IL NUOVO CIMENTO VOL. XLIII A, N. 4 21 Giugno 1966

Embedding Properties of General Relativistic Manifolds (*).

P. SZEKERES

Center for Radiophysics and Space Research, Cornell University - Ithaca, N. Y.

(ricevuto il 10 Gennaio 1966)

Summary. — Integrability conditions for embedding a Riemannian manifold in a pseudo-Euclidean space of higher dimensions are derived. They are used to show that essentially the only physically significant solutions of the Einstein field equations which can be embedded in five dimensions are the Friedmann universes. The relation between symmetry properties and embedding properties of a manifold is discussed, setting upper bounds to the embedding class in certain cases.

1. – Introduction.

General relativity postulates that the world is a four-dimensional Riemannian manifold. It is well known (although the correct proof for indefinite metrics has only been given recently by FRIEDMANN (1)) that such a manifold can always be regarded, at least locally, as embedded in a pseudo-Euclidean space of ten dimensions. This is usually regarded as irrelevant since all geometrical properties can be obtained from the intrinsic structure of the manifold, the metric tensor g_{ab} and its derivatives. However embedding introduces some suggestive features that have led to a resurgence of interest in this problem (2), especially in its possible connection with elementary-particle physics. We may say that a curved space-time introduces in a natural way a pseudo-Euclidean space of as many as ten dimensions (and possibly more if a global embedding is required, although no clear-cut answer is known to this question)

(*) This work was supported by AFOSR Contract AF49(638)-1527.

(1) A. FRIEDMANN: *Journ. Math. Mech.*, **10**, 625 (1961).

(2) I. ROBINSON and Y. NE'EMAN: *Rev. Mod. Phys.*, **37**, 201 (1965), and the articles following this introduction.

and in so far as this space is a consequence of the curvature of space-time it is of physical interest, even if only indirectly.

The higher-dimensional space introduced from embedding has one clear-cut advantage over the original manifold, and that is in its symmetry. JOSEPH (3) points out that we have reduced the parameters in our allowed co-ordinate transformations from an infinite number to 55 by this process, and there is no doubt that it is easier to discuss elementary-particle physics in a flat multi-dimensional space than in a curved four-space. The Minkowskian tangent space at each point of the manifold allows the action of the ordinary Lorentz group, while the normal space is invariant under a (pseudo)-orthogonal group leaving the point in question fixed. This latter group may correspond to an « internal » symmetry or chiral group for elementary particles. NE'EMAN (4) has remarked that embedding in ten dimensions implies the emergence of the six-dimensional orthogonal group, a subgroup of which is the currently popular SU_3.

It is well, however, not to be too carried away with some of these ideas. A principal difficulty is that it is not at all clear what sort of physical interpretation to place on the added dimensions. In particular the question of why nature chooses certain four-dimensional submanifolds as the arena of « ordinary » physics (that is, of particle or photon paths) becomes a particularly puzzling one. In this paper a more modest view is taken and we merely set out to prove some theorems which may have bearing on the above discussion. The main discussion will center around the minimal embedding problem. While a general space-time will need ten dimensions for embedding, most known metrics, on account of their symmetry properties, will need less. Explicit embeddings for almost all known solutions of the field equations have been given by ROSEN (5), and these bear out this assertion. Depending on the kind of theory we envisage, as many as ten dimensions may ultimately prove an embarrassment, and we may be especially interested in those solutions of the field equations which can be embedded in fewer dimensions. In this regard FRONSDAL (6) has pointed out that all the experimental evidence for general relativity has only verified the validity of the Schwarzschild solution, and this can be embedded in six dimensions. It is probable that a good « approximation » to general relativity (whose deviations could not so far be detected by experiment) could be obtained from a six-dimenional theory, in which the gravitational field has the attractive feature of being determined essentially by two scalar fields.

---

(3) D. W. JOSEPH: *Rev. Mod. Phys.*, **37**, 225 (1965).
(4) Y. NE'EMAN: *Rev. Mod. Phys.*, **37**, 227 (1965).
(5) J. ROSEN: *Rev. Mod. Phys.*, **37**, 204 (1965).
(6) C. FRONSDAL: *Nuovo Cimento*, **13**, 988 (1959).

The only theorem one has on minimal embeddings is that no Riemannian manifold R_4 with $R_{ab}=0$ can be embedded in five dimensions. However as we shall see in Sect. **3** the proofs given to date for this theorem are only valid for positive definite metrics, and are therefore not applicable to general relativity. We shall give a correct proof of this theorem for normal hyperbolic manifolds, and show that the method can be extended to find all solutions of the field equations with incoherent matter which can be embedded in five dimensions. These turn out to be the Friedmann universes.

