On the Building of Dual Diagrams from Unitarity.

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1. – Introduction.

The successful extension of the Veneziano amplitude (1) to multiparticle processes (2) gives—to our viewpoint—not only a simple model subject to phenomenological analysis, but a building stone for the construction of a hadron theory. What seems possible is to use unitarity—which the multiparticle Veneziano amplitude lacks—in the same way in which it can be used to construct quantum electrodynamics from the photon and electron poles (propagators) considered as Born terms.

In this way—and contrarily to more simple ways of unitarizing the Veneziano model (3)—crossing and duality are exactly preserved at every step keeping therefore into the theory the accomplishment of the Veneziano model.

We consider, therefore, the real amplitude of the multiparticle Veneziano model (2) as the tree diagram of the theory. The one-loop diagram will be built in such a way that if the loop is cut one finds back the corresponding tree diagram. The further steps would require to find the $(N-1)$-loop diagram in cutting one loop of the N-loop diagram. In this note, we perform only the first step (4) with a simple technique which we believe can be easily extended to the other steps.

The meaning of duality for diagrams with loops has been recently discussed by KIKKAWA, SAKITA and VIRASORO (5). In particular, all Feynman diagrams with one

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(1) G. VENEZIANO: *Nuovo Cimento*, **57** A, 190 (1968).

(2) K. BARDACKI and H. RUEGG: *Phys. Lett.*, **28** B, 342 (1968); M. VIRASORO: *Phys. Rev. Lett.*, **22**, 37 (1969); CHAN HONG MO: *Phys. Lett.*, **28** B, 425 (1969); CHAN HONG MO and TSOU S. TSUN: *Phys. Lett.*, **28** B, 485 (1969); C. GOEBEL and B. SAKITA: *Phys. Rev. Lett.*, **22**, 257 (1969).

(3) G. VENEZIANO: M.I.T. preprint; L. A. P. BALASZ: Imperial College preprint; C. LOVELACE *et al.*: private communication and to be published.

(4) The same program, with similar results, has been independently performed by VENEZIANO, KIKKAWA, SAKITA and VIRASORO (note added to ref. (5,7)) and BARDAKCI, HALPERN and SHAPIRO: Berkeley preprint.

(5) K. KIKKAWA, B. SAKITA and M. VIRASORO: Wisconsin preprint COO-224 (1969) and K. KIKKAWA, S. KLEIN, B. SAKITA and M. VIRASORO: to be published.

loop and a fixed number of external lines are expressed [6] by a single integral symmetrized for all noncyclic permutations of external momenta. That is the structure which will be obtained in the unitarity building we are attempting since duality is preserved at every step.

Before integrating over the loop variable, the unitarity requirement with which we will build the loop of Fig. 1 is that it has a pole if the square momentum k^2 of one vector of the loop is such that $\alpha(k^2) = n$ $(n = 0, 1, \ldots)$. The residue R_n of that pole must be given by

$$R_n = \sum_i \frac{D_{n_i, n_j}}{G\langle n_i|\{P\}\rangle\, G\langle n_j|\{Q\}\rangle}\,, \tag{1}$$

where D_{n_i,n_j} is the residue of the double pole fo the tree diagram of Fig. 2 corresponding to the i and j degenerate states for $\alpha(k^2) = n$ and $G\langle n_i|\{P\}\rangle$, $G\langle n_j|\{Q\}\rangle$ are the corresponding coupling to the external momenta P and Q respectively. The definition of the tree diagram spectrum and its degeneracy have been investigated by FUBINI

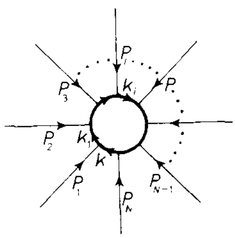

Fig. 1.

