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On a Problem of Maximum in Dispersion Theory.

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Summary. — We present a technique for finding maximal bounds on strengths of interaction when the general inequality of the type arising from dispersion relations is given. It is shown that for quite a large class of functionals involved in such inequalities the solution exists and is unique.

Introduction.

In the problem of finding bounds on strength of interaction one gets conditions of the type [1-3]

$$\Phi[F] = \int_1^\infty g(x, |F(x)|)\,\mathrm{d}x \underset{\mathrm{df}}{=} A \leqslant M\,,$$

$g(x, |F(x)|)$, and $F(x)$—the latter has physical interpretation., *e.g.*, as form factor [2] or scattering amplitude for $t \leqslant 0$ [3]—satisfy the following conditions (comp. ref. [2]):

a) $F(x)$ is an analytic function of x except for a cut from 1 to ∞. Also $F^*(x) = F(x^*)$;

b) $F(x)$ is bounded at ∞ by some power of $x^{\frac{1}{2}}$;

c) $F(x)$ is nonzero on the cut except at a finite number of discrete points;

d) $g(x, |F|)$ is positive;

e) $\varphi[F]$ exist.

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(1) B. V. GESHKENBEIN and B. L. JOFFE: *Žurn. Èksp. Teor. Fiz.*, **44**, 1211 (1963) [English translation: *Sov. Phys. JETP*, **17**, 820 (1964)].

(2) S. D. DRELL, A. FINN and A. HEARN: *Phys. Rev.*, **136**, B 1439 (1964).

(3) A. MARTIN: preprint (Institute of Theoretical Physics, Stanford).

Two types of problem occur:

1) M is given and one wants to get the maximum of $|F|$ at some point outside the cut $x=1 \to x=\infty$. Here we shall restrict ourselves to the problem of finding the maximum of $|F(x=c)|$, $c<1$ and real;

2) M is itself a function of $F(c)$ and one wants to find, by self-consistency, an absolute bound of $F(c)$. Here we shall restrict ourselves to the case $M=F(c)$. We shall deal with these problems in the first and second Section of the paper respectively.

The functions $g(x, |F|)$ for which our technique gives the unique solution have to satisfy the following conditions:

f) when $|F|$ is treated as independent variable, both $g(x, |F|)$ and $\partial g(x, |F|)/\partial |F|$ are continuous and positive functions of x and $|F|$ for $x>1$ and $|F|\geqslant 0$. Moreover, $g(x, |F|)/|F|$ and $\partial g(x, |F|)/\partial |F|$ are increasing with $|F|$ for $x>1$;

g) there exist functions $F(x)$ such that

$$\sqrt{x-1}(x-c)|F|\partial g(x, |F|)/\partial |F| = B > 0|_{x\to 1}$$

and

$$\sqrt{x-1}(x-c)|F|\partial g(x, |F|)/\partial |F| = D > 0|_{x\to \infty} .$$

We demand that they satisfy

$$(x-1)^{\beta} < |F(x)| < (x-1)^{\alpha} \qquad \text{for } x\to 1$$

and

$$x^{\beta'} < |F(x)| < x^{\alpha'} \qquad \text{for } x \to \infty,$$

where B, D, α, β, α', β' are arbitrary constants; $c<1$.

1. – By the change of variables (comp., *e.g.*, ref. [2])

$$Z = -(t-i)/(t+i) \quad \text{with} \quad t = \operatorname{tg}\frac{\theta}{2} = \left[\frac{x-1}{1-c}\right]^{\frac{1}{2}} \quad \text{(notice } x=c, \text{ implies } z=0)$$

one maps the cut x-plane onto the unit disc $|z|\leqslant 1$. Next, denoting $F(z=e^{i\theta})$ by $F(\theta)$ one gets a condition

$$(1) \qquad \Phi[F] = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta, |F(\theta)|)\, d\theta = A \leqslant M ,$$

where f is expressed in a simple way by $g(x, |F|)$ and satisfies condition d). [$f(\theta, |F|)$ will be involved now in equalities $|F|(\partial/\partial F|) f(\theta, |F|)_{\theta\to 0} = B$ and $|F|(\partial/\partial |F|) f(\theta, |F|)_{\theta\to\pi} = D$ in condition g)]. With properties a)–e) the problem of minimum for $\Phi[F]$—which is an essential step in finding bounds—has been solved for the following $\Phi[F]$:

