$ denotes symmetric difference. This statement also holds for all $B \in L(\mathscr{A})$ such that $L(\nu)(B) < +\infty$. He further showed that, if $f: X \to {}^*R$ is $\mathscr{A}$-measurable, then ${}^\circ f : X \to R \cup \{+\infty, -\infty\}$ is $\sigma(\mathscr{A})$-measurable. Finally, he showed that if $f: X \to {}^*[-n, n]$ for some $n \in N$, then ${}^\circ\int_A f d\nu = \int_A {}^\circ f dL(\nu)$ for each $A \in \mathscr{A}$ with $L(\nu)(A) < +\infty$. It follows that if f is finite, then ${}^\circ f$ is $L(\nu)$-integrable and ${}^\circ\int_A f d\nu = \int_A {}^\circ f dL(\nu)$ for each $A \in \mathscr{A}$.

Definition 3 (Cf. [2, p. 185]). $f: X \to {}^*R$ is *S-integrable* if
i) f is $\mathscr{A}$-measurable,
ii) ${}^\circ(\int_X |f| d\nu) < +\infty$,
iii) $A \in \mathscr{A}, \nu(A) \simeq 0 \Rightarrow \int_A |f| d\nu \simeq 0$,
iv) $A \in \mathscr{A}, f(A) \subset \mu(0) \Rightarrow \int_A |f| d\nu \simeq 0$.[‡]

Theorem 4. *Suppose $f: X \to {}^*R$ is $\mathscr{A}$-measurable. Then f is S-integrable if and only if there exists a sequence of finite functions $\{f_n\}_{n \in N}$ such that*

$$ {}^\circ\left(\int_X |f - f_n| \, d\nu\right) \to 0 \quad as \quad n \to \infty. $$

Proof. Suppose f is S-integrable. For $n \in {}^*N$, define

[†] The uniqueness in the case ${}^\circ\nu(X) = +\infty$ was supplied by Ward Henson.

[‡] If $L(\nu)(X) < +\infty$, condition (iii) in Definition 1 and condition (iv) in Definition 3 are redundant. This is the only case required for Sections 3 and 4.

$$f_n(x)=\begin{cases}0 & \text{if } |f(x)|<1/n,\\ +n & \text{if } f(x)>n,\\ f(x) & \text{if } 1/n \leqq |f(x)| \leqq n,\\ -n & \text{if } f(x)<-n.\end{cases}$$

For all $n \in {}^*N$, f_n is $\mathcal{A}$-measurable; for $n \in N$, f_n is finite. For each $\omega \in {}^*N - N$,

$$\nu(\{x : |f(x)| > \omega\}) \leqq \frac{1}{\omega}\int_X |f(x)|\, d\nu \simeq 0.$$

Hence

$$\int_X |f(x) - f_\omega(x)|\, d\nu \leqq \int_{f(x)\neq f_\omega(x)} |f(x)|\, d\nu$$

$$= \int_{|f(x)|>\omega} |f(x)|\, d\nu + \int_{|f(x)|<1/\omega} |f(x)|\, d\nu \simeq 0.$$

Hence ${}^\circ(\int |f(x) - f_n(x)|\, d\nu) \to 0$ as $n \to \infty$.

For the converse, let $\{f_n\}$ have the given properties, and $\sup |f_n| < n$. It is clear that ${}^\circ(\int_X |f|\, d\nu) < +\infty$. Suppose we are given $\varepsilon \in R^+$. Find $n \in N$ such that ${}^\circ(\int_X |f - f_n|\, d\nu) < \varepsilon/2$. Suppose $A \in \mathcal{A}$, $\nu(A) < \varepsilon/2n$. Then

$$\int_A |f|\, d\nu \leqq n\nu(A) + \int_A |f - f_n|\, d\nu < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

In particular, if $\nu(A) \simeq 0$, $\int_A |f|\, d\nu < \varepsilon$ for all $\varepsilon \in R^+$, so $\int_A |f|\, d\nu \simeq 0$.

If $A \in \mathcal{A}$ and $f(A) \subset \mu(0)$, then there exists some $\eta \simeq 0$ such that $|f(x)| < \eta$ for all $x \in A$. Let

$$f'_n(x) = \begin{cases} f_n(x) & \text{if } |f_n(x)| \leqq |f(x)|,\\ f(x) & \text{if } |f_n(x)| > |f(x)|.\end{cases}$$

Then $\int_A |f(x)|\, d\nu \leqq \int_A |f'_n|\, d\nu + \int_A |f - f'_n|\, d\nu$

$$\leqq \eta\nu(\{x : f'_n(x) \neq 0\}) + \int_X |f - f_n|\, d\nu.$$

Since the first term is infinitesimal for all $n \in N$, and the second term tends to zero, $\int_A |f(x)|\, d\nu \simeq 0$.

COROLLARY 5. *Suppose f is S-integrable, g is $\mathcal{A}$-measurable, and $|g(x)| \leqq |f(x)|$ for all $x \in X$. Then g is S-integrable.*

PROOF. Let f_n be a sequence of finite functions such that ${}^\circ\int_X |f-f_n|\,d\nu \to 0$. Let

$$g_n(x)=\begin{cases} g(x) & \text{if} \quad |f_n(x)| \geqq |g(x)| \\ |f_n(x)|\dfrac{g(x)}{|g(x)|} & \text{if} \quad |f_n(x)|<|g(x)|. \end{cases}$$

Then g_n is finite, and $|g(x)-g_n(x)| \leqq |f(x)-f_n(x)|$ for all x. Hence ${}^\circ\int_X |g-g_n|\,d\nu \to 0$.

THEOREM 6. *Suppose $f: X \to {}^*R$ is S-integrable. Then ${}^\circ f$ is $L(\nu)$-integrable and ${}^\circ(\int_A f d\nu) = \int_A {}^\circ f dL(\nu)$ for any $A \in \mathscr{A}$.*

PROOF. Let $\{f_n\}_{n\in N}$ be a sequence of finite functions such that ${}^\circ(\int |f - f_n|\,d\nu) \to 0$ as $n\to\infty$. ${}^\circ f_n$ is $L(\nu)$-integrable by Remark 2. For each $\varepsilon \in R^+$, $L(\nu)(\{x : |{}^\circ f(x) - {}^\circ f_n(x)| > \varepsilon\}) \to 0$ as $n\to\infty$, so ${}^\circ f_n \to {}^\circ f$ in measure (with respect to $L(\nu)$)

$$\int |{}^\circ f_n - {}^\circ f_m|\,dL(\nu) = {}^\circ\left(\int |f_n - f_m|\,d\nu\right) \leqq {}^\circ\left(\int |f_n - f|\,d\nu\right) + {}^\circ\left(\int |f_m - f|\,d\nu\right) \to 0$$

as $m, n \to \infty$.
Therefore, $\{{}^\circ f_n\}$ is a Cauchy sequence in $L^1(X, L(\mathscr{A}), L(\nu))$.
Hence ${}^\circ f \in L^1(X, L(\mathscr{A}), L(\nu))$, and

$$\int_A {}^\circ f dL(\nu) = \lim_{n\to\infty}\int_A {}^\circ f_n dL(\nu) = \lim_{n\to\infty} {}^\circ\left(\int_A f_n d\nu\right) = {}^\circ\int_A f d\nu.$$

THEOREM 7. *Suppose $g: X \to R$ is $L(\nu)$-integrable. Then there is an S-integrable $f: X \to {}^*R$ such that ${}^\circ f = g$ (almost everywhere with respect to $L(\nu)$).*

PROOF. Loeb gives L. C. Moore Jr.'s proof of a result containing a special case of this theorem: it is assumed that g is bounded and $\sigma(\mathscr{A})$-measurable, and that $L(\nu)(\{x : g(x) \neq 0\}) < +\infty$: f is found to be finite. By Remark 2, Moore's proof goes over immediately to the case where g is bounded and $L(\mathscr{A})$-measurable. We now turn to the general case. Let

$$g_n(x)=\begin{cases} +n & g(x)>n, \\ g(x) & 1/n \leqq |g(x)| \leqq n \qquad (n\in N), \\ -n & g(x)<-n, \\ 0 & |g(x)|<1/n. \end{cases}$$

By the Dominated Convergence Theorem, $\int_X |g - g_n| \, dL(\nu) \to 0$ as $n \to \infty$. Since g is integrable, $L(\nu)(\{x: g_n(x) \neq 0\}) < +\infty$. In addition, g_n is bounded and $L(\mathcal{A})$-measurable, so we may find f_n finite such that $°f_n = g_n$ ($L(\nu)$-almost everywhere).

$$°\left(\int_X |f_n - f_m| \, d\nu\right) = \int_X |g_n - g_m| \, dL(\nu) \to 0 \quad \text{as} \quad n, m \to \infty.$$

Using the fact that our enlargement is denumerably comprehensive, we may extend $\{f_n\}_{n \in N}$ to an internal sequence. Then there must exist $\omega \in {}^*N - N$ such that f_ω is $\mathcal{A}$-measurable and $°(\int_X |f_n - f_\omega| \, d\nu) \to 0$ as $n \to \infty$. Write $f = f_\omega$. Thus f is S-integrable, so by Theorem 6 $°f$ is $L(\nu)$-integrable.

$$\int_X |°f - g| \, dL(\nu) \leqq \int_X |°f - °f_n| \, dL(\nu) + \int_X |°f_n - g| \, dL(\nu)$$

$$= °\left(\int_X |f - f_n| \, d\nu\right) + \int_X |g_n - g| \, dL(\nu) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $\int_X |°f - g| \, dL(\nu) = 0$, so $°f = g$ ($L(\nu)$-almost everywhere).

COROLLARY 8. *Let f and g be as in Theorem 7. Then for $A \in \mathcal{A}$, $\int_A f d\nu \simeq \int_A g dL(\nu)$. Moreover, f is uniquely determined in the sense that, if f_1 is S-integrable and $°f_1 = g$ ($L(\nu)$-almost everywhere), then $\int_X |f - f_1| \, d\nu \simeq 0$.*

PROOF. Immediate.

THEOREM 9. *Suppose $f: X \to {}^*R$ is $\mathcal{A}$-measurable. Then the following are equivalent:*

i) *f is S-integrable,*
ii) *$°f$ is $L(\nu)$-integrable and $°\int |f| \, d\nu = \int |°f| \, dL(\nu)$,*
iii) *$°f$ is $L(\nu)$-integrable and $°\int |f| \, d\nu \leqq \int |°f| \, dL(\nu)$.*

PROOF.

(i) $\Rightarrow$ (ii): If f is S-integrable, then so is $|f|$ and the conclusion follows from Theorem 6.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): Suppose $°f$ is $L(\nu)$-integrable. By Theorem 7, we may find g S-integrable such that $°g = °f$ ($L(\nu)$-almost everywhere). By Corollary 5, we may assume there exists $\alpha: X \to {}^*[0,1]$, $g(x) = \alpha(x) f(x)$ for all x. $L(\nu)(\{x: °g(x) \neq °f(x)\}) = 0$. Hence $L(\nu)(\{x: |g(x) - f(x)| > 1/n\}) = 0$ for $n \in N$, so $\nu(\{x: |g(x) - f(x)| > 1/n\}) \simeq 0$. Let $I = \{n: \nu(\{x: |g(x) - f(x)| > 1/n\}) <$

$1/n\}$. Then I is internal, and contains all finite n. Thus, it contains some infinite ω. Let $A = \{x : |g(x) - f(x)| > 1/\omega\}$. $\nu(A) \simeq 0$.

$$\begin{aligned}
{}^{\circ}\int_X |g|\,d\nu &= \int_X |{}^{\circ}g|\,dL(\nu) && \text{(since } g \text{ is } S\text{-integrable)}\\
&= \int_{X-A} |{}^{\circ}g|\,dL(\nu)\\
&= \int_{X-A} |{}^{\circ}f|\,dL(\nu) && \text{(since } {}^{\circ}f = {}^{\circ}g \text{ on } X - A)\\
&= \int_X |{}^{\circ}f|\,dL(\nu)\\
&\geqq {}^{\circ}\int_X |f|\,d\nu && \text{(by (iii))}.
\end{aligned}$$

But since $g(x) = \alpha(x)f(x)$ for all x,

$${}^{\circ}\int_X |g - f|\,d\nu = {}^{\circ}\int_X |f| - |g|\,d\nu \leqq 0.$$

Therefore $\int_X |g - f|\,d\nu \simeq 0$. If g_n is a sequence of finite functions such that ${}^{\circ}\int |g_n - g|\,d\nu \to 0$, then ${}^{\circ}\int |g_n - f|\,d\nu \to 0$. Thus f is S-integrable by Theorem 4.