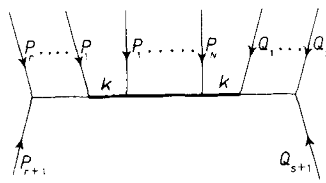

Fig. 2.

and VENEZIANO [7] so that it is possible to use their results [8] to compute R_n from eq. (1) and, therefore, the expression for the loop dual diagram. We prefer, however, to use a different basis to sum over intermediate states which simplifies considerably the computation. Indeed, as noted recently by FUBINI, GORDON and VENEZIANO [9] *one can reformulate the multiparticle Veneziano amplitude* by *using creation and* annihilation operators. The method described before, amounts to use the occupation number basis as the complete set of states; we will see that a coherent-state basis appears much more convenient.

2. – The tree diagram in operator formalism.

FUBINI, GORDON and VENEZIANO have shown [9] that factorization properties of the dual tree diagram of Fig. 3 can be conveniently discussed if one introduces an infinite

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[6] We discuss here the case of scalar neutral particles lying on trajectories with negative intercept.

[7] S. FUBINI and G. VENEZIANO: *Nuovo Cimento* (in press).

[8] We note that the use of the tree diagram spectrum (and not the physical one which will be given by the loop calculations themselves) is inherent of the usual field-theoretic perturbation approach and of the perturbative unitarity we are using. A modification of this approach could perhaps help in removing the divergence we will find in the loop diagram.

[9] S. FUBINI, D. GORDON and G. VENEZIANO: to be published.

set of harmonic oscillators characterized by two integers μ and n, where μ runs from 1 to 4 and n from 1 to infinity. The corresponding creation and annihilation operators $a_\mu^{(n)\dagger}$ and $a_\mu^{(n)}$ obey the following commutation relations:

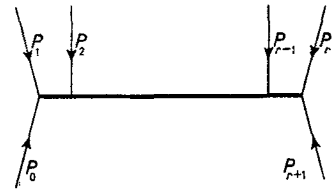

Fig. 3.

$$(2)\qquad [a_\mu^{(n)}, a_\nu^{(n')\dagger}] = \delta_{\mu,\nu}\,\delta_{n,n'}\,.$$

The amplitude corresponding to the dual diagram of Fig. 1 can be written in this formalism as

$$(3)\qquad F_0(p_0, p_1 \dots p_{r+1}) = \langle 0|V(p_r)P(\alpha_{r-1})\,V(p_{r-1})P(\alpha_{r-2})\dots P(\alpha_1)\,V(p_1)|0\rangle$$

with

$$(4)\qquad \alpha_{r-1} = \alpha_0 - \tfrac{1}{2}(p_{r-1} + p_{r-2} + \dots p_0)^2$$

and where $|0\rangle$ is the vacuum state, *i.e.*

$$a_\mu^{(n)}|0\rangle = 0\,.$$

Our metric implies $p^2 = \boldsymbol{p}^2 - p_0^2$ and the energy unit is the inverse of twice the trajectory's slope. The operators $P(\alpha)$ and $V(p)$ are given by

$$(5)\qquad P(\alpha) = \int_0^1 \mathrm{d}x\, x^{-\alpha+H-1}(1-x)^{-c}$$

with

$$(6)\qquad H = \sum_{n,\mu} n a_\mu^{\dagger(n)} a_\mu^{(n)} \qquad \text{and} \qquad c = 1 - \alpha_0\,,$$

$$(7)\qquad V(p) = \exp\left[\sum_n \frac{1}{\sqrt{n}}\, p_\mu \cdot a_\mu^{\dagger(n)}\right] \exp\left[-\sum_n \frac{1}{\sqrt{n}}\, p_\mu \cdot a_\mu^{(n)}\right],$$

where $p_\mu = (\boldsymbol{p}, ip_0)$.

The explicit appearance of $i = \sqrt{-1}$ in the coupling of $a_0^{(n)}$ reflects the presence of ghost states in the theory.