$$\Phi[F] = \frac{1}{2\pi}\int_{-\pi}^{\pi} p(\theta)|F(\theta)|^n \,\mathrm{d}\theta\,, \tag{2}$$

n being an arbitrary positive constant. The solution for $n=2$—found by SZEGÖ (4)—was generalized to the case of several functions F_i by MEIMANN (5). A simpler proof with the additional cut of $F(z)$ from $z=\beta$ $(0<<\beta<1)$ to $z=1$, is given by DRELL *et al.* (2). The last method may be easily applied to any n. One gets for the minimum of (2)

$$\Phi_{\min} = |F_0|^n \exp\left[-\frac{n}{2\pi i}\int_{\beta}^{1} \frac{\operatorname{disc} \ln F(z)}{z}\,\mathrm{d}z\right]\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln p(\theta)\,\mathrm{d}\theta\right]\,, \qquad F_0 \underset{\mathrm{df}}{=} F(z=0)$$

However, in the case of the π-π scattering one is forced (6) to put a more complicated form of $f(\theta, |F|)$ in (1), namely $f_{F\to\infty} \sim \sqrt[4]{|F|}\exp[\alpha(\theta)\sqrt{|F|}]$. Then Meimann's technique does not seem to be convenient. This may be already seen in the comparatively simple case of

$$\Phi[F] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \{p_1(\theta)|F|^2 + p_2(\theta)|F|^4\}\,\mathrm{d}\theta\,, \qquad p_1, p_2 > 0 \tag{3}$$

Using the notation of ref. (3) one could write

$$\Phi[F] = \Phi_1[F] + \Phi_2[F]\,.$$

$$\Phi_1[F] = \frac{1}{2\pi}\int_{-\pi}^{\pi} p_1(\theta)|F(\theta)|^2\,\mathrm{d}0 = \sum_n F_n^{(1)} F_n^{(1)}\,,$$

$$\Phi_2[F] = \frac{1}{2\pi}\int_{-\pi}^{\pi} p_2(\theta)|F(\theta)|^4\,\mathrm{d}\theta = \sum_n F_n^{(2)} F_n^{(2)}\,;$$

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(4) G. SZEGÖ: *Orthogonal Polynomials* (New York, 1959).

(5) N. N. MEIMANN: *Žurn. Èksp. Teor. Fiz.*, **44**, 1228 (1963) [English translation: *Sov. Phys. JETP*, **17**, 830 (1964)].

(6) A. MARTIN: to be published.

$F_n^{(1)}$, $F_n^{(2)}$ are coefficients of expansion $F(\theta)$ and $(F)^2$ in polynomials with weight $p_1(\theta)$ and $p_2(\theta)$, respectively. $\{F_n^{(2)}\}$ are now dependent on $\{F_n^{(1)}\}$ and the dependence between them is by no means expressible in a simple way. The existence of these relations makes the procedure of minimizing $\{F_n\}$ rather unhandy. It is still possible to treat $\{F_n^{(1)}\}$ and $\{F_n^{(2)}\}$ as if they were mutually independent but we do not use then some important information about $\Phi[F]$ in our considerations. For the case of the functional (3) it may be for example explicitly shown that in such an approach there exist no $F_{\text{extr}}(\theta)$ which minimizes simultaneously both Φ_1 and Φ_2 unless $p_2(\theta)$ is proportional to $(p_1(\theta))^2$ (7). It generally means that we get too low a value for $\Phi_{\min}$. We want to propose a technique which was found for the functional (3) by MARTIN (7), and which is efficient for more general forms of $f(\theta, |F|)$. (To illustrate the essential features of our method, we apply it in an Appendix explicitly to a simple particular case of the functional Φ).

Let us consider a functional of the type (1) with *conditions a)–g)* fulfilled. We weaken condition *a)* allowing—as in ref. (2)—for an additional cut from $z=\beta$ to $z=1$ $(0<\beta<1)$.