Definition 10. For $1 \leqq p < \infty$, let $SL^p(X, \mathcal{A}, \nu)$ be the collection of equivalence classes of all $f : X \to {}^*R$ such that f is $\mathcal{A}$-measurable and $|f|^p$ is S-integrable (with respect to ν), under the equivalence relation $f_1 \sim f_2 \Leftrightarrow (\int |f_1 - f_2|^p d\nu)^{1/p} \simeq 0$. We may define a norm by $\|f\|_p = {}^{\circ}(\int |f|^p d\nu)^{1/p}$. As is customary in standard analysis, we shall often think of an element of SL^p as a function, rather than as an equivalence class of functions.

Theorem 11.

i) *Suppose $f : X \to {}^*R$ is $\mathcal{A}$-measurable. Then*

$$\begin{aligned}
f \in SL^p(X, \mathcal{A}, \nu) &\Leftrightarrow {}^{\circ}f \in L^p(X, L(\mathcal{A}), L(\nu)) \quad \textit{and} \quad \|f\|_p = \|{}^{\circ}f\|_p\\
&\Leftrightarrow {}^{\circ}f \in L^p(X, L(\mathcal{A}), L(\nu)) \quad \textit{and} \quad \|f\|_p \leqq \|{}^{\circ}f\|_p.
\end{aligned}$$

ii) *If $g : X \to R$ is in $L^p(X, L(\mathcal{A}), L(\nu))$, there is a unique $f \in SL^p(X, \mathcal{A}, \nu)$ such that ${}^{\circ}f = g$ ($L(\nu)$-almost everywhere).*

iii) *$SL^p(X, \mathcal{A}, \nu)$ and $L^p(X, L(\mathcal{A}), L(\nu))$ are isometrically isomorphic via the standard part map $f \to {}^{\circ}f$.*

iv) *Suppose ${}^{\circ}\nu(X) < +\infty$, $f \in SL^p(X, \mathcal{A}, \nu)$, and $1 \leqq q \leqq p$. Then $f \in SL^q(X, \mathcal{A}, \nu)$.*

PROOF.

i) Assume f is $\mathscr{A}$-measurable. Then $f \in SL^p(X, \mathscr{A}, \nu) \Leftrightarrow |f|^p$ is S-integrable $\Leftrightarrow |{}^\circ f|^p$ is $L(\nu)$-integrable and ${}^\circ\int_X |f|^p d\nu = \int_X |{}^\circ f|^p dL(\nu) \Leftrightarrow {}^\circ f \in L^p(X, L(\mathscr{A}), L(\nu))$ and $\|f\|_p = \|{}^\circ f\|_p$. The proof of the second equivalence is identical.

ii) Let $B = \{x : g(x) > 0\}$. By Remark 2, we may find $A \in \mathscr{A}$ such that $L(\nu)(A \Delta B) = 0$. Hence we may lift $\max(g, 0)$ and $\min(g, 0)$ separately. It is thus sufficient to consider the case where g is positive.

Since $g \in L^p(X, L(\mathscr{A}), L(\nu))$, g^p is integrable. Hence by Theorem 7, there exists f_1 S-integrable such that ${}^\circ f_1 = g^p (L(\nu)$-almost everywhere). Let $f = (\max(f_1, 0))^{1/p}$. f^p is S-integrable by Corollary 5 and ${}^\circ f = g(L(\nu)$-almost everywhere). Uniqueness follows from Corollary 6.

iii) This is now clear.

iv) If $A \in \mathscr{A}$,

$$\int_A |f|^q d\nu = \int_{A, |f| \geqq 1} |f|^q d\nu + \int_{A, |f| < 1} |f|^q d\nu \leqq \int_A |f|^p d\nu + \nu(A).$$

In particular, ${}^\circ\int_X |f|^q d\nu < +\infty$ and, if $\nu(A) \simeq 0$, $\int_A |f|^q d\nu \simeq 0$.

THEOREM 12. *Let P be an internal partition of X with $P \subset \mathscr{A}$, and let $\mathscr{A}'$ be the internal *-algebra generated by P.*

i) *Suppose f is $L(\mathscr{A})$-measurable, $f|_A$ is constant for each $A \in P$, and there exist $C_n \subset R$, $\bigcup_n C_n = R \cup \{+\infty, -\infty\} - \{0\}$, such that $L(\nu)(f^{-1}(C_n)) < +\infty$ for all $n \in N$. Then f is $L(\mathscr{A}')$-measurable.*

ii) *Suppose P is *-finite, $L(\nu)(X) < +\infty$, and $h \in SL^p(X, \mathscr{A}, \nu)$. Then $E(h \mid \mathscr{A}') \in SL^p(X, \mathscr{A}', \nu)$ and ${}^\circ E(h \mid \mathscr{A}') = E({}^\circ h \mid L(\mathscr{A}'))$.*

PROOF.

i) For $x \in X$, let A_x be the element of P such that $x \in A_x$. Let $\alpha \in R$, $B = \{x : f(x) \neq 0,\ f(x) < \alpha\}$. Let $B_n = B \cap f^{-1}(C_n)$; thus $B = \bigcup_n B_n$, $B_n \in L(\mathscr{A})$, $L(\nu)(B_n) < +\infty$, $x \in B_n \Rightarrow A_x \subset B_n$.

Given $\varepsilon \in R^+$, we may find $F, G \in \mathscr{A}$ such that $G \supset B_n \supset F$, $\nu(G - F) < \varepsilon$. Define $G' = \{x : A_x \subset G\}$, $F' = \{x : A_x \cap F \neq \varnothing\}$. Then G', $F' \in \mathscr{A}'$, and $G \supset G' \supset B_n \supset F' \supset F$, so $\nu(G' - F') < \varepsilon$. Since ε is arbitrary, $B_n \in L(\mathscr{A}')$, so $B \in L(\mathscr{A}')$. Thus f is $L(\mathscr{A}')$-measurable.

ii) Let $g = E(h \mid \mathscr{A}')$. Since P is *-finite,

$$\nu\Big(\bigcup_{\substack{A \in P \\ \nu(A) = 0}} A\Big) = 0;$$

thus, we may assume that $\nu(A)>0$ for all $A\in P$. Then for $x\in A$,

$$g(x)=\frac{1}{\nu(A)}\int_A h\,d\nu.$$

If $B\in\mathcal{A}'$, write $B=\bigcup_{n=1}^{\omega}A_n$, where $\omega\in{}^*N$, $A_n\in P$. Then

$$\begin{aligned}\int_B |g|^p d\nu &= \sum_{n=1}^{\omega}\int_{A_n}|g|^p d\nu\\ &=\sum \nu(A_n)\left|\frac{1}{\nu(A_n)}\int_{A_n} h\,d\nu\right|^p\\ &\leqq \sum \nu(A_n)\int_{A_n}|h|^p d\nu\\ &=\int_B |h|^p d\nu.\end{aligned}$$

In particular, ${}^\circ\int_X|g|^p d\nu<+\infty$ and, if $\nu(B)\simeq 0$, $\int_B|g|^p d\nu\simeq 0$. Thus, $g\in SL^p(X,\mathcal{A}',\nu)$.

By Theorem 11(iv), $g,h\in SL^1(X,\mathcal{A},\nu)$. If $D\in L(\mathcal{A}')$, there exists $C\in\mathcal{A}'$ such that $L(\nu)(C\Delta D)=0$. Hence

$$\int_D {}^\circ g\,dL(\nu)=\int_C {}^\circ g\,dL(\nu)={}^\circ\!\int_C g\,d\nu={}^\circ\!\int_C h\,d\nu=\int_C {}^\circ h\,dL(\nu)=\int_D {}^\circ h\,dL(\nu).$$

Since ${}^\circ g$ is $L(\mathcal{A}')$-measurable, ${}^\circ g=E({}^\circ h\mid L(\mathcal{A}'))$.

We now prove a representation theorem for Lebesgue measure on the unit interval. We shall show (Corollary 17) that there exists a *-finite measure space $(X,\mathcal{A},\nu)$ and a measure-preserving transformation from $(X,L(\mathcal{A}),L(\nu))$ onto $[0,1]$. The idea of using the standard part map in the following construction was suggested by Donald J. Brown. Standard part maps have appeared in many guises in nonstandard constructions, and seem to be fundamental tools. See also Corollary 28 and the article by Peter A. Loeb, *Applications of nonstandard analysis to ideal boundaries in potential theory*, in this volume.

NOTATION 13. Let $\eta\in{}^*N-N$, $Y={}^*[0,1]$, with 0 and 1 identified. Let $\mathcal{C}$ be the *-algebra of all internal unions of intervals of the form $[i/\eta,(i+1)/\eta)$, where $0\leqq i<\eta$, $i\in{}^*N$. Let λ be the *-finitely additive set function such that $\lambda([i/\eta,(i+1)/\eta))=1/\eta$. Let st: ${}^*[0,1]\to[0,1]$ denote the standard part map $x\to{}^\circ x$. Finally, let $(W,\mathcal{B},\mu)$ denote the Lebesgue measure space on $[0,1]$, with 0 and 1 identified.

THEOREM 14. *With the notation above,* $\mathrm{st}:(Y, L(\mathscr{C}), L(\lambda)) \to (W, \mathscr{B}, \mu)$ *is measurable and measure-preserving (i.e.* $B \in \mathscr{B} \Rightarrow \mathrm{st}^{-1}(B) \in L(\mathscr{C})$, $L(\lambda)(\mathrm{st}^{-1}(B)) = \mu(B)$).

PROOF. First consider an interval $[a, b)$. Then

$$\mathrm{st}^{-1}([a, b)) = \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \left[\frac{[\eta(a - 1/n)]}{\eta}, \frac{[\eta(b - 1/m)]}{\eta}\right) \in L(\mathscr{C}),$$

since it is a countable union of countable intersections of sets in $\mathscr{C}$. Moreover,

$$L(\lambda)(\mathrm{st}^{-1}([a, b)) = \lim_{m\to\infty} \lim_{n\to\infty} {}^{\circ}\left(b - a + \frac{1}{n} - \frac{1}{m}\right) = b - a = \mu([a, b)).$$

Hence if B is any finite disjoint union of such intervals, $L(\lambda)(\mathrm{st}^{-1}(B)) = \mu(B)$.

Since $L(\mathscr{C})$ is a σ-algebra, $\{B : \mathrm{st}^{-1}(B) \in L(\mathscr{C})\}$ is also a σ-algebra. Thus, if B is any Borel set, $\mathrm{st}^{-1}(B) \in L(\mathscr{C})$. Thus, $\mu_1(B) = L(\lambda)(\mathrm{st}^{-1}(B))$ determines a measure on the Borel sets which agrees with Lebesgue measure on the ring of finite disjoint unions of left-closed, right-open intervals. By the uniqueness portion of the Caratheodory Extension Theorem, $\mu_1(B) = \mu(B)$ for any Borel set B.