In the final Section we derive some relations between the symmetry and conformal properties of a manifold and its embedding class.

2. – The formalism of embedding.

Let R_4 be a Riemannian 4-space embedded in a pseudo-Euclidean space E_n. We may set up normal Gaussian co-ordinates in E_n using R_4 as an initial hypersurfaces as follows. Let x^a $(a=1, ..., 4)$ be co-ordinates in R_4, and at each point of R_4 choose $n-4$ mutually orthogonal straight lines all normal to the initial hypersurface. In the normal E_{n-4} so formed the first four co-ordinates of any point are taken to be x^a, while the remaining $n-4$ co-ordinates x^A $(A=5, ..., n)$ are given with respect to the pseudo-Cartesian frame defined by the selected normals. This defines co-ordinates x^μ $(\mu=1, ..., n)$ in the E_n in a neighborhood of R_4. R_4 is the hypersurface $x^A=0$. Since the straight lines normal to R_4 have equations

$$x^a = \text{const}\,, \qquad x^A = V^A t \qquad\qquad (V^A=\text{const}),$$

we find from the geodesic equations that the Christoffel symbols Γ^μ_{AB} vanish. That is

$$\tfrac{1}{2} g^{\mu\nu}(g_{\nu A,B}+g_{\nu B,A}-g_{AB,\nu}) = 0\,.$$

Now $g_{AB}=\varepsilon_A\delta_{AB}$ $(\varepsilon_A=\pm 1$, the signs depending on the signature of the embedding space $E_n)$ everywhere, so $g_{AB,a}=0$, and we are left with

$$g_{aA,B}+g_{aB,A}=0\,. \tag{1}$$

Thus

$$g_{aA,A}=0$$

and differentiating (1) with respect to x^B gives

$$g_{aA,BB}=0\,.$$

Hence

$$g_{aA} = P_{ABa} x^B , \tag{2}$$

where

$$P_{ABa} = P_{ABa}(x^a) , \qquad P_{ABa} + P_{BAa} = 0 , \tag{3}$$

and the summation convention over the indices A, B is adopted.

The metric $g_{\mu\nu}$ of E_n has the form

$$\begin{pmatrix} g_{ab}(x^a, x^A) & P_{ABa} x^B \\ P_{ABa} x^B & \varepsilon^A \delta_{AB} \end{pmatrix} .$$

At R_4, $x^A = 0$, so the g_{aB} vanish and g_{ab} becomes the metric of R_4.

A calculation of the Christoffel symbols at R_4 gives

$$\Gamma_{b\ c}^{\ a} = \Gamma_{b\ c}^{\ a}(R_4) , \qquad \Gamma_{b\ c}^{\ A} = K^A{}_{bc} , \qquad \Gamma_{A\ b}^{\ a} = - K_{A\ b}^{\ a} ,$$

$$\Gamma_{B\ a}^{\ A} = P^A{}_{Ba} , \qquad \Gamma_{A\ B}^{\ \mu} = 0 ,$$

where

$$K_{Abc} = - \tfrac{1}{2} g_{bc,A}(R_4) , \tag{4}$$

and the indices A, B are raised and lowered by means of the metric g^{AB}, g_{AB}, while a, b are raised and lowered with $g_{ab}(R_4)$.

Finally we calculate the Riemann tensor for E_n,

$$R^\mu{}_{\nu\varrho\sigma} = \Gamma^\mu_{\nu\sigma,\varrho} - \Gamma^\mu_{\nu\varrho,\sigma} + \Gamma^\alpha_{\nu\sigma} \Gamma^\mu_{\alpha\varrho} - \Gamma^\alpha_{\nu\varrho} \Gamma^\mu_{\alpha\sigma} .$$