Let us now assume that we are given some arbitrary positive $\lambda(\theta)$ and look for $F_{\text{extr}}(\theta)$ which minimizes a functional

$$\Psi[F]=\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\{f(\theta,|F(\theta)|)-\lambda(\theta)|F|\}\,\mathrm{d}\theta\,. \tag{4a}$$

We get a condition

$$\frac{\delta\Psi[F]}{\delta|F|}\bigg|_{|F|=F_{\text{extr}}(\theta)}=\frac{\partial f(\theta,|F|)}{\partial|F|}\bigg|_{|F|=F_{\text{extr}}}-\lambda=0\,. \tag{4b}$$

[There may exist only one $F_{\text{extr}}(\theta)$ satisfying (4*b*) for given $\lambda(\theta)$ because of condition *f*]. Combining (4*a*, *b*) with (1) we have

$$A\equiv\Phi[F]\equiv\Psi[F]+\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\lambda|F|\,\mathrm{d}\theta\geqslant\Psi[F_{\text{extr}}]+\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\lambda(\theta)|F(\theta)|\,\mathrm{d}\theta\,,$$

which gives

$$\frac{1}{2\pi}\int\limits_{-\pi}^{\pi}\lambda(\theta)|F(\theta)|\,\mathrm{d}\theta\leqslant A-\Psi[F_{\text{extr}}]\,, \tag{5a}$$

where the equality sign is satisfied for $|F(\theta)|=F_{\text{extr}}(\theta)$. The minimum of the left-hand side may be found by applying the theorem of the geometric and

(7) A. MARTIN: unpublished.

arithmetic means (comp. refs. ([2,4])). One gets then

$$(5b)\quad \begin{cases} \dfrac{|F_0|}{r_1 \dots r_n} \exp\left[\dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} \ln \lambda(\theta)\, d\theta\right] \cdot \exp\left[-\dfrac{1}{2\pi i}\displaystyle\int_{\beta}^{1} \dfrac{\operatorname{disc} \ln F(z)}{z}\right] dz \leqslant \mu\,, \\ \mu \underset{\text{df}}{=} A - \Psi[F_{\text{extr}}] \end{cases}$$

[$r_i < 1$ are moduli of zeros of $F(z = \varrho e^{i\theta})$ inside the circle $|z| < 1$]. The equality sign applies if $|F(\theta)| = F_{\text{extr}}(\theta)$ and $F_{\text{extr}}(\theta) \cdot \lambda(\theta) = \text{const}$. We still have freedom in choosing the function $\lambda(\theta)$. Let us take that $\lambda_{\text{extr}}(\theta)$ which brings inequality (5*b*) closest to equality. It means that we look for the minimum of the functional

$$\chi[\lambda] = \frac{(A - \Psi[F_{\text{extr}}])\, r_1 \dots r_n}{|F_0| \exp\left[1/2\pi \int_{-\pi}^{\pi} \ln \lambda |\theta|\, d\theta\right] \exp\left[-1/2\pi i \int_{\beta}^{1} (\operatorname{disc} \ln F(z)/z)\, dz\right]}\,.$$

This is equivalent to minimizing against $\ln \chi[\lambda]$. We are led then to

$$\frac{1}{\lambda}\bigg|_{\lambda=\lambda_{\text{extr}}} = \frac{1}{\mu}\left[-\frac{\delta\Psi[F_{\text{extr}}]}{\delta\lambda}\right]\bigg|_{\lambda=\lambda_{\text{extr}}} = \frac{1}{\mu}\left[-\frac{\partial f(\theta, |F|)}{\partial |F|}\bigg|_{|F|=F_{\text{extr}}} \cdot \frac{\partial F_{\text{extr}}}{\partial \lambda} + \lambda \frac{\partial F_{\text{extr}}}{\partial \lambda} + F_{\text{extr}}\right]_{\lambda=\lambda_{\text{extr}}}\,.$$

The first two terms of the r.h.s. cancel in virtue of eq. (4*b*) and we get

$$(6a)\qquad \frac{\mu}{\lambda_{\text{extr}}(\theta)} = F_{\text{extr}}(\theta)$$

with

$$\mu = A - \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta, F_{\text{extr}})\, d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} \lambda_{\text{exrt}}(\theta) F_{\text{extr}} d\theta\,.$$

