Dual Self-Energy Diagram in the Operatorial Formalism.

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An operatorial formalism has been obtained which allows simple dual diagrams [1] to be written as a factorized product of propagators and vertex functions [2]. A three-resonance vertex V has been written down by Sciuto [3]; this should permit the calculation of general planar Feynman-like diagrams.

We want to show that one cannot forget about the origin of the vertex and that it is not clear whether in this operatorial formalism, the knowledge of the vertex and of the propagator is enough to calculate general planar diagrams.

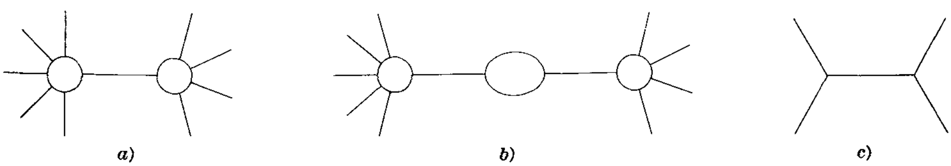

Fig. 1.

We shall illustrate it with the following example: the diagram 1*a*) can be written $\langle V_f|D|V_i\rangle$, where the operator D represents the propagator of a resonance. We shall show that the diagram 1*b*) cannot be written $\langle V_f|D\Sigma D|V_i\rangle$, where Σ is some operator. We shall only be able to cast the one-loop dual diagram under the form $\langle V_f|R|V_i\rangle$. We shall conclude that, in this operatorial formalism, one also needs the four-resonance function (Fig. 1*c*)) before any general diagram can be calculated; it is not the operator VDV if one insists on the propagator of the external resonances to be D.

We shall calculate the self-energy diagram (Fig. 1*b*)) using the operators V and D and compare it with the well-known one-loop dual amplitude.

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(1) S. Fubini and G. Veneziano: *Nuovo Cimento*, **64** A, 881 (1969).

(2) S. Fubini, D. Gordon and G. Veneziano: *Phys. Lett.*, **29** B, 679 (1969).

(3) S. Sciuto: *Lett. Nuovo Cimento*, **2**, 411 (1969).

1. – Calculation of the self-energy diagram.

The operatorial formalism relates resonances to the eigensolutions of harmonic oscillators. We use the notations and results of ref. [2,3]. The three-resonance vertex as written by SCIUTO uses pairs of operators $(c^\dagger, c)$ and $(a^\dagger, a)$ which belong to different spaces. We choose $(c^\dagger, c)$ to refer to particle 3, and $(a^\dagger, a)$ to refer to particles 1 and 2. The vertex reads (Fig. 2)

$$\exp[P_3\cdot a^\dagger - (a^\dagger, c^\dagger)_-]\exp[P_1\cdot c^\dagger]\exp[-P_3\cdot a + (a, c^\dagger)_-]. \tag{1}$$

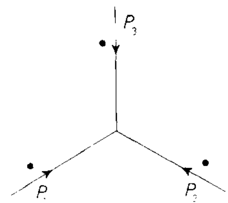

Fig. 2.

If one examines the way the vertex has been obtained one sees that the propagators of resonances 1 and 2 have the classical form D, but the propagator of particle 3 involves some dependence on particle 1. The problem is: can one forget about this dependence? If not, the above vertex should be multiplied on the right-hand side by $(1-z)^{R_a+P_1^2/2}$, where z is the integration variable to be used for the propagator of particle 3. The twisting of line 1 leads to a more symmetrical vertex [4] but does not modify the problem.