Finally, if $B \in \mathscr{B}$, find F, G Borel sets such that $F \subset B \subset G$, $\mu(F) = \mu(G)$. Then $\mathrm{st}^{-1}(F) \subset \mathrm{st}^{-1}(B) \subset \mathrm{st}^{-1}(G)$, and $L(\lambda)(\mathrm{st}^{-1}(F)) = \mu(F) = \mu(G) = L(\lambda)(\mathrm{st}^{-1}(G))$. Since $(Y, L(\mathscr{C}), L(\lambda))$ is complete, $\mathrm{st}^{-1}(B) \in L(\mathscr{C})$, and $L(\lambda)(\mathrm{st}^{-1}(B)) = \mu(B)$.

COROLLARY 15. *Suppose* $f: W \to R$ *is* μ*-integrable. Then* $g = f \circ \mathrm{st}: Y \to R$ *is* $L(\lambda)$*-integrable, and* $\int_Y g dL(\lambda) = \int_W f d\mu$.

PROOF. If f is a simple function, the result is immediate. The general result follows from a routine limit argument.

COROLLARY 16. Theorem 14 and Corollary 15 hold if $(Y, L(\mathscr{C}), L(\lambda))$ is replaced by $(Y, L({}^*\mathscr{B}), L({}^*\mu))$.

PROOF. Immediate.

COROLLARY 17. *Let* $X = \{1, \cdots, \eta\}$, $\mathscr{A}$ *the *-algebra of all internal subsets of* X, *and* ν *the counting measure* $\nu(A) = |A|/\eta$. *Then there is a measure-preserving transformation* $T:(X, L(\mathscr{A}), L(\nu)) \to (W, \mathscr{B}, \mu)$. *Hence if* $f: W \to R$ *is* μ*-integrable,* $f \circ T: X \to R$ *is* $L(\nu)$*-integrable and* $\int_X f \circ T dL(\nu) = \int_W f d\mu$.

PROOF. Since any $\mathscr{C}$-measurable function must be constant on each interval $[i/\eta, (i+1)/\eta)$, there is an obvious measure-preserving transformation from X to $(Y, \mathscr{C}, \lambda)$. Let T be the composition of this with st: $Y \to W$.

REMARK 18. Let

$$\mathcal{B}_1 = \{B \subset [0,1] : \mathrm{st}^{-1}(B) \in \sigma({}^*\mathcal{B})\},$$

$$\mathcal{B}_2 = \{B \subset [0,1] : \mathrm{st}^{-1}(B) \in L({}^*\mathcal{B})\},$$

$$\mathcal{B}_3 = \{\mathrm{st}(B) : B \in \sigma({}^*\mathcal{B})\}.$$

Ward Henson has shown that $\mathcal{B}_1$ is the class of all Borel sets, and that $\mathcal{B}_3$ is the class of all analytic sets. It follows that $\mathcal{B}_2$ is exactly the class of Lebesgue sets.

It is natural to ask what is the relationship between ${}^*f$ and $f \circ \mathrm{st}$. We have the following theorem: Suppose $f : [0,1] \to R \cup \{+\infty, -\infty\}$ is $\mathcal{B}$-measurable. Then ${}^\circ({}^*f(t)) = f({}^\circ t)$ for $L({}^*\mu)$-almost all t. Moreover, if $f \in L^p$, ${}^*f \in SL^p({}^*[0,1], {}^*\mathcal{B}, {}^*\mu)$. Since this result is not required for the development of Brownian Motion and Itô Integration, we shall not prove it here.[†]

We now turn to a non-standard version of the Central Limit Theorem.

DEFINITION 19. A *random variable* on $(X, \mathcal{A}, \nu)$ is a function $x : X \to {}^*R$ which is $\mathcal{A}$-measurable. A collection $\{x_i\}_{i \in I}$ of random variables is **-independent* if, for every *-finite internal subcollection $\{x_1, \cdots, x_m\}$ $(m \in {}^*N)$, and every internal m-tuple $(\alpha_1, \cdots, \alpha_m) \in {}^*R^m$,

$$\nu(\{\omega : x_1(\omega) < \alpha_1, \cdots, x_m(\omega) < \alpha_m\}) = \prod_{k=1}^{m} \nu(\{\omega : x_k(\omega) < \alpha_k\}).$$

The collection $\{x_i\}_{i \in I}$ is *S-independent* if, for every finite subcollection $\{x_1, \cdots, x_m\}$ $(m \in N)$, and every m-tuple $(\alpha_1, \cdots, \alpha_m) \in R^m$

$$\nu(\{\omega : x_1(\omega) < \alpha_1, \cdots, x_m(\omega) < \alpha_m\}) \simeq \prod_{k=1}^{m} \nu(\{\omega : x_k(\omega) < \alpha_k\}).$$

LEMMA 20. *Suppose $\{x_i\}_{i \in I}$ is an S-independent collection of random variables on $(X, \mathcal{A}, \nu)$. Then $\{{}^\circ x_i\}_{i \in I}$ is an independent collection of random variables on $(X, L(\mathcal{A}), L(\nu))$.*

PROOF. Suppose $m \in N, (\alpha_1, \cdots, \alpha_m) \in R^m$.

$$L(\nu)(\{\omega : {}^\circ x_{i_1}(\omega) < \alpha_1, \cdots, {}^\circ x_{i_m} < \alpha_m\})$$

$$= \lim_{n \to \infty} {}^\circ \nu\left(\left\{\omega : x_{i_1}(\omega) < \alpha_1 - \frac{1}{n}, \cdots, x_{i_m}(\omega) < \alpha_m - \frac{1}{n}\right\}\right)$$

[†] This result will be proved in another article, along with a generalization of this representation of Lebesgue measure to Radon measures on arbitrary Hausdorff spaces.

$$= \lim_{n\to\infty} {}^{\circ}\left(\prod_{j=1}^{m} \nu\left(\left\{\omega : x_{i_j} < \alpha_j - \frac{1}{n}\right\}\right)\right)$$

$$= \prod_{j=1}^{m} \lim_{n\to\infty} {}^{\circ}\nu\left(\left\{\omega : x_{i_j} < \alpha_j - \frac{1}{n}\right\}\right)$$

$$= \prod_{j=1}^{m} L(\nu)(\{\omega : {}^{\circ}x_{i_j}(\omega) < \alpha_j\}).$$

THEOREM 21. *Suppose $\{x_n\}_{n\in{}^*N}$ is an internal sequence of *-independent random variables on $(X, \mathcal{A}, \nu)$ with common standard distribution function F. Suppose further that $E(x_n) = 0$, $E(x_n^2) = 1$. Then for any $n \in {}^*N - N$ and $\alpha \in {}^*R$,*

$$\nu\left(\left\{\omega : \frac{1}{\sqrt{n}} \sum_{k=1}^{n} x_k(\omega) \leqq \alpha\right\}\right) \simeq {}^*\psi(\alpha),$$

where $\psi(\alpha)$ is the standard Gaussian distribution.

PROOF. Let G be the distribution function of ${}^{\circ}x_n$ as a random variable on $(X, L(\mathcal{A}), L(\nu))$. We shall show that $G = {}^{\circ}F$, and hence $F = {}^*G$.

Fix $\alpha \in R$, $\varepsilon \in R^+$. F is *-right continuous. Hence there exists $m \in {}^*N$ such that $F(\alpha + 1/m) < F(\alpha) + \varepsilon$. Since F is standard, we may find $m \in N$ such that $F(\alpha + 1/m) < F(\alpha) + \varepsilon$.

$$G(\alpha) = L(\nu)\{\omega : {}^{\circ}x_n(\omega) \leqq a\}) = \lim_{m\to\infty} {}^{\circ}\nu(\{\omega : x_n(\omega) < \alpha + 1/m\})$$

$$= \lim_{m\to\infty} F(\alpha + 1/m) = F(\alpha).$$

Thus, $\{{}^{\circ}x_n\}_{n\in N}$ is a standard sequence of identically distributed random variables. By Lemma 20, it is an independent collection.

$$E({}^{\circ}x_n) = \int_R \alpha dG(\alpha) = \int_{{}^*R} \alpha dF(\alpha) = 0.$$

Similarly, $E({}^{\circ}x_n^2) = 1$. Hence by the standard Central Limit Theorem, with α and ε fixed as before, there exists $n_0 \in N$ such that

$$n > n_0 \Rightarrow \left| L(\nu)\left(\left\{\omega : \frac{1}{\sqrt{n}} \sum_{k=0}^{n} {}^{\circ}x_k(\omega) \leqq \alpha\right\}\right) - \psi(\alpha)\right| < \varepsilon.$$

Since $\{{}^{\circ}x_k\}$ is independent, the distribution function of $\Sigma_{k=0}^{n} {}^{\circ}x_k$ is G^n, the nth convolution product of G. Thus

$$n > n_0 \Rightarrow |G^n(\sqrt{n}\alpha) - \psi(\alpha)| < \varepsilon.$$

But $F = {}^*G$. Thus, by the Transfer Principle, for any $n \in {}^*N - N$ and any $\alpha \in R$, $F^n(\sqrt{n}\,\alpha) \simeq \psi(\alpha)$. But since $\{x_n\}_{n \in {}^*N}$ is a *-independent collection, F^n is the distribution function of $\Sigma_{k=0}^n x_k$. Hence for $\alpha \in R$,

$$\nu\left(\left\{\omega : \frac{1}{\sqrt{n}} \sum_{k=0}^{n} x_k(\omega) \leqq \alpha\right\}\right) \simeq \psi(\alpha).$$

Since ψ is continuous and both sides of the last equation are increasing, and ${}^*\Psi(\alpha) \simeq 1$, ${}^*\Psi(-\alpha) \simeq 0$ whenever ${}^\circ\alpha = +\infty$, the last line holds for all $\alpha \in {}^*R$.

Finally, we study the Loeb spaces of products.

THEOREM 22. *Suppose we are given $(X, \mathcal{A}, \nu)$ and $(X', \mathcal{A}', \nu')$ with $L(\nu)(X) < +\infty$ and $L(\nu')(X') < +\infty$. Consider $(X \times X', L(\mathcal{A} \times \mathcal{A}'), L(\nu \times \nu'))$ and the (complete) product $(X \times X', L(\mathcal{A}) \times L(\mathcal{A}'), L(\nu) \times L(\nu'))$, where $\mathcal{A} \times \mathcal{A}'$ is the internal algebra generated by the Cartesian product of $\mathcal{A}$ and $\mathcal{A}'$. Then $L(\mathcal{A} \times \mathcal{A}') \supset L(\mathcal{A}) \times L(\mathcal{A}')$ and*

$$L(\nu \times \nu')\Big|_{L(\mathcal{A}) \times L(\mathcal{A}')} = L(\nu) \times L(\nu').$$

PROOF. Fix $A' \in \mathcal{A}'$. $\{M \in \sigma(\mathcal{A}) : M \times A' \in \sigma(\mathcal{A} \times \mathcal{A}')\}$ is a σ-algebra, and it contains $\mathcal{A}$. Thus, it must equal $\sigma(\mathcal{A})$. Fix any $M \in \sigma(\mathcal{A})$. $\{M' \in \sigma(\mathcal{A}') : M \times M' \in \sigma(\mathcal{A} \times \mathcal{A}')\}$ is a σ-algebra, and contains $\mathcal{A}'$, so it equals $\sigma(\mathcal{A}')$. Thus $\sigma(\mathcal{A}) \times \sigma(\mathcal{A}') \subset \sigma(\mathcal{A} \times \mathcal{A}')$. By the uniqueness portion of the Caratheodory Extension Theorem,

$$L(\nu \times \nu')\Big|_{\sigma(\mathcal{A}) \times \sigma(\mathcal{A}')} = L(\nu) \times L(\nu')\Big|_{\sigma(\mathcal{A}) \times \sigma(\mathcal{A}')}.$$

Thus, $L(\mathcal{A} \times \mathcal{A}')$ contains the completion (with respect to $L(\nu) \times L(\nu')$) of $\sigma(\mathcal{A}) \times \sigma(\mathcal{A}')$. It is now sufficient to show that $L(\mathcal{A}) \times L(\mathcal{A}')$ is exactly the class of measurable sets in this completion.