Since E_n is flat this must vanish, and calculating successively $R^a{}_{bcd}$, $R^A{}_{bcd}$, $R^A{}_{Bcd}$, $R^A{}_{bBd}$ gives the following integrability conditions for R_4 to be embedded in an E_n:

$$R_{abcd} = K_{Aac} K^A{}_{bd} - K_{Aad} K^A{}_{bc} = \sum_{A=5}^{n} \varepsilon_A (K_{Aac} K_{Abd} - K_{Aad} K_{Abc}) , \tag{5}$$

$$K_{Abc;d} - K_{Abd;c} = K_{Bbc} P^B{}_{Ad} - K_{Bbd} P^B{}_{Ac} , \tag{6}$$

$$P_{ABa;b} - P_{ABb;a} + K_{Ada} K_{B\ b}^{\ d} - K_{Adb} K_{B\ a}^{\ d} + P_{DAa} P^D{}_{Bb} - P_{DAb} P^D{}_{Ba} = 0 , \tag{7}$$

$$K_{Abd,B} + \tfrac{1}{2}(K_{Aad} K_{B\ b}^{\ a} + K_{Aab} K_{B\ d}^{\ a}) + P_{DAb} P^D{}_{Bd} + P_{DAd} P^D{}_{Bb} = 0 . \tag{8}$$

Commas denote ordinary derivatives, while semicolon denotes covariant derivative in R_4. It is clear by reversing the above argument that any R_4 which has tensors K_{Aab}, P_{ABa} satisfying (5), (6), (7) can be embedded in an E_n. The eq. (8) tells nothing about R_4 since it is only an equation for the extrinsic derivatives of K_{Aab}. Since any R_4 can be embedded in ten dimensions we see that there must exist a set of six tensors K_{Aab}, and fifteen vectors $P_{ABa} = -P_{BAa}$ $(A, B = 1, ..., 6)$ satisfying eqs. (5)–(7).

3. – Embedding in five dimensions.

A frequently quoted theorem on minimal embeddings is that no vacuum solution of the Einstein field equations can be embedded in five dimensions (7). It is worth noting however that the proofs given in the literature (8,9) for this theorem depend on diagonalizing a symmetric tensor by a local rotation, a procedure which is invalid for indefinite metrics. We give a correct proof of this theorem for normal hyperbolic manifolds (one minus sign in the signature). It will also be seen that the procedure breaks down completely if the metric has more than one minus sign.

For a five-dimensional embedding the eqs. (5)–(7) reduce to

$$R_{abcd} = 2\varepsilon K_{a[c}K_{d]b} , \tag{9}$$

$$K_{a[b;c]} = 0 , \tag{10}$$

where square brackets denote antisymmetrization over the enclosed indices. The Ricci tensor, $R_{bc} = R^a{}_{bca}$, is given by

$$R^b{}_c = \varepsilon\left(K^b{}_a K^a{}_c - (K^a{}_a) K^b{}_c\right) . \tag{11}$$

THEOREM 1: *No nonflat vacuum metric can be embedded in five dimensions.*

Proof: For vacuum, $R^b{}_c = 0$, and eq. (11) may be written in matrix form

$$K^2 - (\mathrm{Tr}\, K)K = 0 . \tag{12}$$

The usual proof (8,9) involves setting up a local Minkowski frame and diagonalizing K_{ab} by a local rotation. For indefinite metrics this cannot be done in general. For example the tensor $K_{ab} = k_a k_b$, where k_a is a null vector, cannot be diagonalized by a Lorentz transformation. The canonical forms for symmetric tensors in Minkowski space have been given by PIRANI (10), but

(7) L. P. EISENHART: *Riemannian Geometry* (Princeton, 1960).
(8) J. A. SCHOUTEN and D. J. STRUIK: *Am. Journ. Math.*, **43**, 213 (1921).
(9) E. KASNER: *Am. Journ. Math.*, **43**, 130 (1921).
(10) F. A. E. PIRANI: to be published.

we shall refrain from using the full formalism here. We proceed as follows. Equation (22) shows that the eigenvalues of K must satisfy

$$\lambda^2 - \lambda \operatorname{Tr} K = 0 .$$

Hence $\lambda = 0$ or $\operatorname{Tr} K$. Since $\operatorname{Tr} K$ is equal to the sum of the eigenvalues, we see that the only possibilities are that all eigenvalues vanish ($\operatorname{Tr} K = 0$), or that three of the eigenvalues vanish and the fourth is equal to $\operatorname{Tr} K$.