One can construct the bubble in different ways depending on the position of the dots:

i) The propagator of particle 3 is twisted (Fig. 3*i*)):

$$G(c^\dagger, c) = g^2\int dx_1\, dx_2 (x_1 x_2)^{-1-\alpha}(1-x_1)^{-c} g^2 M(c^\dagger, c; x_1, x_2; z, z')(1-x_2)^{-c}$$

and

$$M(c^\dagger, c; x_1 x_2; zz') = \int d^4K\, \mathrm{Tr}\, \{\exp[-P_3\cdot a^\dagger - (a^\dagger, c^\dagger)_-]\cdot \tag{2}$$
$$\cdot\exp[P_1\cdot c^\dagger]\exp[P_3\cdot a + (a, c^\dagger)_-]\, v_1^{R_a+P_1^2/2}|O_c\rangle\langle O_c|\exp[P_3\cdot a^\dagger - (a^\dagger, c)_-]\cdot$$
$$\cdot\exp[-P_2\cdot c]\exp[-P_3\cdot a + (a, c^\dagger)_+]\, v_2^{R_a+P_2^2/2}\}\,.$$

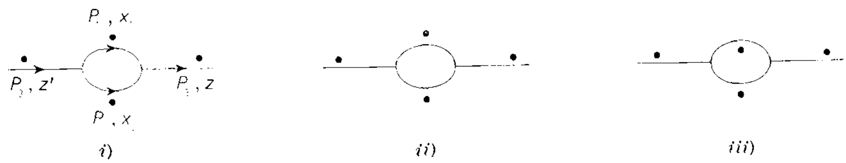

Fig. 3.

The trace has to be performed over a complete set of states of the space where a and $a^\dagger$ operate; $v_1 = x_1$ and $v_2 = x_2$ if one forgets about the origin of the vertex; otherwise one has $v_1 = x_1(1-z)$ and $v_2 = x_2(1-z')$.

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[4] L. CANESCHI, A. SCHWIMMER and G. VENEZIANO: preprint.

ii) The propagator of particle 3 is not twisted (Fig. 3ii)). The last part of formula (2) reads now

$$\langle O_c|(1-z')^{R_a+P_1^2/2}\exp[P_3\cdot a^\dagger-(a^\dagger,c)_+]\exp[-P_1\cdot c]\exp[-P_3\cdot a+(a,c^\dagger)_-]x_2^{R_a+P_2^2/2}\} .$$

iii) The propagator of particle 1 is twisted twice (Fig. 3 iii)). One uses the symmetrical form of the vertex established by CANESCHI, SCHWIMMER and VENEZIANO [4], or the « left and right twisted » propagator of particle 1.

The trace is easily done if one calculates the matrix elements of $M(c^\dagger, c)$ between « coherent states » and uses their algebraic properties [5,6]. The result is, for case i),

$$M=\int d^4 K v_1^{P_1^2/2}v_2^{P_2^2/2}D\exp[Ac^{\dagger 2}+EP_3\cdot c^\dagger+P_1\cdot c^\dagger]B^{c^\dagger c}\exp[Ac^2+FP_3\cdot c-P_2\cdot c]\,, \tag{3}$$

where

$$Ac^{\dagger 2}=\sum_{mn}A_{mn}c^\dagger_{(m)}\cdot c^\dagger_{(n)}\,,\qquad EP_3\cdot c^\dagger=\sum_m E_m P_3\cdot c^\dagger_{(m)}\,,$$

$$B^{c^\dagger c}=\prod_n\exp\Big[\sum_{m\geqslant n}B_{mn}c^\dagger_{(m)}\cdot c_{(n)}\Big]\,,\qquad E_m(v_1,v_2)=-F_m(v_2,v_1)$$

with

$$A_{mn}=-(-1)^{m+n}\sum_i c_{im}c_{-in}\frac{v_1^iv_2^i}{1-v_1^iv_2^i}\,,$$

$$E_m=(-1)^m\sum_i\frac{1}{1-v_1^iv_2^i}[c_{im}v_1^i(1-v_2^i)+c_{-im}v_2^i(1-v_1^i)]\,,$$

$$B_{mn}=-(-1)^m\sum_i\frac{1}{1-v_1^iv_2^i}[v_1^ic_{im}c_{-in}+v_2^ic_{-im}c_{in}]\,,$$

$$D=\exp\Big[P_3^2\sum_i\frac{1}{i}\,\frac{v_1^i+v_2^i-2v_1^iv_2^i}{1-v_1^iv_2^i}+\sum_i\frac{4v_1^iv_2^i}{1-v_1^iv_2^i}\Big]\,,$$

$$c_{im}=\sqrt{\frac{m}{i}}\binom{i}{m}\,,\qquad c_{-im}=\sqrt{\frac{m}{i}}\binom{-i}{m}\,.$$