Since $L(\mathcal{A}) \times L(\mathcal{A}')$ is the σ-algebra of the complete product, it contains any set measurable in the completion of $\sigma(\mathcal{A}) \times \sigma(\mathcal{A}')$ (with respect to $L(\nu) \times L(\nu')$). On the other hand, suppose $M = A \times A'$, where $A \in L(\mathcal{A})$, $A' \in L(\mathcal{A}')$. Then by Remark 2, we may find $B \in \mathcal{A}$, $B' \in \mathcal{A}'$ such that $L(\nu)(A \Delta B) = L(\nu')(A' \Delta B') = 0$, where Δ denotes symmetric difference. Then

$$L(\nu) \times L(\nu')(B \times B' \Delta A \times A') \leqq L(\nu) \times L(\nu')(B \Delta A) \times (A' \cup B') \cup (A \cup B) \times (B' \Delta A')) = 0.$$

Therefore M is measurable with respect to the completion of $(X \times X', \sigma(\mathcal{A}) \times \sigma(\mathcal{A}'), L(\nu) \times L(\nu'))$. Hence any set in $L(\mathcal{A}) \times L(\mathcal{A}')$ is measurable with respect to the completion of $(X \times X', \sigma(\mathcal{A}) \times \sigma(\mathcal{A}'), L(\nu) \times L(\nu'))$.

REMARK 23. The assumption that $L(\nu)(X) < +\infty, L(\nu')(X') < +\infty$, is necessary, as is shown by the following example. Let

$$X = \{x\}, \mathcal{A} = \{\emptyset, \{x\}\}, \nu(\{x\}) = \alpha$$

$$X' = \{x'\}, \mathcal{A}' = \{\emptyset, \{x'\}\}, \nu'(\{x'\}) = 1/\alpha$$

where $\alpha \simeq 0$. Then $L(\nu \times \nu')(X \times X') = {}^{\circ}(\nu \times \nu')(X \times X') = {}^{\circ}(\alpha \frac{1}{\alpha}) = 1$; however $L(\nu) \times L(\nu')(X \times X') = {}^{\circ}\alpha {}^{\circ}(1/\alpha) = 0(+\infty) = 0$. An analogue of Theorem 22 is true, however, for σ-finite subsets of $(X, L(\mathcal{A}), L(\nu))$ and $(X', L(\mathcal{A}'), L(\nu'))$; this will allow one to deal with integrable functions. $(X, L(\mathcal{A}), L(\nu))$ is never σ-finite itself unless $L(\nu)(X) < +\infty$.

THEOREM 24. *Suppose $f: X \to {}^*R$ is $\mathcal{A}$-measurable and $g: X \to R$ is $L(\mathcal{A})$-measurable. If ${}^{\circ}\int_A f d\nu = \int_A g dL(\nu)$ for all $A \in \mathcal{A}$, and X has no atoms of infinite measure, then ${}^{\circ}f = g$ ($L(\nu)$-almost everywhere).*

PROOF. If not, we may find $B \in L(\mathcal{A})$ and $\beta > \alpha \in R$ such that $L(\nu)(B) \in (0, +\infty)$ and ${}^{\circ}f(x) > \beta > \alpha \geqq g(x)$ for all $x \in B$. By Remark 2, we may find $A \in \mathcal{A}$ such that $L(\nu)(A \Delta B) = 0$, and $f(x) \geqq \beta$ for all $x \in A$. Then ${}^{\circ}\int_A f d\nu \geqq {}^{\circ}(\beta \nu(A)) = \beta L(\nu)(B)$, while $\int_A g dL(\nu) = \int_B g dL(\nu) \leqq \alpha L(\nu)(B)$, contradiction.

3. Brownian Motion and Wiener measure

A Brownian Motion on a probability space $(\Omega, \mathcal{D}, P)$ is a function $\beta: [0,1] \times \Omega \to R$ such that

i) β is a stochastic process; i.e. for each $t \in [0,1]$, $\beta(t, \circ)$ is a measurable function of ω.

ii) For $s < t \in [0,1]$, $\beta(t,\omega) - \beta(s,\omega)$ has a normal distribution with mean 0 and variance $t - s$.

iii) β is a differential process; i.e., if

$$s_1 < t_1 \leqq s_2 < t_2 \leqq \cdots \leqq s_n < t_n \in [0,1],$$

then

$$\{\beta(t_1, \omega) - \beta(s_1, \omega), \cdots, \beta(t_n, \omega) - \beta(s_n, \omega)\}$$

is an independent set of random variables.

It is shown in the standard theory that, if Ω is the Lebesgue measure space on $[0,1]$, there is a Brownian Motion defined on Ω.

Two stochastic processes χ, χ' are said to be equivalent if, for all $t \in [0,1]$, $P(\{\omega : \chi(t,\omega) \neq \chi'(t,\omega)\} = 0$. It might happen that, for a Brownian Motion β, the "path" $\beta(\circ,\omega)$ is not continuous for any ω. However, it is shown in the standard theory that any Brownian Motion is equivalent to a Brownian Motion in which all paths are continuous.

We shall construct directly a probability space and a Brownian Motion on this space in which almost all paths are continuous.

NOTATION 25. Let η be a fixed element of $^*N, \Omega = \{-1,1\}^\eta$ = {internal η-tuples of -1's and $+1$'s}. Let $\mathscr{A}$ be the *-algebra of all internal subsets of Ω, and let ν be the counting measure $\nu(A) = |A|/2^\eta$. We set $(\Omega, \mathscr{D}, P) = (\Omega, L(\mathscr{A}), L(\nu))$. Thus, $(\Omega, \mathscr{D}, P)$ is a standard probability space. We define a *-random walk on $(\Omega, \mathscr{A}, \nu)$ by

$$\chi(t,\omega) = \frac{1}{\sqrt{\eta}}\left[\sum_{i=1}^{[\eta t]} \omega_i + (\eta t - [\eta t])\omega_{[\eta t]+1}\right]$$

where $t \in {}^*[0,1]$, $\omega \in \Omega$. Let $\beta(t,\omega) = {}^\circ\chi(t,\omega)$ for $(t,\omega) \in [0,1] \times \Omega$.

THEOREM 26. *If $\eta \in {}^*N - N$, β is a Brownian Motion on $(\Omega, \mathscr{D}, P)$.*

PROOF.

i) Fix $t \in [0,1]$. $\chi(t,\circ)$ is an internal function of ω, so it is ν-measurable. Hence by Remark 2, $\beta(t,\circ)$ is $\mathscr{D} = L(\mathscr{A})$-measurable.

ii) Fix $s < t \in [0,1]$.

$$\begin{aligned}
&P(\{\omega : \beta(t,\omega) - \beta(s,\omega) \leqq \alpha\}) \\
&\quad = P(\{\omega : {}^\circ\chi(t,\omega) - {}^\circ\chi(s,\omega) \leqq \alpha\}) \\
&\quad = P\left(\left\{\omega : {}^\circ\sum_{k=[\eta s]}^{[\eta t]} \frac{\omega_k}{\sqrt{\eta}} \leqq \alpha\right\}\right) \\
&\quad = \lim_{n\to\infty} {}^\circ\nu\left(\left\{\omega : \frac{1}{\sqrt{\lambda}} \sum_{k=[\eta s]}^{[\eta t]} \omega_k \leqq \sqrt{\eta/\lambda}\,(\alpha + 1/n)\right\}\right) \quad (\text{where } \lambda = [\eta t] - [\eta s]) \\
&\quad = \lim_{n\to\infty} {}^\circ\,{}^*\Psi(\sqrt{\eta/\lambda}\,(\alpha + 1/n)) \quad (\text{by Theorem 21}) \\
&\quad = \lim_{n\to\infty} \Psi({}^\circ(\sqrt{\eta/\lambda}(\alpha + 1/n))) \\
&\quad = \lim_{n\to\infty} \Psi\left[\frac{\alpha + 1/n}{\sqrt{t-s}}\right] = \Psi\left[\frac{\alpha}{\sqrt{t-s}}\right].
\end{aligned}$$

Therefore

$$P(\{\omega : \beta(t,\omega) - \beta(s,\omega) < \alpha\sqrt{t-s}\}) = \Psi(\alpha),$$

so $\beta(t,\omega) - \beta(s,\omega)$ has a normal distribution with mean 0 and variance $t-s$.

iii) Suppose $s_1 < t_1 \leqq s_2 < t_2 \leqq \cdots \leqq s_n < t_n \in [0,1]$. Then

$$\left\{\chi(t_1,\circ) - \chi\left(s_1 + \frac{1}{\eta},\circ\right), \cdots, \chi(t_n,\circ) - \chi\left(s_n + \frac{1}{\eta},\circ\right)\right\}$$

is *-independent, and hence S-independent. By Lemma 20,

$$\{\beta(t_1,\circ) - \beta(s_1,\circ), \cdots, \beta(t_n,\circ) - \beta(s_n,\circ)\}$$

is independent.

THEOREM 27. *$\beta(\circ,\omega)$ is continuous and finite for almost all ω. In fact, $\chi(\circ,\omega)$ is near-standard in $^*C[0,1]$ for almost all ω.*

PROOF. If $\eta \in N$, the result is obvious. Now suppose $\eta \in {}^*N - N$. For $m, n \in N$, define

$$\Omega_{mn} = \left\{\omega : \exists_{i<n} \sup_{t\in[i/n,(i+1)/n]} \chi(t,\omega) - \inf_{t\in[i/n,(i+1)/n]} \chi(t,\omega) > \frac{1}{m}\right\}.$$

Note that Ω_{mn} is internal.

$$\nu(\Omega_{mn}) \leqq n\nu\left(\left\{\omega : \left(\sup_{t\in[0,1/n]} - \inf\right)\chi(t,\omega) > \frac{1}{m}\right\}\right)$$

$$\leqq n\nu\left(\left\{\omega : \max_{1\leqq k\leqq\lambda}\left|\sum_1^k \omega_i\right| > \frac{\sqrt{\eta}}{2m}\right\}\right) \qquad (\text{where } \lambda = \eta/n + 1)$$

$$\leqq n\nu\left(\left\{\omega : \max_{k\leqq\lambda}\sum_1^k \omega_i > \frac{\sqrt{\eta}}{2m}\right\}\right) + n\nu\left(\left\{\omega : \min_{k\leqq\lambda}\sum_1^k \omega_i < -\frac{\sqrt{\eta}}{2m}\right\}\right)$$

$$\leqq 2n\nu\left(\left\{\omega : \sum_1^\lambda \omega_i > \frac{\sqrt{\eta}}{2m}\right\}\right) + 2n\nu\left(\left\{\omega : \sum_1^\lambda \omega_i < -\frac{\sqrt{\eta}}{2m}\right\}\right)$$

$$= 4n\nu\left(\left\{\omega : \frac{1}{\sqrt{\lambda}}\sum_1^\lambda \omega_i > \frac{\sqrt{\eta/\lambda}}{2m}\right\}\right)$$

$$\simeq 4n\,{}^*\Psi\left[\frac{\sqrt{\eta/\lambda}}{2m}\right] \simeq 4n\Psi\left[\frac{\sqrt{n}}{2m}\right] \qquad (\text{by Theorem 21})$$

$$= \frac{4n}{\sqrt{2\pi}} \int_{\sqrt{n}/2m}^{\infty} e^{-t^2/2}\, dt$$

$$< 2n \int_{\sqrt{n}/2m}^{\infty} e^{-t/2}\, dt \text{ (provided } \sqrt{n}/2m > 1)$$

$$= 4n e^{-\sqrt{n}/4m}.$$

Let

$$\Omega' = \Omega - \bigcup_{m=1}^{\infty} \bigcap_{n=1}^{\infty} \Omega_{mn}.$$

$$P(\Omega') = 1 - \sup_m \inf_n \nu(\Omega_{mn}) \geqq 1 - \sup_m \inf_n 4n e^{-\sqrt{n}/4m} = 1.$$

Suppose that $s \simeq t \in {}^*[0,1]$ and $\chi(s,\omega) \not\simeq \chi(t,\omega)$. Then if $m > 2/{}^\circ|\chi(s,\omega) - \chi(t,\omega)|$, $\omega \in \Omega_{mn}$ for all n, so $\omega \notin \Omega'$. If ${}^\circ\chi(s,\omega) = +\infty$ for some $s \in {}^*[0,1]$, then $\omega \in \Omega_{mn}$ for all m and n, and again $\omega \notin \Omega'$.