Case a): $\operatorname{Tr} K = 0$. In this case $K^2 = 0$. Then $y^a = K^a{}_b x^b$ is an eigenvector of K for any vector x^a,

$$K^a{}_b y^b = 0 .$$

From the symmetry of K_{ab} we have that

$$y_a K^a{}_b x^b = 0 = y_a y^a .$$

Hence y^a is a null vector. Take any other vector x'^a, and similarly $y'^a = K^a{}_b x'^b$ is a null eigenvector. Furthermore

$$y'_a K^a{}_a x^b = 0 = y'_a y^a .$$

The two null vectors y^a and y'^a are orthogonal, hence for a normal hyperbolic metric they must be proportional and we have that $y_{[a} K_{b]c} x^c = 0$ for any arbitrary vector x^a. Hence $y_{[a} K_{b]c} = 0$, and since K_{ab} is symmetric we must have $K_{ab} = A y_a y_b$. But this implies that $R_{abcd} = 0$ from eq. (9).

Case b): $\operatorname{Tr} K \neq 0$. In this case the only nonvanishing eigenvalue is $\lambda = \operatorname{Tr} K$, and this eigenvalue is nondegenerate. There exists a unique eigenvector k^a (determined up to a factor) corresponding to λ. Now for any vector x^a, by eq. (12) the vector $y^a = K^a{}_b x^b$ is an eigenvector corresponding to λ. Hence $y^a \propto k^a$, and an identical argument to that above shows that $K_{ab} \propto k_a k_b$, and $R_{abcd} = 0$.

Hence if a vacuum metric can be embedded in five dimensions it must be flat (Minkowski space).

It is also clear now that the theorem breaks down if the metric is not normal hyperbolic, for then we may find two distinct orthogonal null vectors k^a, m^a, and the tensor $K_{ab} = k_a k_b + m_a m_b$ gives $R_{ab} = 0$ and $R_{abcd} \neq 0$.

We can use similar arguments to find canonical forms for K_{ab} if a solution of the field equations with cosmological constant

$$R_{ab} = -\Lambda g_{ab} \tag{13}$$

is to be embedded in five dimensions. The matrix K must now satisfy

$$K^2 - (\operatorname{Tr} K)K = -\varepsilon\Lambda I . \tag{14}$$

The eigenvalues of K then satisfy

$$\lambda^2 - \lambda \operatorname{Tr} K + \varepsilon\Lambda = 0 ,$$

And since $\operatorname{Tr} K = \sum_i \lambda_i$, we get

$$\lambda_j \sum_{i \neq j} \lambda_i = \varepsilon\Lambda .$$

Hence none of the λ_i vanish and subtracting the various equations gives three equations

$$(\lambda_1 - \lambda_2)(\lambda_3 + \lambda_4) = 0 ,$$

$$(\lambda_1 - \lambda_3)(\lambda_2 + \lambda_4) = 0 ,$$

$$(\lambda_1 - \lambda_4)(\lambda_2 + \lambda_3) = 0 .$$

It is easily seen that if $\lambda_1 \neq \lambda_2$ then $\lambda_3 = \lambda_1$ or λ_2. Hence some pair of λ's must be equal and we end up with two possibilities:

a) All λ's are equal, $\lambda = \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, $\varepsilon\Lambda = 3\lambda^2$,

b) $\lambda_1 = \lambda_2 = \lambda = -\lambda_3 = -\lambda_4$, $\varepsilon\Lambda = -\lambda^2$.

In case *a*) it can be proved that $K = \lambda I$. This is most easily seen from the Cayley-Hamilton equation for K, which takes the form

$$K^4 - 4\lambda K^3 + 6\lambda^2 K^2 - 4\lambda^3 K + \lambda^4 I = 0 .$$

Substitute eq. (14) with $\operatorname{Tr} K = 4\lambda$, $\varepsilon\Lambda = 3\lambda^2$, and then we have

$$3\lambda^2 K^2 - 4\lambda^3 K + \lambda^4 I = 0 .$$

A final comparison with (14) gives the result $K = \lambda I$.

In case *b*) we have $\operatorname{Tr} K = 0$ and the Cayley-Hamilton equation gives nothing new. Equation (14) becomes $K^2 = \lambda^2 I$. Let x^a be an arbitrary vector, then

$$K^a{}_b(\pm \lambda x^b + K^b{}_c x^c) = \pm \lambda(\pm \lambda x^a + K^a{}_b x^b) ,$$

so $y^a = \pm\lambda x^a + K^a{}_b x^b$ is an eigenvector corresponding to the eigenvalue $\pm\lambda$. Now if x^a is a unit timelike vector, $x_a x^a = -1$, then $y_a y^a = -2\lambda^2 \pm 2\lambda K_{ab} x^a x^b$ Clearly one of the two eigenvectors y^a must be timelike. Let u^a be the unit timelike vector in this direction, and suppose it corresponds to the eigenvalue $-\lambda$ (the choice $+\lambda$ would of course be quite similar). We can now set up a local Minkowski frame with u^a as time axis. K_{ab} has the form

$$\begin{pmatrix} & & & 0 \\ & & & 0 \\ & K'_{\mu\nu} & & 0 \\ 0 & 0 & 0 & +\lambda \end{pmatrix}$$

and by a rotation in the Euclidean three-space orthogonal to u^a, we can diagonalize K', giving for K_{ab} the final form

$$\begin{pmatrix} -\lambda & & & \\ & +\lambda & & \\ & & +\lambda & \\ & & & +\lambda \end{pmatrix}$$