Case ii) is very similar. The main change is that $v_1=x_1(1-z)(1-z')$ and $v_2=x_2$ if one does not forget about the origin of the vertex. Also $P_2\cdot c$ is now $P_1\cdot c$, A_{mn} and E_m stay the same and

$$F'_m(v_1,v_2)=-E_m(v_1,v_2)\,,\qquad B_{mn}=-(-1)^{m+n}\sum_i\frac{1}{1-v_1^iv_2^i}[v_1^ic_{im}^2+v_2^ic_{-im}^2]\,.$$

Case iii) leads to a very complicated expression for the A, B .... One has to replace the propagator of particle 1 inside the loop $\int dx_1x_1^{R_a-1-a}(1-x_1)^{-c}$, by a propagator which is twisted left and right: $\int dx_1x_1^{-1-a}(1-x_1)^{-c}\,\Omega^\dagger(P_1)x_1^{R_a}\Omega(P_1)$, where $\Omega(P_1)$ is the

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(5) D. AMATI, C. BOUCHIAT and J. L. GERVAIS: *Lett. Nuovo Cimento*, **2**, 399 (1969).

(6) J. R. KLAUDER and E. C. G. SUDARSHAN: *Fundamentals of Quantum Optics* (New York, 1968).

twisting operator ([4]). This new propagator can be written

$$\int \mathrm{d}x_1 x_1^{-1-\alpha}(1-x_1)^{-c-P_1^2} \exp[\widetilde{P}_1\cdot a^\dagger]\left(:\exp\left[\sum_{i,j} f_{ij} a^\dagger_{(i)} a_{(j)}\right]:\right)\exp[\widetilde{P}_1\cdot a]$$

with

$$\widetilde{P}_1\cdot a = P_1\cdot\sum_{ln} a_{(n)} c_{ln}\frac{x_1^l}{\sqrt{l}}\,,$$

$$f_{ij} = (-1)^{i+j}\sum_l c_{lj}c_{li}x_1^l - \delta_{ij}\,.$$

The dependence on the four-vector P_1 is rather complicated and this complexity is reflected on the dependence of $M(c^\dagger, c)$ on the loop momentum.

2. – Comparison with the one-loop dual amplitude.

The single-loop dual amplitude (Fig. 4*a*)) is well known in its symmetrical form ([5-7]). An appropriate change of variables makes explicit the presence of the diagram of Fig. 1*b*) in the dual amplitude. The easiest comparison with formula (3) is done with a six-point amplitude but the method is easily generalized to a n-point amplitude.

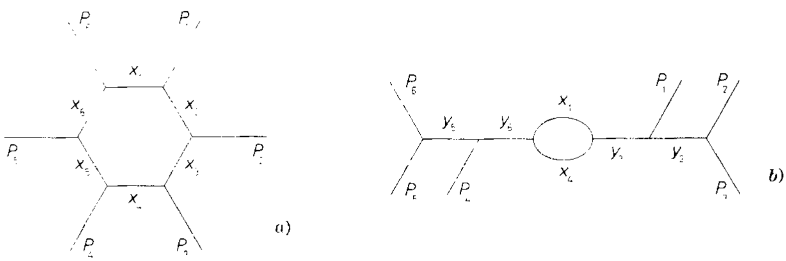

Fig. 4.