In this paper we shall make constant use of the coherent states. These are defined as direct products of coherent states for each mode which, at its turn, is defined by

$$(8)\qquad |z\rangle = \exp[za^\dagger]|0\rangle\,,$$

where $a^\dagger$ is the creation operator for the mode in question. We list some useful properties of the coherent states defined by (8):

$$(9)\qquad \begin{cases} \langle z'|z\rangle = \exp[z'^* \cdot z]\,, \\ a|z\rangle = z|a\rangle\,, \\ \exp[z' a^\dagger]|z\rangle = |z + z'\rangle\,, \\ x^{a^\dagger a}|z\rangle = |xz\rangle\,. \end{cases}$$

As a simple illustration, we sketch how we can see that the expression (3) is indeed identical with that of RUEGG and BARDAKCI in the particular case $r = 3$.

The five-point amplitude, using (3), can be written as

$$F(p_0, p_1, \ldots, p_4) = \int \mathrm{d}x_1 \mathrm{d}x_2 x_1^{-\alpha_1 - 1} x_2^{-\alpha_2 - 1} [(1-x_1)(1-x_2)]^{-c} \prod_n M_n(p_1, p_2, p_3) , \tag{10}$$

where the amplitude for the mode n is given by

$$M_n = \langle 0| \exp\left[-\frac{p_3^\mu a_\mu^{(n)}}{\sqrt{n}} x_2^{h^{(n)}}\right] \exp\left[\frac{p_2^\mu a_\mu^{\dagger(n)}}{\sqrt{n}}\right] \exp\left[-\frac{p_2^\mu a_\mu^{(n)}}{\sqrt{n}} x_1^{h^{(n)}}\right] \exp\left[\frac{p_1^\mu a_\mu^{\dagger(n)}}{\sqrt{n}}\right] |0\rangle \tag{11}$$

with

$$h^{(n)} = n a_\mu^{(n)\dagger} a_\mu^{(n)} . \tag{12}$$

Using the relations (9) one gets immediately

$$M_n = \exp\left[-\frac{1}{n}(x_1^n p_2 \cdot p_1 + x_2^n p_3 \cdot p_2 + x_1^n x_2^n p_3 \cdot p_1)\right] .$$

Performing the product over n one recovers the expression given by BARDAKCI and RUEGG [2].

To exhibit the factorization properties of the dual tree diagrams it is convenient to rewrite formula (3) in a slightly different manner. We introduce a new oscillator mode $a^{(0)}, a^{(0)\dagger}$ in order to transform the product $[(1-x_1)(1-x_2)(1-x_{r-1})]^{-c}$ into a vacuum expectation value. Let us define a vertex operator $V^{(0)}$ as

$$V^{(0)} = f_c(a^{(0)\dagger}) Q^{(0)} f_c(a^{(0)}) , \tag{13}$$

where $Q^{(0)}$ is the projection operator onto the vacuum state and $f_c(u)$ is given by the series expansion

$$f_c(u) = 1 + \sum_{l>0} \frac{\sqrt{c(c+1)(c+l-1)}}{l!} u^l . \tag{14}$$

It is then a trivial matter to verify the identity

$$[(1-x_1) \ldots (1-x_{r-1})]^{-c} = \langle 0| V^{(0)} x_1^{h^{(0)}} V^{(0)} x_2^{h^{(0)}} \ldots x_{r-1}^{h^{(0)}} V^{(0)} |0\rangle , \tag{15}$$

where $h^{(0)} = a^{(0)\dagger} a^{(0)}$.

Introducing the total vertex operator $V'(p) = V(p) V^{(0)}$ and the total Hamiltonian $H' = a^{(0)\dagger} a^{(0)} + \sum_{n,\mu} n a_\mu^{(n)\dagger} a_\mu^{(n)}$ in eq. (3) and performing the integration over x_i one obtains a new expression for the tree diagram amplitude:

$$F(p_0, \ldots, p_r) = \langle 0| V'(p_r) \frac{1}{H' - \alpha_{r-1}} V'(p_{r-1}) \ldots V'(p_2) \frac{1}{H' - \alpha_1} V'(p_1) |0\rangle . \tag{16}$$

The factorization properties of F can be read out from this formula. The (mass)^2 spectrum and degeneracies of the intermediate states correspond to the level structure of the total operator H' in agreement with the findings of FUBINI and VENEZIANO.

3. – One-loop dual diagram.