Thus we have two possible canonical forms for K_{ab}

$$a)\quad K_{ab} = \lambda g_{ab}\,, \qquad \varepsilon\Lambda = 3\lambda^2\,,$$

$$b)\quad K_{ab} = \lambda g_{ab} + 2\lambda u_a u_b - 2\lambda s_a s_b\,, \qquad \varepsilon\Lambda = -\lambda^2\,,$$

where u_a, s_a are orthogonal unit vectors, timelike and spacelike respectively, $-u_a u^a = s_a s^a = 1$, $u_a s^a = 0$.

A similar analysis may be carried out for the case of a perfect fluid solution of the field equations, which can be written in the form

$$R_{ab} = -(p+\mu)u_a u_b + \tfrac{1}{2}(p - \mu - 2\Lambda)g_{ab}\,, \tag{15}$$

where p is the pressure, μ the density of the fluid. This yields an equation for K:

$$(K^2 - (\mathrm{Tr}\,K)K)^a{}_b = -Xu^a u_b + Y\delta^a{}_b\,, \tag{16}$$

where

$$X = \varepsilon(p+\mu)\,, \qquad Y = \tfrac{1}{2}\varepsilon(p - \mu - 2\Lambda)\,.$$

We first show that u^a is an eigenvector of $K^a{}_b$. Suppose $K^a{}_b u^b = v^a$. Then

$$K^a{}_b v^b = K^a{}_c K^c{}_b u^b = (\mathrm{Tr}\, K) v^a + (X + Y) u^a$$

by (16). Applying $K^a{}_b$ again to this equation gives

$$K^a{}_c K^c{}_b v^b = (\mathrm{Tr}\, K) K^a{}_b v^b + (X + Y) v^a ,$$

while from (16) we have

$$K^a{}_c K^c{}_b v^b = (\mathrm{Tr}\, K) K^a{}_b v^b - X(u_b v^b) u^a + Y v^a .$$

Comparison of these two equations shows that

$$X v^a = - X(u_b v^b) u^a .$$

Thus $x^a \propto u^a$ and u^a is an eigenvector of $K^a{}_b$ (assuming $X \neq 0$ — $X = 0$ reduces to the previous example of vacuum with cosmological constant). Let λ_4 be the eigenvalue corresponding to the eigenvector u^a. We then have equations for the eigenvalues of $K^a{}_b$:

$$- \lambda_4(\lambda_1 + \lambda_2 + \lambda_3) = X + Y ,$$

$$- \lambda_i \sum_{j \neq i} \lambda_j = Y \qquad (i, j = 1, 2, 3).$$

Solving these equations in the same way as for the vacuum case we finally get the following two canonical forms for K_{ab}:

(a) $$K_{ab} = \lambda g_{ab} + (\lambda - \nu) u_a u_b ,$$

$$X = 2\lambda(\lambda - \nu) , \qquad Y = - \lambda(2\lambda + \nu) ,$$

(b) $$K_{ab} = \lambda g_{ab} + 2\lambda u_a u_b + (\lambda - \nu) s_a s_b , \qquad s_a s^b = 1 , \qquad s_a u^a = 0 ,$$

$$X = 2\lambda(\lambda + \nu) , \qquad Y = - \lambda\nu .$$

For $p = \mu = 0$, we have $X = 0$ and these reduce to the vacuum forms derived above.