The change of variables $(x_2, x_3)\to(y_2, y_3)$ gives (Fig. 4*b*))

$$x_2x_3 = 1 - y_3C - O(y_3^2)\,,\qquad x_3 = 1 - y_3y_2C + O(y_3^2)$$

with $C = (1-x_1)(1-x_4)/(1-x_1x_4)$, and similar expressions for x_5, x_6. The one-loop dual amplitude is

$$\int \mathrm{d}^4K\prod_{i=1}^{6}[\mathrm{d}x_i x_i^{-\alpha(-k_i^2)-1}(1-x_i)^{-1-\alpha}]\prod_{n=1}^{\infty}\exp\left[\frac{4w^n - \sum\limits_{i,j=1}^{6} C^n_{ij}P_i\cdot P_j}{n(1-w^n)}\right].$$

(7) K. BARDAKÇI, M. HALPERN and J. SHAPIRO: Berkeley preprint No. 69-1417.

The method is the following: in the exponential, separate the $P_i \cdot P_j$ terms into terms involving the contraction of the right-hand side momenta with the left-hand side momenta, and terms involving the contraction of one-side momenta.

The last group gives

$$(1-x_2)^{P_1\cdot P_2}(1-x_2x_3)^{P_1\cdot P_3}(1-x_3)^{P_2\cdot P_3}\exp\left[\sum_n\left[-\frac{w^n(P_1+P_2+P_3)^2}{n(1-w^n)}\right]\right]$$

up to terms $O(y_3^2)$ and a similar term for (P_4, P_5, P_6).

The first group can be written

$$x_1^n[P_1+x_2^nP_2+x_2^nx_3^nP_3][P_6+x_6^nP_5+x_5^nx_6^nP_4^n]+\\ +x_4^n[P_3+x_3^nP_2+x_2^nx_3^nP_1][P_4+x_5^nP_5+x_5^nx_6^nP_4]$$

and gives, up to terms $O(y_3^2)$, $O(y_6^2)$,

$$\exp\left[-\frac{1}{n(1-w^n)}\right]\{-\Pi^2\left(x_1^n(1-ny_3C)+x_4^n(1-ny_6C)\right)+\\ +Q^1\cdot\Pi ny_3C[x_1^n-x_4^n(1-ny_6C)]+\mathscr{P}^1\cdot\Pi ny_6C[x_1^n(1-ny_3C)-x_4^n]-\\ -n^2y_3y_6C^2(x_1^n+x_4^n)\,\mathscr{P}^1\cdot Q^1\}\,, \tag{4}$$

where

$$\Pi=P_4+P_5+P_6,\quad \mathscr{P}^1=P_4+y_5P_5,\quad Q^1=P_1+y_2P_2 \quad\text{and}\quad w=x_1x_4(1-y_3C)(1-y_6C)\,.$$

Now one recognizes the five-point function (Fig. 5*a*))

$$(1-x_2)^{P_1\cdot P_2-1-a}\,(1-x_3)^{P_2\cdot P_3-1-a}\,(1-x_2x_3)^{P_1\cdot P_3}x_2^{-\alpha(-k_2^2)-1}x_3^{-\alpha(-k_3^2)-1}\,\mathrm{d}x_2\,\mathrm{d}x_3\,.$$

The change of variables $x_2x_3=1-y_3'$, $x_3=1-y_2y_3'$ $(y_3'=y_3C)$ makes explicit the presence of the diagram of Fig. 5*b*), and the y_3' dependence is separated out:

$$\mathrm{d}y_3'\,y_3'^{-1-a-\Pi^2/2}(1-y_3')^{-1+k_1^2/2}\exp\left[-\sum_n y_3'^n k_1\cdot\frac{Q^{(n)}}{n}\right]\mathrm{d}y_2\,\varphi(y_2,\,P_1P_2P_3)\,.$$

The expression we have obtained in this Section is only valid up to terms of order y_3^2 and y_6^2. The comparison with expression (3) can only be made for the first two lowest resonances. So we study the case when only the mode $n=1$ is excited, *i.e.* we sandwich $M(c^\dagger, c)$ between the coherent states $\langle -zQ^1|$ and $|z'\mathscr{P}^1\rangle$. The comparison is straightforward and it is clear that in formula (3) $v_1=x_1(1-z)$ and $v_2=x_2(1-z')$ rather than $v_1=x_1$, $v_2=x_2$. So the (z, z') dependence of $M(c^\dagger, c)$ is established.