Because of (6*a*) we may change the above condition for μ to

$$(6b)\qquad A = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta, F_{\text{extr}}(\theta))\, d\theta\,.$$

Making use of (4*b*) one may rewrite (6*a*, *b*) in the following way:

$$(7a)\qquad \lambda_{\text{extr}} F_{\text{extr}} = F_{\text{extr}} \frac{\partial f(\theta, F_{\text{extr}})}{\partial F_{\text{extr}}} = \mu = \text{const}\,,$$

$$(7b)\qquad A = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta, F_{\text{extr}}(\theta))\, d\theta\,.$$

Equation ($7a$) fixes F_{extr} for given μ. Equation ($7b$) fixes μ. Our inequality ($5b$) becomes, after inserting λ from ($7a$)

$$(7c')\qquad \frac{1}{r_1 \dots r_n}\,|F_0|\exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln F_{\text{extr}}(\theta)\,\mathrm{d}\theta\right]\exp\left[-\frac{1}{2\pi i}\int_{\beta}^{1}\frac{\operatorname{disc}\ln F(z)}{z}\,\mathrm{d}z\right]\leqslant 1\,.$$

For any μ the r.h.s. integral in ($7b$) exists, because $|F|(\partial f/\partial|F|)>f$ (from the fact that $(1/|F|)f$ is an increasing function of $|F|$); also the integral $\int\ln F_{\text{extr}}(\theta)\,\mathrm{d}\theta$ in ($7c'$) exists [comp. condition (g)].

Next, it is evident that for $|F|=F_{\text{extr}}$ the equality sign in ($7c'$) holds. Because, on the other hand, $F_{\text{extr}}(\theta, A)$ satisfies eq. (1) [comp. ($7b$)] and may be expressed—in virtue of condition g)—as the modulus of an analytic function satisfying conditions a)–e), it is impossible to make the r.h.s. of ($7c'$) smaller than unity.

On the other hand [8] one has $(1/r_1 \dots r_n)\geqslant 1$. Further, if for $\beta<y<1$ $\operatorname{Im}Fz=(y+i\varepsilon)>0$ we may write instead of ($7c'$)

$$(7c)\qquad |F_0|\exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln F_{\text{extr}}(\theta)\,\mathrm{d}\theta\right]\leqslant 1/\beta\,.$$

Now let us come to the problems of uniqueness and existence of $F_{\text{extr}}(\theta)$. As regards the first, it is easy to prove the following statement:

> if for a given A there exist $F_{\text{extr}}(\theta)$ and μ, satisfying ($7a, b$) then they are unambiguously determined by these equations.

When μ is increased, $F_{\text{extr}}(\theta)$ gets larger for any θ because of eq. ($7a$) and condition f). It means that $F_{\text{extr}}(\theta)$ is uniquely determined by μ. Next, increased $F_{\text{extr}}(\theta)$ makes the value of A larger [comp. ($7b$) and f)] which means that there is only one μ, $F_{\text{extr}}(\theta)$ for given A, which proves our statement.

Equation ($7a$) has always a solution; the danger at points $\theta=0, \pi$ ($x=1, \infty$) is avoided because of condition g).

Next, we want to point out the following inequality resulting from eq. (1) and condition f):

$$A=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta, |F|)\,\mathrm{d}\theta>\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta, 0)\,\mathrm{d}\theta\,.$$

It results from ($7a$) that the above condition is equivalent to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\theta, F_{\text{extr}})\,\mathrm{d}\theta<A \text{ for } \mu=0 \qquad (F_{\text{extr}}=0 \text{ for } \mu=0)\,.$$

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[8] See, *e.g.*, E. C. TITCHMARSH: *The Theory of Functions*, 2nd ed. (London, 1939), p. 125.

Now

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta, F_{\text{extr}})\, d\theta$$

is an increasing function of μ. Making μ sufficiently large it is always possible to reach the value A for I.