So far we have only used the properties of K_{ab} at a single point of the manifold. To get further information we must use the differential identities (10). These will allow us to prove some valuable theorems. Let us apply (10) to the form (b) obtained above for K_{ab}. We set up an orthonornal tetrad consisting of u^a, s^a obtained from the canonical form of K_{ab}, and completing it with two further unit orthogonal spacelike vectors e^a and f^a. Now we can write the second covariant derivatives of the tetrad vectors in the form

$$u_{a;b} = s_a A_b + e_a B_b + f_a C_b \,, \tag{17a}$$

$$s_{a;b} = u_a A_b + e_a D_b + f_a E_b \,, \tag{17b}$$

$$e_{a;b} = u_a B_b - s_a D_b + f_a G_b \,, \tag{17c}$$

$$f_{a;b} = u_a C_b - s_a E_b - e_a G_b \,. \tag{17d}$$

Substituting these into (10) and contracting with the different tetrad vectors gives, after a somewhat tedious calculation, the following equations:

$$A_b = (\lambda_{,d} s^d u_b - \nu_{,d} u^d s_b + P e_b + Q f_b)/(\lambda + \nu) \,, \tag{18a}$$

$$B_b = (P s_b - \lambda_{,d} u^d e_b)/2\lambda \,, \tag{18b}$$

$$C_b = (Q s_b - \lambda_{,d} u^d f_b)/2\lambda \,, \tag{18c}$$

$$D_b = (P u_b + \nu_{,d} e^d s_b + \lambda_{,d} s^d e_b)/(\nu - \lambda) \,, \tag{18d}$$

$$E_b = (Q u_b + \nu_{,d} f^d s_b + \lambda_{,d} s^d f_b)/(\nu - \lambda) \,. \tag{18e}$$

There is no equation for G_b and we have assumed that λ, $\nu - \lambda$, $\nu + \lambda$ all do not vanish. P and Q are arbitrary scalar functions.

We shall also make use of the Ricci identities

$$u_{b;[cd]} = \tfrac{1}{2} R^a{}_{bcd} u_a \,.$$

Substituting from (9) and contracting with e^b, f^b gives two equations

$$B_{[c;d]} + A_{[c} D_{d]} + G_{[c} C_{d]} = \varepsilon \lambda^2 e_{[c} u_{d]} \,, \tag{19a}$$

$$C_{[c;d]} + A_{[c} E_{d]} + G_{[c} B_{d]} = \varepsilon \lambda^2 f_{]c} u_{p]} \,. \tag{19b}$$

We could obtain equations for the curls of all vectors $A, .., G$ but these are the only ones we shall need.

THEOREM 2: *The only vacuum solutions of the Einstein field equations with cosmological constant which can be embedded in five dimensions are spaces of constant curvature.*

Proof: Putting $p = \mu = 0$ in eq. (b) gives $\nu + \lambda = 0$, $\varepsilon \Lambda = -\lambda^2$. Hence eqs. (18) are not valid as they stand. We get instead

$$B_a = C_a = D_a = E_a = 0 \,.$$

When substituted in ($19a$) or ($19b$) these give $\lambda = 0$, and hence $K_{ab} = 0$. Thus case b) is eliminated and we are left with case a),

$$K_{ab} = \lambda g_{ab}\,, \qquad \varepsilon\Lambda = 3\lambda^2\,.$$

In this case (9) gives for the Riemann tensor

$$R_{abcd} = 2\varepsilon\lambda^2 g_{a[c}g_{d]b}\,.$$

Thus the space is a space of constant curvature. It is one of the two deSitter spaces (the usual deSitter universe has positive constant curvature and we must take $\varepsilon = +1$, but there is another one with negative curvature having the unphysical property of possessing closed timelike lines ([11]).

THEOREM 3: *The only solutions of the field equations with incoherent matter* ($p = 0$) *which can be embedded in five dimensions are the Friedmann cosmological models.*

Proof: Suppose K_{ab} has the form (b) with $\nu - \lambda \neq 0$. Then eqs. (18) are valid and we can substitute them into ($19a$) and ($19b$). Contracting ($19a$) with $f^c u^d$ results in

$$-\frac{PQ}{\nu^2 - \lambda^2} = 0\,.$$

Hence either P or Q vanishes. Assume $P = 0$ (the argument is identical for $Q = 0$). Contracting ($19a$) with $e^c u^d$, and ($19b$) with $f^c u^d$, results in two equations,

$$(\lambda_{,d}\mu^d/2\lambda)_{,c}u^c - (\lambda_{,b}u^b/2\lambda)^2 - (\lambda_{,b}s^b)^2/(\nu^2 - \lambda^2) - \varepsilon\lambda^2 = 0\,,$$

$$(\lambda_{,b}u^b/2)_{,c}u^c - (\lambda_{,b}u^b/2\lambda)^2 - (\lambda_{,b}s^b)^2/(\nu^2 - \lambda^2) - \varepsilon\lambda^2 + 2Q^2/(\nu^2 - \lambda^2) = 0\,.$$

Comparing these two equations we see that $Q = 0$.