Suppose $\omega \in \Omega'$. By the preceding paragraph, $\beta(t,\omega)$ is finite for all t. Fix $t \in [0,1]$ and suppose we are given $\varepsilon \in R^+$.

$\{n : |t-s| < 1/n \Rightarrow |\chi(t,\omega) - \chi(s,\omega)| < \varepsilon/2\}$ is internal and contains all infinite n. Hence it contains some finite n. Thus if $|t-s| < 1/n$, $|\chi(t,\omega) - \chi(s,\omega)| < \varepsilon/2$, so $|\beta(t,\omega) - \beta(s,\omega)| < \varepsilon$. Hence $\beta(\circ,\omega)$ is continuous.

It remains to show that $\chi(\circ,\omega)$ is near-standard. Since $\beta(\circ,\omega)$ is continuous, $t \simeq s \Rightarrow {}^*\beta(t,\omega) \simeq {}^*\beta(s,\omega)$. Hence, for $t \in {}^*[0,1]$,

$$|{}^*\beta(t,\omega) - \chi(t,\omega)| \leqq |{}^*\beta(t,\omega) - \beta({}^\circ t,\omega)| + |\beta({}^\circ t,\omega) - \chi({}^\circ t,\omega)| + |\chi({}^\circ t,\omega) - \chi(t,\omega)| \simeq 0.$$

Hence $\|{}^*\beta(\circ,\omega) - \chi(\circ,\omega)\|_\infty \simeq 0$, so $\beta(\circ,\omega)$ is near-standard in ${}^*C[0,1]$. This completes the proof.

Wiener measure is defined to be the unique Borel measure on $C[0,1]$ such that the following two conditions are satisfied:

i) The measure of $\{f : f(t) < \alpha\} = \Psi(\alpha/\sqrt{t})$,

ii) If $s_1 < t_1 \leqq \cdots \leqq s_n < t_n \in [0,1]$, then the random variables $\{f(t_1) - f(s_1), \cdots, f(t_n) - f(s_n)\}$ are independent.

Sets in the ring generated by sets of the form $\{f : f(t_1) < \alpha_1, \cdots, f(t_n) < \alpha_n\}$ are called finite cylinder sets. The uniqueness of Wiener measure follows from the fact that the finite cylinder sets generate the Borel σ-algebra on $C[0,1]$.[†]

[†] See [3].

Because of Theorem 27, the space $(\Omega, \mathcal{D}, P)$ induces a measure space $(C[0,1], \mathcal{E}, P')$ by $E \in \mathcal{E} \Leftrightarrow \{\omega : \beta(\circ, \omega) \in E\} \in \mathcal{D}$ and $P'(E) = P(\{\omega : \beta(\circ, \omega) \in E\})$ whenever $E \in \mathcal{E}$. Since $\mathcal{E}$ contains all the finite cylinder sets, it must contain all the Borel sets. Moreover, the conditions in the definition of Wiener measure are satisfied. Hence, we have the following result:

COROLLARY 28. $(C[0,1], \mathcal{E}, P')$ *is an extension of Wiener measure if* $\eta \in {}^*N - N$.[†]

We now wish to prove a special case of Donsker's Theorem. We introduce a subscript η to indicate the dependence of Ω_η, β_η, P_η, etc. on η.

THEOREM 29. $\{P'_\eta\}_{\eta \in N}$ *converges weakly to* P' (*i.e. if* F *is a bounded continuous function from* $C[0,1]$ *to* R, *then* $\int F dP'_\eta \to \int F dP'$).

PROOF. Let $\eta_1, \eta_2 \in {}^*N - N$, and let $\mathcal{F}$ be the Borel σ-algebra on $C[0,1]$. We have shown that $P'_{\eta_1}|_{\mathcal{F}} = P'_{\eta_2|\mathcal{F}} = P'$. Hence, if F is a real-valued bounded continuous function on $C[0,1]$,

$$\int_{C[0,1]} F dP'_{\eta_1} = \int_{C[0,1]} F dP'_{\eta_2} = \int_{C[0,1]} F dP'.$$

Therefore

$$\int_{\Omega_{\eta_1}} F(\beta_{\eta_1})(\circ, \omega)) dP_{\eta_1}(\omega) = \int_{\Omega_{\eta_2}} F(\beta_{\eta_2}(\circ, \omega)) dP_{\eta_2}(\omega) = \int_{C[0,1]} F dP'.$$

Since F is continuous and $\chi_{\eta_i}(\circ, \omega)$ is near-standard for P_{η_i}-almost all ω, ${}^*F(\chi_{\eta_i}(\circ, \omega)) \simeq {}^*F({}^*\beta_{\eta_i}(\circ, \omega)) = F(\beta_{\eta_i}(\circ, \omega))$ for P_{η_i}-almost all ω. Since F is bounded, ${}^*F(\chi_{\eta_i}(\circ, \omega)) \in SL^1(\Omega_{\eta_i})$. Hence, by Theorem 6,

$$\begin{aligned}\int_{\Omega_{\eta_1}} {}^*F(\chi_{\eta_1}(\circ, \omega)) d\nu_{\eta_1}(\omega) &\simeq \int_{\Omega_{\eta_1}} F(\beta_{\eta_1}(\circ, \omega)) dP_{\eta_1}(\omega) \\ &= \int_{\Omega_{\eta_2}} F(\beta_{\eta_2}(\circ, \omega)) dP_{\eta_2}(\omega) \\ &\simeq \int_{\Omega_{\eta_2}} {}^*F(\chi_{\eta_2}(\circ, \omega)) d\nu_{\eta_2}(\omega).\end{aligned}$$

Therefore

[†] Ward Henson's result referred to in Remark 18 shows that $\mathcal{E}$ is in fact the completion of the Borel σ-algebra on $C[0,1]$.

$$\lim_{\substack{\eta\to\infty\\ \eta\in N}} \int_{\Omega_\eta} F(\chi_\eta(\circ,\omega))d\nu_\eta(\omega)$$

exists and equals

$$^\circ\int_{\Omega_{\eta_1}} {}^*F(\chi_{\eta_1}(\circ,\omega))d\nu_{\eta_1}(\omega) = \int_{C[0,1]} FdP'.$$

4. Stochastic integration and Itô's Lemma

In a sense, the stochastic integral with respect to Brownian Motion may be thought of as a generalized Stieltjes integral. However, the standard approach to stochastic integration masks this analogy; the problem is that almost all Brownian Motion paths are of unbounded variation. However, the random walk χ defined in Section 3 is a function of bounded variation in the non-standard sense. Hence, one can define a Stieltjes integral with respect to χ. We shall show that this Stieltjes integral is the same as Itô's stochastic integral. While this process is of difficulty comparable to that of giving Itô's original definition, subsequent computations (including the proof of Itô's Lemma) are substantially simplified.

The standard Itô integral is defined in the following way ([8], [11]). Suppose β is a Brownian Motion on any probability space $(\Omega, \mathcal{D}, P)$. Let $\{\mathcal{D}_t\}_{t\in[0,1]}$ be a collection of σ-algebras such that

i) $\mathcal{D}_t \supset \mathcal{D}_s$ if $t > s$.

ii) $\beta(t,\circ)$ is $\mathcal{D}_t$-measurable.

iii) if $t \leqq s_1 < t_1 \leqq s_2 < t_2 \leqq \cdots \leqq s_n < t_n$, then $\mathcal{D}_t$ is independent of the σ-algebra generated by

$$\{\beta(t_1,\circ) - \beta(s_1,\circ), \cdots, \beta(t_n,\circ) - \beta(s_n,\circ)\}.$$

Let $\mathcal{G}_0$ be the set of all g such that

i) $g \in L^2([0,1]\times\Omega)$ (the complete product).

ii) For each $t\in[0,1]$, $g|_{[0,t]\times\Omega}$ is $\mathcal{B}\times\mathcal{D}_t$-measurable ($g$ is said to be progressively measurable).†

Let $\mathcal{G}_0'$ be the subset of $\mathcal{G}_0$ satisfying the additional assumption

iii) There exist $0 = t_0 < t_1 < \cdots < t_n = 1$ such that, for each ω, $g(t,\omega) = g(t_i,\omega)$ for $t\in[t_i, t_{i+1})$.

For $g \in \mathcal{G}_0'$, the stochastic integral is defined by

† The assumption $g\in L^2$ can be weakened to $g(\circ,\omega)\in L^2([0,1])$ for almost all ω. For if one defines $g_n(t,\omega) = g(t,\omega)$ if $\|g(\circ,\omega)\|_2 \leqq n$, and 0 otherwise, then $g_n \in L^2$. One then defines $\int g d\beta(\omega) = \int g_n d\beta(\omega)$ where $n \geqq \|g(\circ,\omega)\|_2$.

$$\int_0^t g(\tau,\omega)d\beta(\tau,\omega)=\sum_{i=0}^{k-1} g(t_i,\omega)(\beta(t_{i+1},\omega)-\beta(t_i,\omega))$$
$$+g(t_k,\omega)(\beta(t,\omega)-\beta(t_k,\omega)),$$

where $k=\max\{i: t_i \leqq t\}$. Thus, the stochastic integral is a stochastic process. One shows that, for $g\in\mathscr{G}_0'$, $\|\int g d\beta\|_2=\|g\|_2$, and that $\mathscr{G}_0'$ is L^2-dense in $\mathscr{G}_0$. Hence one can define $\int g d\beta$ for all $g\in\mathscr{G}_0$ by extension. However, in order to establish several important properties of the stochastic integral (including continuity in t), one must give additional arguments involving uniform convergence on subsequences.

We shall deal with the Brownian Motion β and the probability space $(\Omega,\mathscr{D},P)$ defined in Section 3. However, in order to allow integration of functions g which depend on information not contained in β, it is convenient to modify the definition slightly.

Let $\Omega=\{-1,1\}^{\{-\eta,\cdots,\eta\}}=\{$internal $(2\eta+1)$-tuples of $+1$'s and -1's indexed by the set $\{-\eta,\cdots,\eta\}\}$. As before, $\mathscr{A}$ is the *-algebra of internal subsets of Ω, ν counting measure, and $(\Omega,\mathscr{D},P)=(\Omega,L(\mathscr{A}),L(\nu))$. β and χ are as defined before. Thus, $\beta(\circ,\omega)$ and $\chi(\circ,\omega)$ depend only on ω_i for $i>0$. Let $\Omega'=\{\omega:(\omega_1,\cdots,\omega_\eta)$ belongs to Ω' as defined in Section 3$\}$.