We shall use the results of the previous Sections to compute the one-loop dual diagram of Fig. 1. We denote internal momenta by $k_0, k_1, \ldots, k_{N-1}$. The energy-momentum conservation at each vertex gives

$$(17)\qquad k_0 = k\,, \qquad k_1 = k + p_1\,, \qquad k_2 = k + p_1 + p_2, \ldots, k_i = k + p_1 + \ldots + p_i\,.$$

Let us first express the dual amplitude as an integral over the internal momenta k:

$$(18)\qquad F_1(p_1, p_2, \ldots, p_N) = \int \mathrm{d}^4k\, M(k, k_1, \ldots, k_{N-1})\,.$$

We require that M should have a pole each for $\alpha(k^2) = j$ $(j = 0, 1, \ldots)$. Following the prescription of Sect. 1 we write down M in the following way:

$$(19)\qquad M = \sum_n \frac{1}{\alpha(k^2) - j} R_j(k, p_1 \ldots p_N)\,,$$

where $R_j(k, p_1 \ldots p_N)$ is obtained from the residue of the double pole at $\alpha(k^2) = j$ of the tree diagram of Fig. 2. A compact form is readily obtained in the operator formalism of Sect. 2:

$$(20)\qquad M = \mathrm{Tr}\left[\frac{1}{\alpha(k^2) - H'} V'(p_1) \frac{1}{\alpha(k_1^2) - H'} \ldots V'(p_{N-1}) \frac{1}{\alpha(k_{N-1}^2) - H'} V'(p_N)\right].$$

The trace has to be performed over a complete set of states of the Hilbert space obtained by taking the direct product of the energy eigenstates of the oscillators $a^{(0)}$, $a_\mu^{(n)}$. To compute this trace, it is more convenient to use an expression where the integration over the variables x_i has not been performed explicitly. M can then be written as an integral over a product of traces of operators acting on only one oscillator mode states:

$$(21)\qquad M = \int \mathrm{d}x_1 \ldots \mathrm{d}x_N \left\{\mathrm{Tr}\,(x_1^{h^{(0)}} V^{(0)} \ldots x_N^{h^{(0)}} V^{(0)}) \cdot \right.$$
$$\left. \cdot \prod_n \mathrm{Tr}\,(x_1^{h^{(n)}} V^{(n)}(p_1) \ldots x_N^{h^{(n)}} V^{(n)}(p_n))\right\} \prod_{i=1}^{i=N} x_i^{\alpha(k_{i-1}^2)}\,,$$

where

$$V^{(n)}(p) = \exp\left[\frac{1}{\sqrt{n}} p_\mu a_\mu^{(n)\dagger}\right] \exp\left[-\frac{1}{\sqrt{n}} p_\mu a_\mu^{(n)}\right].$$

The trace over the model (0) is readily performed using the occupation number basis $|l\rangle$

by noting that

$$V^{(0)}|l\rangle = f_c(a^{(0)\dagger})|0\rangle\langle 0|f_c(a^{(0)})|l\rangle = V^{(0)}|0\rangle\sqrt{\frac{c(c+1)(c+l-1)}{l!}} \tag{22}$$

and obviously gives

$$[(1-x_1)\dots(1-x_N)]^{-c}\,. \tag{23}$$

As far as the trace over the modes (n) is concerned the coherent-states basis $|z\rangle$ appears to be more convenient. They form a complete basis (in fact an over-complete one). The decomposition of the unit operator for one mode reads

$$1 = \frac{1}{\pi}\int \mathrm{d}(\mathrm{Re}\, z)\,\mathrm{d}(\mathrm{Im}\, z)\exp\left[-|z|^2\right]|z\rangle\langle z|\,. \tag{24}$$

The trace to be performed over each mode (n) is of the form

$$T = \frac{1}{\pi}\int \exp\left[-|z|^2\right]\langle z|O_N O_k \dots O_1|z\rangle\,\mathrm{d}(\mathrm{Re}\, z)\,\mathrm{d}(\mathrm{Im}\, z)\,. \tag{25}$$