Now $X + 2Y = 2\varepsilon(p - \Lambda) = \lambda^2$. Hence if $p = 0$, $\lambda^2 = -2\varepsilon\Lambda = \text{constant}$. This means that $\lambda_{,a} = 0$ and from the first of the above equations we get that $\lambda = 0$. But then

$$K_{ab} = \nu s_a s_b$$

([11]) J. L. SYNGE: *Relativity, the General Theory* (Amsterdam, 1960).

and the Riemann tensor vanishes. Thus we see that K_{ab} must be of the form (*a*), or (*b*) with $\nu - \lambda = 0$. In either of these cases K_{ab} takes the form

$$K_{ab} = \lambda g_{ab} + \beta u_a u_b .$$

Equation (10) for a K_{ab} of this form implies that $u_{a;q}$ has the form

$$u_{a;b} = \theta(e_a e_b + f_a f_b + s_a s_b) + q_a u_b .$$

Hence the streamlines of the fluid have no shear and rotation. But this is just the characterization of the Friedmann cosmological models (12), hence our theorem is proved. The converse, that the Friedmann solutions can be embedded in five dimension is settled at once since ROSEN (5) has given the explicit embeddings.

THEOREM 4: *No perfect fluid solution with rotating matter can be embedded in five dimensions.*

Proof. We have shown in the preceding theorem even without the assumption $p = 0$, that $p = Q = 0$. Substituting into (17*a*) from (18) now gives that

$$u_{[a;b}u_{c]} = 0 .$$

Hence the fluid trajectories are normal (orthogonal to hypersurfaces), which is the relativistic statement of no rotation (11).

4. – Embedding in higher dimensions.

In the previous Section we showed that except for the Friedmann universes, essentially all physically meaningful solutions have a lower bound of 2 on their embedding class (embedding class being defined as the minimum number of extra dimensions needed to embed the manifold). It appears impossible to derive any general theorems of this kind for higher-dimensional embeddings. The integrability conditions (5-7), reduce for a six-dimensional embedding to

$$\begin{aligned} R_{abcd} &= 2(\varepsilon_1 K_{a[c}K_{d]b} + \varepsilon_2 H_{a[c}H_{d]b}) , \\ K_{a[b;c]} &= -\varepsilon_2 H_{a[b}P_{c]} , \\ H_{a[b;c]} &= \varepsilon_1 K_{a[b}P_{c]} , \\ P_{[a;b]} &+ K_{d[a}H^d{}_{b]} = 0 . \end{aligned}$$

(12) O. HECKMANN and E. SCHÜCKING: article in *Gravitation, and Introduction to Current Research*, edited by L. WITTEN (New York, 1962).

Given any specific form for the Riemann tensor R_{abcd}, there is in general a family of tensor pairs K_{ab}, H_{ab} which will satisfy the first of these equations, and this family cannot be characterized in any straightforward canonical fashion as we managed to do in the previous Section. Hence the remaining differential identities are quite inapplicable. In fact there appears to be nothing significant that an embedding class greater than 1 says about the manifold.

There are however a few remarks one can make with regard to fixing an *upper bound* to the embedding class in the case when certain symmetries are present.

An R_4 possessing a group of symmetries with normal trajectories (hypersurface orthogonal Killing vector field), can be embedded in eight dimensions.

Proof: If the trajectories are spacelike or timelike with x^4 a co-ordinate labelling normal hypersurfaces, we can write the metric in the form

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \varepsilon\varphi^2(\mathrm{d}x^4)^2 \,,$$

where $\mu, \nu = 1, 2, 3$, $\varphi = \varphi(x^\mu)$, $g_{\mu\nu} = g_{\mu\nu}(x^\mu)$, and $\varepsilon = \pm 1$ depending on whether the trajectories are spacelike or timelike. Put

$$u = \varphi \cos x^4 \,, \qquad v = \varphi \sin x^4 \,.$$

Then

$$\varepsilon(\mathrm{d}u^2 + \mathrm{d}v^2) = \varepsilon\big(\mathrm{d}\varphi^2 + \varphi^2(\mathrm{d}x^4)^2\big) \,,$$

and our problem has been reduced to finding an embedding of the three-dimensional metric form $g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu - \mathrm{d}\varphi^2$, which can always be done in at most six dimensions.