For $i\in\{0,1,\cdots,\eta\}$, let $\sim_i$ be the equivalence relation on Ω defined by $\omega\sim_i\omega'\Leftrightarrow\omega_j=\omega_j'$ for all $j\leqq i$. Let $\mathscr{A}_i$ be the *-algebra generated by the partition of Ω into equivalence classes with respect to $\sim_i$, and let $\mathscr{H}_i$ be the external algebra of all unions of these equivalence classes. By abuse of notation, we shall let $L(\mathscr{A}_i)$ denote the intersection of $\mathscr{H}_i$ and the completion of $\sigma(\mathscr{A}_i)$. For $t\in[0,1]$, let $\mathscr{D}_t$ be the σ-algebra generated by

$$\bigcap_{\substack{i\geqq 0\\ i/\eta\simeq t}} L(\mathscr{A}_i)$$

and $\beta(t,\circ)$. Then $\{\mathscr{D}_t\}_{t\in[0,1]}$ is a family of σ-algebras as outlined above.

Define $\mathscr{G}$ to be the set of functions g such that

i) $g\in L^2([0,1]\times\Omega)$.

ii) For each $t\in[0,1]$, $g(t,\circ)$ is $\mathscr{D}_t$-measurable.

Thus, $\mathscr{G}\supset\mathscr{G}_0$. The *-finiteness of the measure spaces Ω and $(Y,\mathscr{C},\lambda)$ will make the assumption of progressive measurability superfluous.

We shall employ Notation 13. We shall define the stochastic integral for the following class $\mathscr{F}$ containing $\mathscr{G}$.

Definition 30. Suppose $f:[0,1]\times\Omega\to R\cup\{+\infty,-\infty\}$. We say that g is a p-lifting of f if

i) $g \in SL^p({}^*[0,1] \times \Omega, \mathscr{C} \times \mathscr{A}, \lambda \times \nu)$.
ii) ${}^\circ g(t,\omega) = f({}^\circ t, \omega)$ for $L(\lambda \times \nu)$-almost all (t,ω).
iii) For all $t \in {}^*[0,1]$, $g(t,\circ)$ is $\mathscr{A}_{[\eta t]}$-measurable.

Let $\mathscr{F}$ be the class of all f having 2-liftings.

LEMMA 31. *Suppose $f \in L^p([0,1] \times \Omega)$ and, for each $t \in [0,1]$, $f(t,\circ)$ is $\mathscr{D}_t$-measurable. Then f has a p-lifting. In particular, $\mathscr{F} \supset \mathscr{G}$.*

PROOF. Suppose $f \in L^p([0,1] \times \Omega)$ and $f(t,\circ)$ is $\mathscr{D}_t$-measurable ($t \in [0,1]$). Let

$$\mathscr{D}'_t = \bigcap_{\substack{i \geq 0 \\ {}^\circ i/\eta = t}} L(\mathscr{A}_i).$$

We shall first modify f on a set of P-measure 0 to find a function f' such that $f'(t,\circ)$ is $\mathscr{D}'_t$-measurable.

Define an equivalence relation $\sim_t$ on Ω by $\omega \sim_t \omega' \Leftrightarrow \omega_j = \omega'_j$ for all $j \leqq 0$ and all j such that ${}^\circ(j/\eta) < t$. Let

$$f'(t,\omega) = \begin{cases} f(t,\omega') \text{ if there exists } \omega' \in \Omega' \text{ such that } \omega \sim_t \omega' \\ 0 \quad \text{otherwise.} \end{cases}$$

In order to see that f' is well-defined, suppose $\omega', \omega'' \in \Omega'$, and $\omega' \sim_t \omega''$. Then $\beta(s,\omega') = \beta(s,\omega'')$ for all $s < t$; since $\omega', \omega'' \in \Omega'$, $\beta(\circ,\omega')$ and $\beta(\circ,\omega'')$ are continuous; thus, $\beta(t,\omega') = \beta(t,\omega'')$. Since f is $\mathscr{D}_t$-measurable, and $\omega' \sim_t \omega''$, $f(t,\omega') = f(t,\omega'')$.

Suppose $i \in {}^*N$, $i \geqq 0$, $(i/\eta) \simeq t$. Then, for any $\omega', \omega'' \in \Omega$, $\omega' \sim_i \omega'' \Rightarrow f'(t,\omega') = f'(t,\omega'')$. By Theorem 12, $f'(t,\circ)$ is $L(\mathscr{A}_i)$-measurable. Hence $f'(t,\circ)$ is $\mathscr{D}'_t$-measurable. Since $f'(\circ,\omega) = f(\circ,\omega)$ whenever $\omega \in \Omega'$, we may substitute f' for f and assume without loss of generality that f is $\mathscr{D}'_t$-measurable.

For $(t,\omega) \in {}^*[0,1] \times \Omega$, let $f_1(t,\omega) = f({}^\circ t,\omega)$. By Theorem 14,

$$f_1 \in L^p(({}^*[0,1] \times \Omega, L(\mathscr{C}) \times \mathscr{D}, L(\lambda) \times P));$$

by Theorem 22,

$$f_1 \in L^p({}^*[0,1] \times \Omega, L(\mathscr{C} \times \mathscr{A}), L(\lambda \times \nu)).$$

By Theorem 11, there exists $h \in SL^p({}^*[0,1] \times \Omega, \mathscr{C} \times \mathscr{A}, \lambda \times \nu)$ such that ${}^\circ h(t,\omega) = f_1(t,\omega) = f({}^\circ t,\omega)$ $L(\lambda \times \nu)$-almost everywhere. Let $\mathscr{C}'$ be the $*$-subalgebra of $\mathscr{C} \times \mathscr{A}$ generated by $\{[i/\eta,(i+1)/\eta) \times A : A \in \mathscr{A}_i, 0 \leqq i < \eta\}$. Let $g = E(h \mid \mathscr{A}')$. By Theorem 12(ii), $g \in SL^p({}^*[0,1] \times \Omega, \mathscr{C}', \lambda \times \nu)$ and ${}^\circ g = E({}^\circ h \mid L(\mathscr{C}')) = E(f_1 \mid L(\mathscr{C}'))$ ($L(\lambda \times \nu)$-almost everywhere). But f_1 is $L(\mathscr{C} \times \mathscr{A})$-

measurable, and $f_1 \big|_{[i/\eta,(i+1)/\eta)\times A}$ is constant if A is any equivalence class mod $\sim_i$. By Theorem 12(i), f is $L(\mathscr{C}')$-measurable, so $E(f_1 \mid L(\mathscr{C}')) = f_1(L(\lambda \times \nu)$-almost everywhere). Since g is $\mathscr{C}'$-measurable, $g(t,\circ)$ is $\mathscr{A}_{[\eta t]}$-measurable. Hence g is a p-lifting of f.

Definition 32. Suppose $f \in \mathscr{F}$, and g is a 2-lifting of f. Define

$$I(t,\omega) = \int_0^t f(\tau,\omega)d\beta(\tau,\omega) = {}^\circ\!\int_0^t g(\tau,\omega)d\chi(\tau,\omega),$$

where the last integral is a *-Stieltjes integral.

Theorem 33. *Definition 32 is independent of the choice of g, and coincides with Itô's definition of the stochastic integral on $\mathscr{G}_o$. $I(t,\circ)$ is $\mathscr{D}$-measurable for all $t \in [0,1]$.*

Proof. It is clearly sufficient to establish the result for $t = 1$. Let $G(\omega) = \int_0^1 g(t,\omega)d\chi(t,\omega)$. G is $\mathscr{A}$-measurable, so $I(1,\circ)$ is $\mathscr{D}$-measurable.

$$^\circ\!\int_\Omega G(\omega)^2 d\nu = {}^\circ\!\int_\Omega \left(\sum_{k=0}^{\eta-1} g(k/\eta,\omega)\frac{\omega_{k+1}}{\sqrt{\eta}}\right)^2 d\nu$$

$$= {}^\circ\!\int_\Omega \sum_{k=0}^{\eta-1} \frac{g^2(k/\eta,\omega)}{\eta}d\nu + 2\,{}^\circ\!\int_\Omega \sum_{j<k} \frac{g(k/\eta,\omega)g(j/\eta,\omega)\omega_{j+1}\omega_{k+1}}{\eta}d\nu$$

$$= {}^\circ\!\int_\Omega \int_0^1 g^2(t,\omega)d\lambda d\nu \quad \text{(since } g(\circ,\omega) \text{ is } \mathscr{C}\text{-measurable)}$$

$$+ 2\,{}^\circ\!\sum_{j<k} \int_\Omega \omega_{k+1}\, d\nu \int_\Omega \frac{g(k/\eta,\omega)g(j/\eta,\omega)\omega_{j+1}}{\eta}d\nu$$

(since $g(k/\eta,\circ)$, $g(j/\eta,\circ)$, and ω_{j+1} are all $\mathscr{A}_k$-measurable, while ω_{k+1} is independent of $\mathscr{A}_k$) $= \| g \|_2^2 + 0$ (since $\int_\Omega \omega_{k+1}\, d\nu = 0$). Therefore $\int |{}^\circ G(\omega)|^2 dL(\nu) \leqq \| g \|_2^2$. Thus, if g_1 and g_2 are liftings of f, and G_1 and G_2 are the resulting integrals,

$$\int_\Omega |{}^\circ G_1 - {}^\circ G_2|^2 dL(\nu) \leqq {}^\circ\!\int_\Omega |G_1 - G_2|^2 d\nu = \| g_1 - g_2 \|_2^2 = 0.$$

Thus, ${}^\circ G_1 = {}^\circ G_2(L(\nu)$-almost everywhere), so the integral is well-defined.

We will now show that Definition 32 is the same as Itô's definition. Suppose first that $f \in \mathscr{G}_0'$. Then we have $0 = t_0 < t_1 < \cdots < t_n = 1$ such that $f(t,\omega) = f(t_i,\omega)$ for $t \in [t_i, t_{i+1})$. Thus, we may take g a lifting of f such that $g(t,\omega) = g(t_i,\omega)$ for $t \in {}^*[t_i, t_{i+1})$, and ${}^\circ g(t_i,\omega) = f(t_i,\omega)$ for almost all ω (with respect to P). Thus, for such ω,

$$G(\omega)=\sum_{i=0}^{n-1} g(t_i,\omega)(\chi(t_{i+1},\omega)-\chi(t_i,\omega))\simeq\sum_{i=0}^{n-1} f(t_i,\omega)(\beta(t_{i+1},\omega)-\beta(t_i,\omega)).$$

Therefore, ${}^{\circ}G$ is the Itô integral of f. Since $\mathscr{G}_0'$ is L^2-dense in $\mathscr{G}_0$ and $\| G\|_2\leqq\|f\|_2$ for $f\in\mathscr{G}_0$ the integrals coincide on $\mathscr{G}_0$.

We shall now give an example illustrating the utility of this characterization of the stochastic integral.