We have proved then that there always exists *one and only one* solution of our equation for $F_{\text{extr}}(\theta)$. What we do know about A is $A \leqslant M$, M being a given constant. However, because $F_{\text{extr}}(\theta)$ increase with A [it results from eqs. (7*a*, *b*) and condition *f*)], the maximal value of F_0 is obtained when one puts the maximal value of A in (7*c*), *i.e.*, $A = M$. This means that we have only to solve (7*a*, *b*, *c*) for $A = M$ and the obtained F_0 will be a maximal one satisfying eq. (1).

2. – In cases of physical interest one is not given the value of M but usually an inequality of the type $A \leqslant M = F_0$. This leads to the same equations as (7*a*, *b*, *c*) with A replaced by F_0:

$$F_{\text{extr}}(\theta) \cdot \frac{\partial f(\theta, F_{\text{extr}})}{\partial F_{\text{extr}}} = \mu(F_0) = \text{const}\,, \tag{8a}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta, F_{\text{extr}}(\theta))\, d\theta = F_0\,, \tag{8b}$$

$$F_0 \exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln F_{\text{extr}}(\theta)\, d\theta\right] \leqslant \frac{1}{\beta}. \tag{8c}$$

The condition for F_0 is contained now implicitly in eqs. (8*a*, *b*, *c*) and it is difficult to see what values of F_0 will be forbidden.

We shall prove now that eqs. (8*a*, *b*, *c*) *give a maximum for* F_0. In the first we want to show that the l.h.s. of inequality (8*c*) increases when $F_{\text{extr}}(\theta)$ increases. To this end we have to show that the functional derivative of the l.h.s. with respect to $F_{\text{extr}}(\theta)$ is positive:

$$\exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln F_{\text{extr}}\, d\theta\right]\left[\frac{\delta F_0}{\delta F_{\text{extr}}(\theta)} + F_0\,\frac{\delta}{\delta F_{\text{extr}}(\theta)}\left(-\frac{1}{2\pi}\int_{-\pi}^{\pi} \ln F_{\text{extr}}(\theta)\, d\theta\right)\right] > 0\,.$$

This, after usign (8*b*) and dropping the «exp» term, is equivalent to

$$\frac{\partial f(\theta, F_{\text{extr}})}{\partial F_{\text{extr}}} - F_0 \cdot \frac{1}{F_{\text{extr}}} > 0\,.$$

With the help of (8a) it gives

$$\frac{1}{F_{\text{extr}}}\{\mu(F_0)-F_0\}>0\ .$$

Now, $\mu(F_0)-F_0=-\Psi[F_{\text{extr}}]$ [comp. (4a), (8b)] and is larger than zero because $(1/|F|)\,\partial f/\partial|F|$ increases with $|F|$ [condition f)]. This proves that the l.h.s. of (8c) is increasing with $F_{\text{extr}}(\theta)$. But $F_{\text{extr}}(\theta)$ itself increases (for any θ) when F_0 is made larger [as seen from Eqs. (8a, b)]. So we proved that really the l.h.s. of (8c) is growing with F_0, which simultaneously gives our statement. We also want to make this remark that the obtained maximum will be larger than zero if

$$\lim_{|F|\to 0}\frac{\partial f(\theta,|F|)}{\partial|F|}=0\ .$$

In the case of finite limit, unless

$$\beta\cdot\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\frac{\partial f(\theta,|F|)}{\partial|F|}\bigg|_{|F|\to 0}\mathrm{d}\theta\right]<1\ ,$$

We obtain $F_0=0$ or no F_0 satisfying (8a, b, c).

In cases of physical interest, where $f(\theta,|F|)\sim|F|^2$ for small $|F|$, the above-mentioned condition is of course fulfilled and always $F_0>0$.