The operators O_k are defined as

$$O_k = (\exp[q_k a^\dagger]\exp[-q_k a])\,u_k^{a^\dagger a}\,, \tag{26}$$

where

$$q_k = \frac{p_{k\mu}}{\sqrt{n}}\,,\qquad u_k = x_k^n \qquad \text{and} \qquad a = a_\mu^{(n)}\,. \tag{27}$$

From eq. (9) it follows that

$$O_k|y\rangle = \exp[-q_k y u_k]|u_k y + q_k\rangle\,. \tag{28}$$

This leads to

$$O_l O_{l-1}\dots O_1|z\rangle = \exp[-\beta_l]|z_l\rangle\,, \tag{29}$$

where β_l and z_l are determined by

$$\begin{cases} z_l = u_l z_{l-1} + q_l & \text{with } z_0 = z\,,\\ \beta_l = q_l u_l z_{l-1} + \beta_{l-1} & \text{with } \beta_0 = 0\,. \end{cases} \tag{30}$$

Those recurrence equations are easily solved and give

$$\begin{cases} z_l = z\left(\prod\limits_{i=1}^{l} u_i\right) + \sum\limits_{i=1}^{l-1} q_i(u_{i+1}\dots u_l) + q_l\,,\\ \beta_l = z\sum\limits_{k=1}^{l} q_k\left(\prod\limits_{i=1}^{k} u_i\right) + \sum\limits_{j>k}^{l} q_k q_j(u_{k+1}u_{k+2}\dots u_j)\,. \end{cases} \tag{31}$$

Using (29) and (25) we obtain the following expression for T:

$$T = \frac{1}{\pi}\int \mathrm{d}(\mathrm{Re}\, z)\, \mathrm{d}(\mathrm{Im}\, z) \exp\left[-|z|^2\right]\langle z|z_N\rangle \exp\left[-\beta_N z\right].$$

The scalar product $\langle z|z_N\rangle = \exp[z^* z_N]$ is easily computed with the help of (31) leading to

$$T = \frac{1}{\pi}\int \mathrm{d}(\mathrm{Re}\, z)\, \mathrm{d}(\mathrm{Im}\, z) \exp\left[-(1-w)|z|^2 + Bz + Cz^* + D\right], \tag{32}$$

where

$$\begin{cases} B = -\sum\limits_{l=1}^{N} q_l \left(\prod\limits_{i=1}^{l} u_i\right), \\ C = \sum\limits_{l=1}^{N-1} q_l \left(\prod\limits_{i=l+1}^{N} u_i\right) + q_N, \\ D = -\sum\limits_{l=1}^{N}\sum\limits_{i=1}^{l-1} q_i q_l (u_{i+1} \dots u_l) \end{cases} \tag{33}$$

and $w = \prod\limits_{i=1}^{i=N} u_i$.

The Gaussian integral appearing in (32) is easily performed, leading to the final result for T:

$$T = \frac{1}{1-w} \exp\left[\frac{-1}{1-w}\sum_{i,j=1}^{N} q_i q_j C_{ij}\right], \tag{34}$$

where

$$C_{ij} = \begin{cases} u_1 \dots u_i & \text{if } j = N, \\ (u_1 \dots u_i)(u_{j+1} \dots u_N) & \text{if } i \leqslant j \leqslant N-1, \\ (u_{j+1} \dots u_i) & \text{if } i > j. \end{cases} \tag{35}$$

Let us now perform the product $\mathscr{T}$ of the traces of type T relative to all modes $(n \neq 0; \mu)$. We obtain

$$\mathscr{T} = \prod_{n=1}^{\infty}\left\{\left[\frac{1}{1-w^n}\right]^4 \exp\left[\frac{-\sum\limits_{i,j=1}^{N} p_i \cdot p_j (C_{ij})^n}{n(1-w^n)}\right]\right\} = \prod_{n=1}^{\infty} \exp\left[\frac{4w^n - \sum\limits_{i,j=1}^{N} (C_{ij})^n p_i \cdot p_j}{n(1-w^n)}\right]. \tag{36}$$

In eq. (36) $p_i \cdot p_j$ is now the usual scalar product of Lorentz vectors and C_{ij} is obtained from formula (35) by replacing u_i by x_i and, again, $w = \prod\limits_{i=1}^{n} x_i$.