EXAMPLE 34. Let $f(t,\omega)=\beta(t,\omega)$, Then in Definition 32, we may take $g(t,\omega)=\chi([t\eta],\omega)$. Then

$$\int_0^1 f(t,\omega)d\beta(t,\omega)$$

$$={}^{\circ}\sum_{k=0}^{\eta-1}\chi(k/\eta,\omega)\frac{\omega_{k+1}}{\sqrt{\eta}}={}^{\circ}\sum_{k=0}^{\eta-1}\sum_{j=1}^{k}\frac{\omega_j\omega_{k+1}}{\eta}={}^{\circ}\sum_{j<k}\frac{\omega_j\omega_k}{\eta}$$

$$=\frac{1}{2}{}^{\circ}\Big(\sum_{j,k}\frac{\omega_j\omega_k}{\eta}-\sum_{j=1}^{\eta}\frac{\omega_j^2}{\eta}\Big)=\frac{1}{2}{}^{\circ}\Big(\Big(\sum_{j=1}^{\eta}\frac{\omega_j}{\sqrt{\eta}}\Big)^2-\sum_{j=1}^{\eta}\frac{1}{\eta}\Big)$$

$$=\frac{1}{2}{}^{\circ}(\chi^2(1,\omega)-1)=\frac{1}{2}(\beta^2(1,\omega)-1).$$

THEOREM 35. *Let $f\in\mathscr{F}$, and let g be a 2-lifting of f. Then for P-almost all ω,*

$$G(t,\omega)=\int_0^t g(\tau,\omega)d\chi(\tau,\omega),$$

viewed as a function of $t\in{}^*[0,1]$, is near-standard in ${}^*C[0,1]$. Hence, for P-almost all ω,

$$I(t,\omega)=\int_0^t f(\tau,\omega)d\beta(\tau,\omega),$$

viewed as a function of $t\in[0,1]$, is continuous.

PROOF. Let ρ be any element of ${}^*N$ such that $\rho/\eta\simeq 0$. Let $\mathscr{C}_\rho$ be the internal *-subalgebra of $\mathscr{C}$ consisting of internal unions of intervals from the collection

$$\{[i\rho/\eta,(i+1)\rho/\eta):0\leqq i\leqq[\eta/\rho]-1\}\cup\{[[\eta/\rho]\rho/\eta,1]\}.$$

Then, since f_1 (defined by $f_1(t,\omega)=f({}^{\circ}t,\omega)$) is $L(\mathscr{C}_\rho\times\mathscr{A})$-measurable, we may find g_ρ' which is $\mathscr{C}_\rho\times\mathscr{A}$-measurable and which is a 2-lifting of f. Then $\|g_\rho'-g\|_2=0$.

By Theorem 27, $\omega\in\Omega'\Rightarrow(s\simeq t\Rightarrow\chi(s,\omega)\simeq\chi(t,\omega))$. Thus, for any $m\in N$, the set $A^\rho_m = \{\omega : |s-t| < \rho/\eta \Rightarrow |\chi\ (s,\omega) - \chi(t,\omega)| < 1/m\}$ is internal and contains Ω'; therefore $\nu(A^\rho_m) \simeq 1$, so $\nu(A^\rho_m) > 1 - 1/m$. Then we may find $m_\rho \in {}^*N - N$, and $A^\rho \in \mathcal{A}$ such that

$$\nu(A^\rho) > 1 - 1/m_\rho \simeq 1$$

$$\omega \in A^\rho \Rightarrow (|s-t| < \rho/\eta \Rightarrow |\chi(s,\omega) - \chi(t,\omega)| < 1/m_\rho \simeq 0).$$

Let

$$g_\rho = \begin{cases} \sqrt{m_\rho} & g'_\rho > \sqrt{m_\rho} \\ g'_\rho & |g'_\rho| \leqq \sqrt{m_\rho} \\ -\sqrt{m_\rho} & g'_\rho < -\sqrt{m_\rho} \end{cases}$$

Since $g'_\rho \in SL^2$, $g_\rho \in SL^2$ by Corollary 5 and

$$\| g_\rho - g \|_2 \leqq \| g_\rho - g'_\rho \|_2 + \| g'_\rho - g \|_2 = \| {}^\circ g_\rho - {}^\circ g'_\rho \|_2 + \| g'_\rho - g \|_2 = 0.$$

Define

$$G_\rho(t,\omega) = \int_0^t g_\rho(\tau,\omega) d\chi(\tau,\omega) \quad \text{for} \quad t \in {}^*[0,1].$$

If $s, t \in {}^*[0,1], |s-t| < \rho/\eta$, and $\omega \in A^\rho$,

$$\begin{aligned} &\int_s^t g_\rho(\tau,\omega) d\chi(\tau,\omega) \\ &\quad = g_\rho(s,\omega)\left(\chi\left(\left[\frac{t\eta}{\rho}\right]\frac{\rho}{\eta},\omega\right) - \chi(s,\omega)\right) \\ &\quad + g_\rho(t,\omega)\left(\chi(t,\omega) - \chi\left(\left[\frac{t\eta}{\rho}\right]\frac{\rho}{\eta},\omega\right)\right) \\ &\quad \leqq 2\sqrt{m_\rho}\frac{1}{m_\rho} \simeq 0. \end{aligned}$$

For $j = 0, 1, , \cdots, \eta$,

$$\left| \int_0^{j/\eta} (g_\rho(\tau,\omega) - g(\tau,\omega)) d\chi(\tau,\omega) \right|$$

is a positive submartingale. Hence by Doob's Inequality [4, p. 317]

$$\int_\Omega \sup_{t\in{}^*[0,1]} (G(t,\omega)-G_\rho(t,\omega))^2 d\nu \leqq 4\int_\Omega (G(1,\omega)-G_\rho(1,\omega))^2 d\nu$$

$$\simeq 4\| g - g_\rho \|_2^2 = 0.$$

Therefore, for P-almost all ω, $G(t,\omega)\simeq G_\rho(t,\omega)$ for all t.

Fix $m\in N$. Let $B_\rho^m = \{\omega : |s-t|<\rho/\eta \Rightarrow |\int_s^t g(\tau,\omega)d\chi(\tau,\omega)|<1/m\}$. B_ρ^m is internal, and contains a set of P-measure 1. Therefore $\nu(B_\rho^m)\simeq 1$.

Fix $n\in N$. For any $\rho\in{}^*N$ such that $\rho/\eta\simeq 0$, $\nu(B_\rho^m)>1-1/n$. Hence, for every pair $(m,n)\in N^2$, there exists $\rho_{mn}\in{}^*N$, $\rho_{mn}/\eta\not\simeq 0$, and $C_{mn}\in\mathcal{A}$, such that

$$\nu(C_{mn})>1-\frac{1}{n}$$

$$\omega\in C_{mn}, |s-t|<\rho_{mn}/\eta \Rightarrow |G(t,\omega)-G(s,\omega)|<\frac{1}{m}.$$

Let $C=\bigcap_{m=1}^\infty\bigcup_{n=1}^\infty C_{mn}$. $P(C)=1$. $\omega\in C\Rightarrow(s\simeq t\Rightarrow G(s,\omega)\simeq G(t,\omega))$. Then, as in Theorem 27, we conclude that $\omega\in C\Rightarrow G(\circ,\omega)$ is near-standard in ${}^*C[0,1]$, and $I(\circ,\omega)$ is continuous.

COROLLARY 36. *Suppose $f\in\mathcal{F}$. Then $I\in\mathcal{F}$, I is $\mathcal{B}\times\mathcal{D}$-measurable, and ${}^\circ G$ is $L(\mathcal{C})\times\mathcal{D}$-measurable.*

PROOF.

$$G(t,\omega)=\int_0^t g(\tau,\omega)d\chi(\tau,\omega)$$

$$=\sum_{k=0}^{[\eta t]-1} g\left(\frac{k}{\eta},\omega\right)\frac{\omega_{k+1}}{\sqrt{\eta}}+g\left(\frac{[\eta t]}{\eta},\omega\right)\frac{\omega_{[\eta t]+1}}{\sqrt{\eta}}(\eta t-[\eta t]).$$

Thus $G(t,\circ)$ is $\mathcal{A}_{[\eta t]+1}$-measurable, so $I(t,\circ)$ is $L(\mathcal{A}_{[\eta t]+1})$-measurable. Let

$$I'(t,\omega)=\begin{cases} I(t,\omega') & \text{if } \exists\omega'\in C,\quad \omega\sim_t\omega' \\ 0 & \text{otherwise.}\end{cases}$$

Then as in the first paragraph of the proof of Lemma 31, $I'(t,\circ)$ is $\mathcal{D}'_t$-measurable.

For P-almost all ω, $G(\circ,\omega)$ is near-standard in ${}^*C([0,1])$; hence, for such ω, ${}^\circ G(t,\omega)=I({}^\circ t,\omega)$ for all t. Since $I(\circ,\omega)$ is continuous for these ω, and $I(t,\circ)$ is $\mathcal{D}$-measurable for all t, I is $\mathcal{B}\times\mathcal{D}$-measurable. Therefore I' is $\mathcal{B}\times\mathcal{D}$-measurable, $I'\in L^2([0,1]\times\Omega)$, and $I'(t,\circ)$ is $\mathcal{D}_t$-measurable, so $I'\in\mathcal{G}$. Therefore $I \in \mathscr{F}$. Since I is $\mathscr{B} \times \mathscr{D}$-measurable, ${}^{\circ}G$ is $L(\mathscr{C}) \times \mathscr{D}$-measurable by Theorem 14.

We can now give a proof of Itô's Lemma ([7]).

THEOREM 37. *Suppose* $h: R^m \to R$ *is* C^2, $f_j^l \in \mathscr{F}$ $(1 \leqq j \leqq n, 1 \leqq l \leqq m)$, $\beta_1, \cdots, \beta_n$ *are Brownian Motions arising from *-independent random walks* $\chi_1, \cdots, \chi_n$ *with increments* $\omega_{ji}/\sqrt{\eta}$ $(1 \leqq j \leqq n, 1 \leqq i \leqq \eta)$ *in the manner of Section* 3. *Suppose* $a_l \in L^1([0,1] \times \Omega, \mathscr{B} \times \mathscr{D}, \mu \times P)$, *and* $a_l(t, \circ)$ *is* $\mathscr{D}_t$*-measurable. Let*

$$I_l(t,\omega) = \int_0^t a_l(\tau,\omega)d\tau + \sum_{j=1}^n \int_0^t f_j^l(\tau,\omega)d\beta_j(\tau,\omega) \quad (1 \leqq l \leqq m)$$

$$I(t,\omega) = (I_1(t,\omega), \cdots, I_m(t,\omega))$$

$$H(t,\omega) = h(I(t,\omega)).$$

Then H *is a stochastic integral; in fact, if* h_l *and* h_{lk} *denote partial derivatives,*

$$\begin{aligned} H(t,\omega) = H(0,\omega) &+ \sum_{l=1}^m \int_0^t h_l(I(\tau,\omega))a_l(\tau,\omega)d\tau \\ &+ \frac{1}{2}\sum_{jkl} \int_0^t h_{lk}(I(\tau,\omega))f_j^l(\tau,\omega)f_j^k(\tau,\omega)d\tau \\ &+ \sum_{lj} \int_0^t h_l(I(\tau,\omega))f_j^l(\tau,\omega)d\beta_j(\tau,\omega). \end{aligned}$$

PROOF. The essential difficulty in the standard proof is assigning meaning to, and proving, the heuristic formula $d\beta_i d\beta_j = \delta_{ij} dt$. In our formulation, over the interval $[(i-1)/\eta, i/\eta)$, we have

$$(d\chi_j)^2 = \left(\frac{\omega_{ji}}{\sqrt{\eta}}\right)^2 = 1/\eta = dt.$$

The proof consists of an elaboration of this observation.

For simplicity of notation, we shall assume that $m = 1$ and $n = 2$; the proof extends immediately to the general case. We shall write f_1, f_2 for f_1^1 and f_2^1 respectively, and let g_1, g_2, G_1, and G_2 correspond as in Definition 32 and Theorem 33. Using Lemma 31, let b be a 1-lifting for a $(= a_1)$.