If the trajectories are null lines, we can write the metric in the form ([13])

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + 2\,\mathrm{d}x^3\,\mathrm{d}x^4 \,,$$

$g_{\mu\nu} = g_{\mu\nu}(x^\mu)$. Put $u = 1/\sqrt{2}(x^3 + x^4)$, $v = 1/\sqrt{2}(x^3 - x^4)$ and we have

$$\mathrm{d}u^2 - \mathrm{d}v^2 = 2\,\mathrm{d}x^3\,\mathrm{d}x^4 \,.$$

Hence we can again find an eight-dimensional embedding of the manifold.

If the trajectories in the above theorem are spacelike or timelike and are geodesic we can further reduce the embedding class by one, since the metric

---

([13]) Cf. I. ROBINSON and A. TRAUTMAN: *Phys. Rev. Lett.*, **4**, 431 (1960).

can be written in the form

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu + (dx^4)^2 \,.$$

This can clearly be embedded in seven dimensions. The statement is of no use for vacuum metrics, since any vacuum metric with such a group of symmetries must be a p.p. wave ([14]), and the group trajectories are null lines. However the Gödel universe

$$ds^2 = a^2\left[(dx^1)^2 + \tfrac{1}{2}\exp[2x^1]^2(dx^2)^2 + (dx^3)^2 - \left(dx^4 + \exp[x^1]\,dx^2\right)^2\right]$$

does possess such a group (trajectories along the x^3 lines), hence it can be embedded in seven dimensions. We know furthermore from Theorem 3 or 4 of the previous Section that the Gödel universe cannot be embedded in five dimensions, but whether its embedding class is 2 or 3 remains undecided.

Since most known solutions of the field equations possess a group of motions with normal trajectories, we can accordingly put an upper bound of 4 on the embedding class. When further groups are available the embedding class can frequently be reduced still further. For the Schwarzschild solution (or any spherically symmetric static metric) the embedding class can be reduced to 2. For the Weyl cylindrically symmetric static solutions it can be reduced to 3, and in special cases to 2.

A final theorem, a generalization of a theorem of Schouten ([7]), relates embedding class to the conformal structure of the manifold.

If R_4 has embedding class p, and S_4 is conformally related to R_4, then S_4 has embedding class at most $p+2$, and can be embedded in the null cone of an E_{p+6}.

Proof: Let R_4 have metric tensor g_{ab}. Then S_4 has metric tensor $\varphi^2 g_{ab}$, for some scalar function φ. Since R_4 has embedding class p, there exist $p+4$ functions $y^\mu = y^\mu(x^a)$ $(\mu = 1, \ldots, p+4)$ satisfying

$$\sum_\mu \varepsilon_\mu y^\mu{}_{,a} y^\mu{}_{,b} = g_{ab} \,.$$

Put now

$$\begin{aligned} z^\mu &= \varphi\, y^\mu(x^a)\,, \\ z^{p+5} &= \varphi\left(\sum \varepsilon_\mu (y^\mu)^2 - \tfrac{1}{4}\right), \\ z^{p+6} &= \varphi\left(\sum \varepsilon_\mu (y^\mu)^2 + \tfrac{1}{4}\right). \end{aligned}$$

([14]) W. Kundt and J. Ehlers: article in *Gravitation, and Introduction to Current Research*, edited by L. Witten (New York, 1962).

Then $\sum \varepsilon_\mu z^\mu{}_{,a} z^\mu{}_{,b} + z^{p+5}{}_{,a} z^{p+5}{}_{,b} - z^{p+6}{}_{,a} z^{p+6}{}_{,b} = \varphi^2 g_{ab}$. Thus S_4 is of embedding class at most $p+2$. Furthermore, we have

$$\sum_\mu \varepsilon_\mu (z^\mu)^2 + (z^{p+5})^2 - (z^{p+6})^2 = 0 \,,$$

so that S_4 has been embedded in the null cone of E_{p+6}.

RIASSUNTO (*)

Si discutono le condizioni di integrabilità per adagiare una molteplicità riemanniana su uno spazio pseudoeuclideo. Si usano queste condizioni per dimostrare che essenzialmente le sole soluzioni fisicamente significative delle equazioni di campo di Einstein che possono essere adagiate su cinque dimensioni sono gli universi di Friedmann. Si discute la relazione fra le proprietà di simmetria e le proprietà di inserzione di una molteplicità, ponendo in alcuni casi limiti superiori alla classe inserente.

(*) *Traduzione a cura della Redazione.*