Collecting the results we arrive to the following expression for the one-loop dual amplitude:

$$F_1^{(N)}(p_1, \dots, p_N) = \int \mathrm{d}^4k\, \mathrm{d}x_1 \dots \mathrm{d}x_N x_1^{-\alpha(k^2)-1} x_N^{-\alpha(k_{N-1}^2)-1} \cdot \prod_{i=1}^{i=N}(1-x_i)^{-c} \prod_{n=1}^{\infty}\left\{\frac{1}{(1-w^n)^4}\exp\left[-\frac{\sum\limits_{i,j=1}^{N} C_{ij}^n p_i \cdot p_j}{n(1-w^n)}\right]\right\}. \tag{37}$$

As in ref. [5] the integration over the loop variable k presents no difficulty if it is defined with the Euclidean metric (*i.e.* after Wick rotation). The x_i integrations present however an end-point divergence for all x_i equal to one (*i.e.* $w = 1$). Indeed, in the vicinity of such a point $C_{ij} \to 1$ and therefore

$$\sum_{i,j=1}^{N} C_{ij}^{n} p_i \cdot p_j \to \left(\sum_{i}^{N} p_i\right)^2 = 0$$

due to momentum conservation. Therefore, from eq. (36) we see that for $w = 1 - \varepsilon$

$$\mathscr{T} \propto \exp \frac{4}{\varepsilon} \frac{\pi^2}{6} ,$$

which implies an exponential divergence for $\varepsilon \to 0$.

Equation (34) shows that every mode n contributes to this exponential divergence with a power divergence of order 4 (or order $4 + \epsilon$ if we do not quantize the scalar mode), the 4 representing the dimensions of space-time (possible values of μ). The origin of such a divergence can be traced back to the fact that the trace of the identity operator is undefined even for the one-mode case. This reflects the fact that even with one mode the density of states remains constant with the squared mass.

The vacuum fluctuations present already such a divergence. Indeed, the one-loop contribution to it—given by a closed loop without external lines—is given by

$$\frac{1}{\pi} \int \mathrm{d}^4 k \int \mathrm{d}x\, x^{-\alpha(k^2)-1} (1-x)^{-\epsilon} \prod_{n=1}^{\infty} \prod_{\mu=1}^{4} T_{(\mu)}^{(n)} , \tag{38}$$

where (there is no summation over μ)

$$\begin{aligned} T_{(\mu)}^{(n)} &= \int \mathrm{d}\,\mathrm{Re}\, z\, \mathrm{d}\,\mathrm{Im}\, z \exp\left[-|z|^2\right] \langle z | x^{n a_\mu^{(n)\dagger} a_\mu^{(n)}} | z \rangle = \\ &= \frac{1}{\pi} \int \mathrm{d}\,\mathrm{Re}\, z\, \mathrm{d}\,\mathrm{Im}\, z \exp\left[|-|z|^2 (1 - x^n)\right] = \frac{1}{1 - x^n} . \end{aligned} \tag{39}$$

Unfortunately the divergence of eq. (36) is not factorizable so that a naive renormalization of the vacuum does not remove the infinity.

This divergence represents a substantial difficulty that must be somehow circumvented. A usual renormalization program seems rather unapplicable due to the nondual structure of the contribution to be lumped into renormalization counter terms. *Ad hoc* modification to the loop rules are indeed possible [10] but they break the nice Feynman-like structure of closed-loop computation and some asymptotic properties of amplitudes. To change the spectrum of intermediate states (whose abundance gives the divergence) it is possible to modify the tree diagram input [8]. These modifications [11] seem however to increase the divergency instead of reducing it.

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We have derived much benefit from discussions of one of us (D.A.) with S. FUBINI and G. VENEZIANO.

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[10] See the second ref. of [4].
[11] CHAN HONG MO: private communication to the first author named (D.A.).