If t is finite and $\delta \simeq 0$, $|h''(t) - h''(t+\delta)| \simeq 0$. Therefore there exists $M \in {}^*N - N$ such that

$$\varepsilon = \sup_{|t|<M, |\delta|<3\eta^{-1/3}} |h''(t) - h''(t+\delta)| \simeq 0.$$

Let $G(t,\omega)=\int_0^t b(\tau,\omega)d\lambda(\tau)+G_1(t,\omega)+G_2(t,\omega)$. By truncation, using Corollary 5, we may assume $|b(t,\omega)|,|g_j(t,\omega)|<\min\{\eta^{1/6},\varepsilon^{-1/3}\}$. Therefore

$$|G(i/\eta,\omega)-G((i-1)/\eta,\omega)|<\min\{3\eta^{-1/3},3\varepsilon^{-1/3}\eta^{-1/2}\}.$$

Let $M_\omega=\sup_{t\in{}^*[0,1]}\max\{|G(t,\omega)|,|{}^*h'(G(t,\omega))|,|{}^*h''(G(t,\omega))|\}$. For P-almost all ω, $G(\circ,\omega)$ is near-standard, so M_ω is finite.

Suppose $t\in[0,1]$. Then

$$\begin{aligned}
&H(t,\omega)-H(0,\omega)\\
&={}^\circ\left({}^*h\left(G\left(\frac{[\eta t]}{\eta},\omega\right)\right)-{}^*h(G(0,\omega))\right)\\
&={}^\circ\left(\sum_{i=1}^{[\eta t]}\left({}^*h\left(G\left(\frac{i}{\eta},\omega\right)\right)-{}^*h\left(G\left(\frac{i-1}{\eta},\omega\right)\right)\right)\right)\\
&={}^\circ\sum_{i=1}^{[\eta t]}{}^*h'\left(G\left(\frac{i-1}{\eta},\omega\right)\right)\left(G\left(\frac{i}{\eta},\omega\right)-G\left(\frac{i-1}{\eta},\omega\right)\right)\\
&\quad+{}^\circ\frac{1}{2}\sum_{i=1}^{[\eta t]}\left({}^*h''\left(G\left(\frac{i-1}{\eta},\omega\right)\right)+\varepsilon_i\right)\left(G\left(\frac{i}{\eta},\omega\right)-G\left(\frac{i-1}{\eta},\omega\right)\right)^2
\end{aligned}$$

where the ε_i's are error coefficients.

$$\begin{aligned}
&\sum_{i=1}^{[\eta t]}|\varepsilon_i|(G(i/\eta,\omega)-G((i-1)/\eta,\omega))^2\\
&\leqq\eta\sup_{|\delta|<3\eta^{-1/3}}|h''(G(i/\eta,\omega)+\delta)-h''(G(i/\eta,\omega))|(9\varepsilon^{-2/3}\eta^{-1})\\
&\leqq 9\varepsilon^{1/3}\simeq 0,\quad\text{for almost all}\quad\omega.
\end{aligned}$$

Hence we may neglect the ε_i's.

The first summation yields

$$\begin{aligned}
&{}^\circ\sum_{i=1}^{[\eta t]}{}^*h'(G((i-1)/\eta,\omega))(G(i/\eta,\omega)-G((i-1)/\eta,\omega))\\
&\quad={}^\circ\sum_{i=1}^{[\eta t]}\frac{{}^*h'(G((i-1)/\eta,\omega))b((i-1)/\eta,\omega)}{\eta}\\
&\qquad+{}^\circ\sum_{i=1}^{[\eta t]}{}^*h'(G((i-1)/\eta,\omega))g_1((i-1)/\eta,\omega)\omega_{1i}/\sqrt{\eta}\\
&\qquad+{}^\circ\sum_{i=1}^{[\eta t]}{}^*h'(G((i-1)/\eta,\omega))g_2((i-1)/\eta,\omega)\omega_{2i}/\sqrt{\eta}.
\end{aligned}$$

Let $n \in N$, $A_n = \{\omega : M_\omega < n\}$. A_n is internal, and $P(\bigcup_{n=1}^{\infty} A_n) = 1$. For any $B \in \mathcal{A}$, let $B_n = A_n \cap B$. Since ${}^*h' \circ G \,|_{B_n}$ is bounded by n and $b \in SL^1$, $b\,{}^*h' \circ G \,|_{B_n} \in SL^1$ by Corollary 5.

$$
\begin{aligned}
{}^\circ\int_{B_n} \sum_{i=1}^{[\eta t]} {}^*h'(G((i-1)/\eta, \omega))b((i-1)/\eta, \omega)/\eta \, d\nu
&= {}^\circ\int_{[0,t]\times B_n} {}^*h'(G([\eta\tau]/\eta, \omega))b([\eta\tau]/\eta, \omega)d(\lambda \times \nu) \\
&= \int_{[0,t]\times B_n} {}^\circ({}^*h'(G([\eta\tau]/\eta, \omega))){}^\circ b([\eta\tau]/\eta, \omega)dL(\lambda \times \nu) \\
&= \int_{B_n} \int_0^t h'(I({}^\circ\tau, \omega))a({}^\circ\tau, \omega)dL(\lambda)dP \\
&\qquad \text{(since } h' \text{ is continuous, and } {}^\circ G, {}^\circ b \text{ are } L(\mathscr{C}) \times \mathscr{D}\text{-measurable)} \\
&= \int_{B_n} \int_0^t h'(I(\tau, \omega))a(\tau, \omega)d\tau dP \quad \text{(by Theorem 14).}
\end{aligned}
$$

Thus, by Theorem 24,

$$
{}^\circ\sum_{i=1}^{[\eta t]} {}^*h'(G((i-1)/\eta, \omega))b((i-1)/\eta, \omega)/\eta = \int_0^t h'(I(\tau, \omega))a(\tau, \omega)d\tau
$$

for P-almost all ω in each A_n, and hence for P-almost all ω.

Restricting to A_n, ${}^*h'(G([\eta\tau]/\eta, \omega))g_j([\eta\tau]/\eta, \omega)$ is a 2-lifting of $h'(I(\tau, \omega))f_j(\tau, \omega)$. Hence

$$
{}^\circ\sum_{i=1}^{[\eta t]} {}^*h'(G((i-1)/\eta, \omega))g_j((i-1)/\eta, \omega)\omega_{ji}/\sqrt{\eta} = \int_0^t h'(I(\tau, \omega))f_j(\tau, \omega)d\beta_j(\tau, \omega)
$$

for P-almost all ω, by Theorem 33.

It remains to evaluate

$$\left(G\left(\frac{i}{\eta},\omega\right)-G\left(\frac{i-1}{\eta},\omega\right)\right)^2$$

$$=\left(\frac{b\left(\frac{i-1}{\eta},\omega\right)}{\eta}+\frac{g_1\left(\frac{i-1}{\eta},\omega\right)\omega_{1i}}{\sqrt{\eta}}+\frac{g_2\left(\frac{i-1}{\eta},\omega\right)\omega_{2i}}{\sqrt{\eta}}\right)^2$$

$$=\frac{b^2\left(\frac{i-1}{\eta},\omega\right)}{\eta^2}+\frac{g_1^2\left(\frac{i-1}{\eta},\omega\right)}{\eta}+\frac{g_2^2\left(\frac{i-1}{\eta},\omega\right)}{\eta}$$

$$+\frac{2b\left(\frac{i-1}{\eta},\omega\right)}{\eta\sqrt{\eta}}\left(g_1\left(\frac{i-1}{\eta},\omega\right)\omega_{1i}+g_2\left(\frac{i-1}{\eta},\omega\right)\omega_{2i}\right)$$

$$+\frac{2g_1\left(\frac{i-1}{\eta},\omega\right)g_2\left(\frac{i-1}{\eta},\omega\right)\omega_{1i}\omega_{2i}}{\eta}.$$

$$\left|\sum_{i=1}^{[\eta t]}{}^*h''\left(G\left(\frac{i-1}{\eta},\omega\right)\right)\frac{b^2\left(\frac{i-1}{\eta},\omega\right)}{\eta^2}\right|$$

$$\leqq \eta M_\omega\frac{\eta^{1/3}}{\eta^2}=\frac{M_\omega}{\eta^{2/3}}\simeq 0 \quad \text{(almost everywhere).}$$

$$\left|\sum_{i=1}^{[\eta t]}\frac{{}^*h''\left(G\left(\frac{i-1}{\eta},\omega\right)\right)b\left(\frac{i-1}{\eta},\omega\right)g_j\left(\frac{i-1}{\eta},\omega\right)\omega_{ji}}{\eta\sqrt{\eta}}\right|$$

$$\leqq\frac{\eta M_\omega\eta^{1/3}}{\eta^{3/2}}=\frac{M_\omega}{\eta^{1/6}}\simeq 0.$$

$$\frac{\left|\left({}^*h''\left(G\left(\frac{i-1}{\eta},\omega\right)\right)g_1\left(\frac{i-1}{\eta},\omega\right)g_2\left(\frac{i-1}{\eta},\omega\right)\right|}{\sqrt{\eta}}\leqq\frac{M_\omega}{\eta^{1/6}}\simeq 0.$$

If we define a new random walk χ' using $\omega_1' = \omega_{1i}\omega_{2i}$, we see that this integrand, when restricted to A_n, is a 2-lifting of the identically zero integrand. Then by Theorem 33, for almost all ω,

$$\sum_{i=1}^{[\eta t]}{}^*h''\left(G\left(\frac{i-1}{\eta},\omega\right)\right)g_1\left(\frac{i-1}{\eta},\omega\right)g_2\left(\frac{i-1}{\eta},\omega\right)\omega_{1i}\omega_{2i}/\eta$$

$$\simeq\int_0^t d\chi'(\tau,\omega)=0.$$

Finally,

$$
{}^{\circ}\int_{B_n} \sum_{i=1}^{[\eta t]} \frac{{}^*h''\left(G\left(\frac{i-1}{\eta}, \omega\right)\right) g_j^2\left(\frac{i-1}{\eta}, \omega\right)}{\eta} d\nu
$$

$$
= {}^{\circ}\int_{[0,t]\times B_n} {}^*h''\left(G\left(\frac{[\eta\tau]}{\eta}, \omega\right)\right) g_j^2\left(\frac{[\eta\tau]}{\eta}, \omega\right) d(\lambda \times \nu)
$$

$$
= \int_{[0,t]\times B_n} {}^{\circ*}h''\left(G\left(\frac{[\eta\tau]}{\eta}, \omega\right)\right) {}^{\circ}g_j^2\left(\frac{[\eta\tau]}{\eta}, \omega\right) dL(\lambda \times \nu)
$$

$$
= \int_{[0,t]\times B_n} h''(I({}^{\circ}\tau, \omega)) f_j^2({}^{\circ}\tau, \omega) dL(\lambda \times \nu)
$$

$$
= \int_{B_n} \int_0^t h''(I({}^{\circ}\tau, \omega)) f_j^2({}^{\circ}\tau, \omega) dL(\lambda) dP.
$$

By Theorem 24, for almost all $\omega \in A_n$,

$$
{}^{\circ}\sum_{i=1}^{[\eta t]} \frac{{}^*h''\left(G\left(\frac{i-1}{\eta}, \omega\right)\right) g_j^2\left(\frac{i-1}{\eta}, \omega\right)}{\eta}
$$

$$
= \int_0^t h''(I({}^{\circ}\tau, \omega)) f_j^2({}^{\circ}\tau, \omega) dL(\lambda)
$$

$$
= \int_0^t h''(I(\tau, \omega)) f_j^2(\tau, \omega) d\tau \quad \text{by Theorem 14.}
$$

Putting these computations together, we have the desired result.

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