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APPENDIX

As an illustration let us consider a functional of the form:

$$\Phi[F]=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{p_1(\theta)|F|+p_2(\theta)|F|^2\}\,\mathrm{d}\theta=A\ . \tag{A.1}$$

Assume that $F(z)$ has no cut apart from the integration region. What we want to obtain is a maximum for $F(z=0)=F_0$:

The first step is the replacement of $\Phi[F]$ by a linear functional in F by means of minimizing the functional

(A.2) $$\Psi[F]=\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\{p_1(\theta)|F|+p_2(\theta)|F|^2-\lambda(\theta)|F|\}\,d\theta\,.$$

The extremum condition

$$\frac{\delta\Psi[F]}{\delta[F]}\Big|_{|F|F_{\text{extr}}}=0$$

leads to

(A.3) $$F_{\text{extr}}(\theta)=\frac{\lambda(\theta)-p_1(\theta)}{2p_2(\theta)}\,.$$

Now, from $\Psi[F]\geqslant\Psi[F_{\text{extr}}]$ we have

$$A=\Phi[F]\geqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\lambda(\theta)|F|d\theta+\Psi[F_{\text{extr}}]\,,$$

which, after using (A.3), leads to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\lambda(\theta)|F|d\theta\leqslant A+\frac{1}{2\pi}\int_{-\pi}^{\pi}d\theta\,\frac{(\lambda-p_1)^2}{4p_2}\,d\theta\underset{\text{df}}{=}\mu\,.$$

Applying to the l.h.s. the theorem of the geometric and arithmetic means we come to

(A.4) $$F_0\exp\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\lambda(\theta)d\theta\right]\leqslant\mu\,.$$

Now we choose a $\lambda_{\text{extr}}(\theta)$ which makes the two sides of the inequality as close to each other as possible This means, one has to find the minimum of a functional

$$\chi[\lambda]=\frac{\mu}{F_0\exp\left[(1/2\pi)\int_{-\pi}^{\pi}\ln\lambda\,d\theta\right]}$$

or, which is equivalent, the minimum of

$$\ln\chi[\lambda]=\ln\mu-\ln F_0-\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln\lambda\,d\theta$$

with respect to $\lambda(\theta)$.

The condition

$$\frac{\delta \ln \chi[\lambda]}{\delta \lambda(\theta)}\bigg|_{\lambda=\lambda_{\text{extr}}} = \frac{1}{\mu}\,\frac{(\lambda-p_1)}{2p_2} - \frac{1}{\lambda}\bigg|_{\lambda=\lambda_{\text{extr}}} = 0 \tag{A.5}$$

together with the definition of μ [comp. (A.4)]:

$$\mu = A + \frac{1}{2\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta\,\frac{(\lambda_{\text{extr}} - p_1)^2}{4p_2}\,\mathrm{d}\theta \tag{A.6}$$

give us a transcendental equation for μ which, when solved, allows to find $\lambda_{\text{extr}}(\theta)$ from (A.5) and maximal F_0 from (A.4).

In order to obtain our eqs. ($7a$, b, c) for F_{extr}, it is enough to express $\lambda(\theta)$ by $F_{\text{extr}}(\theta)$ in eqs. (A.4), (A.5), (A.6). One has from (A.5) and (A.3)

$$\text{const} = \mu = \big(p_1(\theta) + 2p_2(\theta)F_{\text{extr}}\big)\cdot F_{\text{extr}}\,, \tag{A.7}$$

from (A.6) and (A.7)

$$A = \frac{1}{2\pi}\int_{-\pi}^{\pi}\big(|F|^2_{\text{extr}}p_2(\theta) + |F|_{\text{extr}}p_1(\theta)\big)\,\mathrm{d}\theta \tag{A.7$'$}$$

and from (A.4) using (A.3) and (A.7)

$$F_0 \exp\left[-\frac{1}{2\pi}\int_{-\pi}^{\pi}\ln F_{\text{extr}}\,\mathrm{d}\theta\right] \leqslant 1\,. \tag{A.7$''$}$$

Equations (A.7), (A.7′) and (A.7″) are just those which one could obtain directly from ($7a$, b, c) applied to functional (A.1) ($\beta = 1$).

RIASSUNTO (*)

Si presenta una tecnica per trovare i limiti massimi delle forze di interazione quando è data una ineguaglianza generale del tipo che deriva dalle relazioni di dispersione. Si dimostra che per una classe abbastanza ampia di funzionali comparenti in tali ineguaglianze la soluzione esiste ed è unica.

(*) *Traduzione a cura della Redazione.*