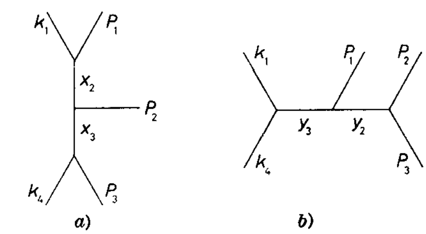

Fig. 5.

Starting from the one-loop dual amplitude, one obtains case ii) with a different change of variables $(x_5, x_6)\to(z_5, z_6)$. One easily verifies that, to the lowest order in

y_3 and z_6, the replacement $\mathscr{P}^1 = -\mathscr{P}'^1 + \Pi$ with $\mathscr{P}'^1 = P_6 + z_5 P_5$ in formula (4) gives the corresponding formula for case ii). In the left-hand five-point function k_4 is replaced by k_1. This new expression agrees with formula (3) for case ii) if $v_1 = x_1(1-z)(1-z')$ and $v_2 = x_2$. Let us do one remark about the way this comparison works: it would be interesting to know whether the operators $M_{\text{ii)}}(c^\dagger, c)$ and $M_{\text{i)}}(c^\dagger, c)\,\Omega(\Pi)$ have the same matrix elements between the external states. (Ω is the twisting operator.)

One does not know to which graph corresponds case iii).

3. – Qualitative features of the first-order correction.

3'1. *Need for a four-resonance function.* – The result of Sect. **1** and **2** shows that the vertex of the operatorial formalism is indeed unsymmetrical in its present form. Two resonances possess the standard propagator, the third one has a more complicated propagator. In this operatorial formalism it seems impossible to separate the vertex out of the third propagator.

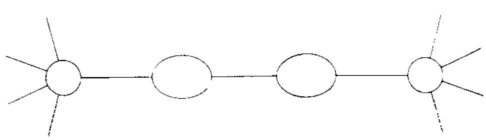

Fig. 6.

In order to calculate n-loop planar diagrams ($n \geqslant 2$), for example the diagram of Fig. 6, one needs a formula for the four-resonance function: it is *not* VDV if one insist on the propagator of the external resonances to be the usual one.

If one studies the infinite sum of all loop diagrams, an interesting question is whether this sum could be written formally in a compact way, or whether it would satisfy an integral equation with a kernel similar to $M(c^\dagger, c)$.

3'2. *Features of the first-order correction.* – We study the implications of formula (3). The resonances are eigenvectors of the operators $c^\dagger_{(n)}\cdot c_{(n)}$. The operator $M(c^\dagger, c)$ has matrix elements between any two resonances, but the contribution from the off-diagonal matrix elements to the mass renormalization will appear only to the next order in g^2. Formula (3) allows us to calculate the renormalization of any trajectory to the lowest order in g^2.

Let us examine the diagonal matrix elements of $M(c^\dagger, c)$ ignoring the unpleasant singularity for $x_1 = x_2 = 1$:

A term $c^\dagger_{(m)}\cdot c^\dagger_{(n)}$ means the contraction of two indices of the final vertex $\langle V_f|$; it does not operate on traceless tensors.

The Ward identities, if they can be generalized, will relate a term $P_3\cdot c^\dagger_{(m)}$ to the mode $m-1$.

When one eigenvalue has been chosen for each $c^\dagger_{(n)}\cdot c_{(n)}$, one has chosen the spin (loosely speaking) of the exchanged particle between the initial and final vertices. The diagonal matrix element may now be expanded in powers of z and z'. To choose a (common) power for z and z' is to specify the mass of the exchanged resonance.

In this model, the high-energy elastic scattering of two stable zero-spin particles will look different from the inelastic-scattering processes. To lowest order in g^2 instead of exhibiting the splitting of the second trajectory, it will show up a mean second trajectory; the reason is that the various excitation modes m cannot be separated out when an external vertex has only two legs.

4. – The imaginary part of the first-order correction to the leading trajectory.

Let us concentrate on the correction for the resonances which belong to the leading trajectory. The properties of the leading trajectory are the following:

i) only the mode $m = 1$ is excited;

ii) the vertices $|V_i\rangle$ and $\langle V_f|$ involve completely symmetric traceless tensors;

iii) the dependence on $(1-z)$ and $(1-z')$ is dropped.

The consequences are: on both sides of $M(c^\dagger, c)$, there appear the propagator of the resonances of the leading trajectories:

$$\int \mathrm{d}z\, z^{c^\dagger_{(1)}\cdot c_{(1)}-1-a-P_3^2/2}\,.$$

The product of operators in $M(c^\dagger, c)$ reduces to

$$\left(-\frac{1}{\log x_1 x_2}+B_{11}\right)^{c^\dagger_{(1)}\cdot c_{(1)}}$$

after the K integration has been performed. Each eigenvalue λ of the operator $c^\dagger_{(1)}\cdot c_{(1)}$ is a resonance whose spin is λ and mass $-P_3^2 = 2(\lambda - a)$. The first-order correction to the leading trajectory is then

$$g^2F(s) = g^2\int \mathrm{d}x_1\,\mathrm{d}x_2(x_1x_2)^{-1-a}(1-x_1)^{-c}(1-x_2)^{-c}[\log x_1x_2]^{-2}\cdot$$

$$\cdot\exp\left[-2bs\sum_n\frac{1}{n}\frac{x_1^n+x_2^n-2x_1^nx_2^n}{1-x_1^nx_2^n}+4\sum_n\frac{1}{n}\frac{x_1^nx_2^n}{1-x_1^nx_2^n}-bs\frac{\log x_1\log x_2}{\log x_1x_2}\right]\cdot$$

$$\cdot\left[-\frac{1}{\log x_1x_2}+\sum_n n\frac{x_1^n+x_2^n}{1-x_1^nx_2^n}\right]^{bs+a}.$$

The last sum over n can be rearranged, and one recognizes in its first terms the expression obtained by KIKKAWA, SAKITA, VIRASORO ([8]). This correction is infinite, due to the singularity at $x_1 = x_2 = 1$. The hope is that the infinity is purely real: a dispersion relation could be written down if the asymptotic behaviour of the imaginary part of $F(s)$ would allow it.

The imaginary part of $F(s)$ can be defined in the usual way by turning around the poles corresponding to the internal lines of the loop (Fig. 3). To exhibit these poles,

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([8]) K. KIKKAWA, B. SAKITA and M. VIRASORO: Madison preprint No. COO-224.

it is convenient to retain the momentum integration. We notice that

$$\frac{1}{[-\log x_1x_2]^{q+2}}\exp\left[-bs\,\frac{\log x_1\log x_2}{\log x_1x_2}\right]=\int\limits_{-\infty}^{+\infty}\mathrm{d}k_0\int\limits_0^{\infty}k^2\,\mathrm{d}k\,x_1^{(K+\Pi)^2/2}x_2^{(K-\Pi)^2/2}\,\frac{2k^{2q}}{\Gamma(\frac{1}{2})\Gamma(q+\frac{3}{2})}\,;$$

here two-dimensional vectors are used $K(k_0, k)$, $\Pi(\sqrt{2bs}, 0)$. In terms of four-vectors, this means that we are in the rest system of the incoming resonance and that we have performed the angular integration for the internal momentum. If one defines the following expansion:

$$H(x_1, x_2; s, P)=[(1-x_1)(1-x_2)]^{-c}\exp\left[-2bs\sum_n\frac{1}{n}\,\frac{x_1^n+x_2^n-2x_1^nx_2^n}{1-x_1^nx_2^n}+4\sum_n\frac{x_1^nx_2^n}{1-x_1^nx_2^n}\right]\cdot$$

$$\cdot\left[\sum_n n\,\frac{x_1^n-x_2^n}{1-x_1^nx_2^n}\right]^P=\sum_{i>P}\sum_{j\leqslant i}(x_1^ix_2^j+x_2^ix_1^j)\,C_{ij}(s, P)\,.$$

the integration over x_1, x_2 is easily done and the poles of $F(s)$ are explicit:

$$F(s)=\int\limits_{-\infty}^{+\infty}\mathrm{d}k_0\int\limits_0^{\infty}k^2\,\mathrm{d}k\sum_{P=0}^{\infty}\binom{\alpha(s)}{P}\frac{2[k^2]^{\alpha(s)-P}}{\Gamma(\frac{1}{2})\Gamma(\alpha(s)-P+\frac{3}{2})}\cdot$$

$$\cdot\sum_{ij}\frac{c_{ij}(s, P)}{[(K+\Pi)^2/2+i-a][(K-\Pi)^2/2+j-a]}\,.$$

The integers i and j indicate the mass of the intermediate resonances of the loop, and $\alpha(s)-P$ the orbital angular momentum of the intermediate system.

The imaginary part is obtained with the usual prescription of replacing propagators by δ-functions:

$$k^2=\frac{1}{4bs}\left[bs-(\sqrt{i+bM^2}+\sqrt{j+bM^2})^2\right]\left[bs-(\sqrt{i+bM^2}-\sqrt{j+bM^2})^2\right].$$

For fixed s, there is a finite number of intermediate states, the sum over (i, j) stops at $bs-(\sqrt{i+bM^2}+\sqrt{j+bM^2})^2=0$

$$\operatorname{Im}F(s)=\sum_{i=0}^{i_{\max}(s)}\sum_{j<i}^{j_{\max}}\sum_{P=0}^{i}\frac{(k^2)^{\alpha(s)-P+\frac{1}{2}}}{\sqrt{\pi s}}\,\frac{\Gamma(\alpha(s)+1)}{\Gamma(P+1)\Gamma(\alpha(s)-P+1)\Gamma(\alpha(s)-P+\frac{3}{2})}\,c_{ij}(s, P)\,.$$

We are interested in the asymptotic behaviour of $\operatorname{Im}F(s)$. If one studies the variation of $\operatorname{Im}F(s)$ when s increases from s to $s+\mathrm{d}s$, one sees that the opening of new channels gives no contribution to the lowest order in $\mathrm{d}s$, due to the factor $[k^2]^{\alpha(s)-P+\frac{1}{2}}$.

For s large, one can forget about the terms bM^2:

$$\operatorname{Im}F(s)=\sum_{(i-j)=0}^{bs/2}\;\sum_{(i+j)=i-j}^{(i-j)^2/2bs+bs/2}\;\sum_{P=0}^{i}\left\{\left(1-\frac{2(i+j)}{bs}+\frac{(i-j)^2}{b^2s^2}\right)\left(\frac{bs}{bs-P}\right)^2\frac{e}{4}\right\}^{bs-P}\cdot$$

$$\cdot\left(\frac{bs}{e}\right)^P\frac{1}{P!}\,C_{ij}(s, P)\,.$$

What is needed is the behaviour of $C_{ij}(s, P)$ for s large. One has

$$\sum_{i,j} C_{ij} x_1^i x_2^j = H(x_1, x_2; sP) .$$

H is a decreasing function of s when P is fixed, so the terms C_{ij} may be either positive or negative. H has a singularity for $x_1 = x_2 = 1$ and it would be possible to know, for fixed s, the behaviour of C_{ij} for large i and j. But the asymptotic behaviour in s for i and j smaller than s, is a more difficult problem. All that we can say is that it is not obvious that $\operatorname{Im} F(s)$ increases faster than a polynomial.

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